A comparative numerical study of fractional derivatives and their role in capturing memory effects in viscoelastic materials

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Fractional calculus is a crucial mathematical tool for understanding complex systems across various fields, such as physics, engineering, and biology. This work delves into the intricate task of computing fractional-order derivatives, which typically necessitates numerical methods due to their inherent complexity. We present a comprehensive comparative study of three significant fractional derivatives-Riemann-Liouville, Grünwald-Letnikov, and Caputo-computed using appropriate numerical techniques. The study begins with a thorough examination of fractional-order derivatives, highlighting their fundamental importance in modern scientific research. We investigate basic transcendental functions -sine, cosine, and exponential- due to their vital role in modeling real-world systems. Our research reveals the complex relationship between sensitivity and precision by consistently computing fractional derivatives across various orders. Principal Component Analysis (PCA) is employed for statistical validation to enhance the credibility of our findings, demonstrating that fractional derivatives can reliably represent underlying data patterns. Consequently, this study significantly advances our understanding of fractional-order derivatives and underscores the importance of carefully handling fractional-orders in diverse applications. These findings support the reliability of data analytics and deepen our comprehension of the behavior of complex systems. An important application of this work lies in the modeling and control of viscoelastic materials, where fractional calculus provides a powerful framework for accurately capturing memory-dependent behaviors and complex stress-strain relationships.

Keywords: Fractional calculus; Riemann-Liouville derivative; caputo derivative; Grünwald-Letnikov derivative; numerical methods; transcendental functions; principal component analysis; viscoelastic materials; memory-dependent systems; stress-strain modeling.

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1. Introduction

Fractional calculus has emerged as a powerful mathematical tool with important implications for understanding complex systems defying traditional integer-order calculus. These complex systems, which may be found in many disciplines including biology, engineering, and physics, have dramatically increased scientific attention [1]. The main challenge in employing fractional calculus is computing fractional-order derivatives, which often defy analytical solutions. This challenge has led to the development of several numerical methods, including the Grünwald-Letnikov, Riemann-Liouville, Caputo, Adams-Bashforth-Moulton, and predictor-corrector approaches [2], designed to compute fractional-order derivatives efficiently and accurately. The primary objective of this research is to present a comprehensive comparative analysis of widely used numerical methods for computing fractional-order derivatives [3]. This study does not derive new analytical solutions; instead, it evaluates numerical performance by comparing the computed results with established analytical benchmark formulations available in the literature [4]. Three basic types of fractional derivatives are studied here: firstly, the Riemann-Liouville derivative, secondly, the Grünwald-Letnikov derivative, and thirdly, the Caputo derivative [5]. The fractional derivative concept is introduced by modifying the general integer order derivative framework using the most commonly used definition of fractional derivatives known as the Riemann-Liouville derivative [6]. However, the Grünwald-Letnikov derivative also provides a discrete manner of obtaining the fractional derivatives by using the weighted-sum structural form [7]. The Caputo derivative, also known as the fractional derivative, on initial conditions,

improves on the traditional derivative by including initial conditions within its framework [8]. In the process of comparing the two, this research seeks to uncover some of the strengths and weaknesses of each numerical approach in estimating fractional-order derivatives. Scientists and engineers willing to apply fractional calculus in their field of specialization will find this knowledge very vital. The journey undertaken in this research analysis starts with the introduction of fractional-order derivatives as mathematical concepts and explores their importance to current scientific discovery [9]. As will be seen when analyzing the variant trigonometric functions subdivided principally into the exponential function and the sine and cosine functions, this knowledge is presupposed [10]. Previous studies have established reliable numerical schemes and analytical benchmarks for fractional-order derivatives, including standard test functions such as sine, cosine, and exponential functions [11-13]. The study computes fractional-order derivatives for orders 0.5, 0.6, 0.7, 0.8, 0.9, and 1.0 for the functions $\sin(x)$, $\cos(x)$, and $\exp(x)$, serving as a valuable probe into the accuracy and sensitivity of various numerical approaches [14]. While numerous past finals can be solely hypothetical, this one focuses on how fractional calculus is applied to deal with the complicated kinetic systems in the real world. The fascinating feature of fractional-order derivatives is that they provide a better and more accurate description of phenomena that are difficult to describe using standard integer order derivatives. While ordinary calculus does not capture the details present in systems described by non-local responses, fractal features, and memory effects, fractional calculus can do so. As a result, it has the potential to transform many areas, namely, physics, engineering, biology, economics, and more, there is a considerable application of fractional calculus [15-18]. A significant aspect of this study effort is the inclusion of fractional-order derivative issues related to trigonometric functions. The goal here is to approximate the fractional-order derivatives of sine, cosine, and exponential functions at various fractional-orders to compare the sensitivity of the functions and the differences and other features required when taking fractional-order derivatives. This analytical approach has the potential to significantly enhance the knowledge of fractional calculus and pave the way for new creative ideas concerning innovative application areas of fractional calculus where accurate models and analyses are required [19]. To sum up, while theoretical investigation also forms the theoretical framework for this study, the research proposed here is far from a purely theoretical investigation. This article aims to fill this gap by implementing the fractional-order derivatives in complex systems analysis and with the real data. This research work extends the fractional-order derivative and its real-life application to a variety of scientific and applicative domains by establishing a platform for subsequent research studies that may enrich and broaden our understanding of this useful mathematical manipulator. In particular, fractional calculus plays an important role in areas such as viscoelastic material modeling, anomalous diffusion, signal processing, and control theory, where memory-dependent behavior must be accurately represented [20-22]. Finally, to guide the reader, the structure of this manuscript is outlined as follows: Section 2 presents the mathematical background of the fractional-order derivatives used; Section 3 describes the numerical methods and computational procedures; Section 4 provides the comparative results and discussion; and Sec. 5 concludes the study and highlights future research directions. It is also established that to further enhance the progression of this topic, future investigations may examine more complex techniques, other types of numerical solutions, or deeper involvement of the fractional-order derivatives in different disciplines.

2. Problem background literature

Fractional calculus was, in fact, first introduced at the beginning of the eighteenth century when Euler, Johann Bernoulli, as well as Laplace, considered extending the differentiation or integration order to a non-integer value [23]. Yet, the topic remained almost dormant until the late 1960s when some mathematicians reawakened interest in the topic by analyzing it, including Anatoly N. Kolmogorov, Alexander M. Ostrowski, and Anatoly M. Samoilenko . Interest in fractional calculus has recently begun experiencing a surge due to its versatility. Notably, applications of the concept include biosystem dynamics, control and automation systems, and viscoelastic properties. Moreover, this paper has demonstrated that the fractional-order (FO) integral operator has applications in signal processing, image processing, and data analysis [24]. Several numerical methods have been developed to compute fractional-order derivatives, reflecting their importance and broad applicability, including Grünwald-Letnikov, Riemann-Liouville, Caputo, Adams-Bashforth-Moulton, and predictor-corrector methods [25-28]. However, perhaps the most intimidating difficulty in the study of fractional calculus is the approximation of the fractional-order derivatives since these are often very difficult to come by in their closed-form exact forms. As a result, several numerical methods have been proposed for the computation of fractional-order derivatives. These methods can be broadly classified into two categories: non-spectral methods and spectral methods. Spectral techniques provide approximate solutions for the fractional-order derivatives and involve the expansion of the function in the spectral domain [29]. The methods that use this spectrum are: the Legendre polynomial method, the Chebyshev polynomial method, and the Fourier series method [30]. However, non-spectral approaches differ in that finite differences and finite element methods are used for the approximation of fractional-order derivatives. Of all the non-spectral techniques, the fractional-order derivatives used most often include the Riemann-Liouville, Grünwald-Letnikov, and Caputo derivatives [31,32]. The integral operator sets up the Riemann-Liouville derivative, which is an easier concept to understand and is but an extension of the normal integer order derivative. The Grünwald-Letnikov fractional differential is a discrete definition of the fractional-order derivative, uti-

lizing a weighted sum of function values. Caputo's differentiation gets to a great extent of sophistication of the classical one, which includes beginning circumstances. Many papers have been published in which various numerical methods have been compared concerning their effectiveness and efficiency in approximating the fractional-order derivatives [33-35]. Researchers considered the computing complexity, stability, and convergence rates of the described techniques as outstanding. These works have provided insights as to the advantages and disadvantages of various numerical methods and have enriched the initiation of improved techniques for approximating fractional derivatives. Therefore, in recent years, fractional calculus has received significant attention due to the versatility of its applications. However, even with the emergence of some numerical algorithms to deal with this problem, the computation of fractional-order derivatives is still a cumbersome task. Out of these methods, the Riemann-Liouville, Grünwald-Letnikov, Letnikov, and Caputo derivatives are the most used definitions of fractional-order derivatives. A lot of work has been done to understand how effective different numerical methods are for solving different problems, and hence, the area of fractional calculus has not been left out.

3. Methodlogy

3.1. Materials and methods

1. **Test Functions**: Give the names $\sin(x)$, $\cos(x)$, and e^x to the three test functions in that sequence. The numerical algorithms used in this work are evaluated considering the following test functions

   (a) Function 1: $f(x) = \sin(x)$.

   (b) Function 2: $f(x) = \cos(x)$.

   (c) Function 3: $f(x) = e^x$.

2. **Numerical Illustrations**: In this work, we investigate numerical methods for computing fractional-order derivatives and conduct a comparative study of their performance, including the Riemann-Liouville, Grünwald-Letnikov, and Caputo approaches.

3.2. Riemann-Liouville fractional-order derivative

The Riemann-Liouville fractional-order integral operator extends the concept of Cauchy's formula to incorporate multiple integrals [36].

$$\int_a^x dx_1 \int_a^{x_1} dx_2 \cdots \int_a^{x_{n-1}} f(x_n)dx_n = \frac{1}{(n-1)!}\int_a^x \frac{f(\eta)}{(x-\eta)^{1-n}}d\eta, \quad (1)$$

if $f(\eta) \in C^1[a,b]$ and $\alpha > 0$, then

$$I^{\alpha}_{a^+} f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \frac{f(\eta)}{(x-\eta)^{1-\alpha}}d\eta\,; \qquad x > a, \quad (2)$$

$$I^{\alpha}_{b^-} f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b \frac{f(\eta)}{(x-\eta)^{1-\alpha}}d\eta\,; \qquad x < b. \quad (3)$$

The left and right sides use the Riemann-Liouville fractional integral operator of order α. For the numerical computation of the Riemann-Liouville fractional derivatives used in this study, we employ a quadrature-based approximation in which the integral kernel $(x-\eta)^{-\alpha}$ is discretized using a uniform grid. The function values are approximated by a piecewise linear interpolant, and the resulting weighted sum provides a stable numerical estimation of the Riemann-Liouville operator. This discretization parallels the modified trapezoidal rule and is consistent with the implementation described later in Sec. 3.4.

Definition 1. *Let $\alpha \in (0,1)$, then*

$$D^{\alpha}_a f(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x \frac{f(\eta)}{(x-\eta)^{\alpha}}d\eta = DI^{1-\alpha}_a f(x). \quad (4)$$

The phrase "left side" is commonly used when both the right side and the Riemann-Liouville fractional derivative of order α are present.

Definition 2. *Let $n-1 < \alpha \leq n$ then for order α, the left and right-sided Riemann−Liouville fractional-order derivatives are defined as:*

$$D^{\alpha}_{a^+} f(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d^n}{dx^n}\int_a^x \frac{f(\eta)}{(x-\eta)^{\alpha+1-n}}d\eta = D^n I^{n-\alpha}_{a^+} f(x) \qquad x > a, \quad (5)$$

$$D^{\alpha}_{b^-} f(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d^n}{dx^n}\int_x^b \frac{f(\eta)}{(x-\eta)^{\alpha+1-n}}d\eta = D^n I^{n-\alpha}_{b^-} f(x) \qquad x < b. \quad (6)$$

The constant's fractional Riemann-Liouville derivative is greater than zero.

$$D^{\alpha}C = \frac{C\eta^{-\alpha}}{\Gamma(1-\alpha)} \neq 0, \quad (7)$$

An initial value problem (IVP) with a Riemann-Liouville fractional derivative must have initial conditions of the form $D^{\alpha-j}f(0)$, *i.e.,*

$$I^{\alpha}\left(D^{\alpha}f(\eta)\right) = f(\eta) - \sum_{j=1}^{n} D^{\alpha-j}f(0)\frac{\eta^{\alpha-j}}{\Gamma(\alpha-j+1)}, \quad n-1 \leq \alpha < n \quad (8)$$

To transcend these limitations, which rule out any chance of referring to factual occurrences, M. Caputo introduced a revolutionary derived concept. It solely employs boundary values of the full-order derivative at the lower terminal to obtain the initial conditions in the fractional IVPs.

3.3. Grünwald-Letnikov fractional-order derivative

Also well known in many application fields, the Grünwald-Letnikov derivative is a highly effective method to estimate fractional-order derivatives of functions. Due to the applicability of numerical computations, the dynamics of systems with non-integer order is possible, for instance, viscoelastic materials and the anomalous diffusion processes. This method is very useful in most areas of applied mathematics, including fractional differential equations in physics and engineering, control of fractional systems, fractal geometry, and fractals. The Grünwald-Letnikov fractional-order derivative provides a flexible tool for modeling and describing non-integer order processes and remembering in simulations. This can be used in different fields that may concern complicated heat conduction patterns, the control of a set of robotic systems, tracking of pollutant dispersion, or the study of natural fractals. The Grünwald-Letnikov fractional derivative is defined using a finite difference quotient and is expressed as follows: [37]

$$f^{(1)}(\eta) = \lim_{h\to 0} \frac{f(\eta+h) - f(\eta)}{h}, \tag{9}$$

$$\begin{aligned} f^{(2)}(\eta) &= \lim_{h\to 0} \frac{f^{(1)}(\eta+h) - f^{(1)}(\eta)}{h} \\ &= \lim_{h\to 0} \frac{f(\eta+2h) - 2f(\eta+h) + f(\eta)}{h}, \end{aligned} \tag{10}$$

and so on generally,

$$\begin{aligned} f^{(n)}(\eta) &= D^n f(\eta) \\ &= \lim_{h\to 0} \frac{1}{h^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(\eta - kh), \end{aligned} \tag{11}$$

where,

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \tag{12}$$

is a coefficient of a binomial. We can write a non-integer $\alpha > 0$ as

$$\binom{\alpha}{k} = \frac{\Gamma(\alpha+1)}{k!\Gamma(\alpha-k+1)}. \tag{13}$$

The Grünwald-Letnikov definition is the extension of the concept to a non-integer $\alpha > 0$.

$$\begin{aligned} D_a^{\alpha} f(\eta) = \lim_{h\to 0} \frac{1}{h^{\alpha}} \sum_{k=0}^{\frac{\eta-a}{h}} \\ \times (-1)^k \frac{\Gamma(\alpha+1)}{k!\Gamma(\alpha-k+1)} f(\eta - kh). \end{aligned} \tag{14}$$

Fractional integral of order $\alpha > 0$ is defined by

$$D_a^{\alpha} f(\eta) = \lim_{h\to 0} \frac{1}{h^{\alpha}} \sum_{k=0}^{\frac{\eta-a}{h}} (-1)^k \frac{\Gamma(\alpha+1)}{k!\Gamma(\alpha)} f(\eta - kh). \tag{15}$$

In this work, the Grünwald-Letnikov derivative is computed numerically using its standard discrete form:

$$D^{\alpha} f(\eta) \approx \frac{1}{h^{\alpha}} \sum_{k=0}^{N} (-1)^k \binom{\alpha}{k} f(\eta - kh), \tag{16}$$

where h is the step size and $\binom{\alpha}{k}$ denotes the generalized binomial coefficients. This finite-difference-based scheme is well suited for uniform grids and provides a direct numerical realization of the Grünwald-Letnikov definition used throughout our simulations [38].

3.4. Caputo fractional-order derivative

An illustration of a Caputo fractional-order Derivative is an example of a Caputo Fractional-order derivative. Let $f \in C^n[a,b]$ and $n-1 < \alpha < n$, then

$$\begin{aligned} D_a^{\alpha} f(x) &= DI^{n-\alpha} D^n f(x) \\ &= \frac{1}{\Gamma(\alpha-n)} \int_a^x \frac{f^{(n)}(\eta)}{(x-\eta)^{(}\alpha-n+1))} d\eta, \\ & a < x < b. \end{aligned} \tag{17}$$

Properties

$$\begin{aligned} D_a^{\alpha} C = 0, \qquad C \quad \text{is constant,} \\ \lim_{\alpha\to n} cD_a^{\alpha} f(x) = f^n(x) \end{aligned}$$

The numerical algorithm described in this subsection is consistent with the discretization strategies introduced in subsections 3.2 and 3.3.

The algorithm for the approximate calculation of Caputo fractional derivatives of any non-negative order $\alpha > 0$ associated with a given function is described. The estimation of the function values and their first derivative is performed using a weighted sum at predetermined points. The fundamental definition of the Caputo derivative and the numerical approximation method employed are explicitly based on the modified trapezoidal rule [39].

$$\int_a^b f(\xi) d\xi \approx \mathcal{T}(f,h), \tag{18}$$

where,

$$\mathcal{T}(f,h) = \frac{h}{2}\left(f(a) + f(b)\right) + h \sum_{k=1}^{M-1} f(\xi_k). \tag{19}$$

This is an estimate of the integral of $f(\xi)$ over $[a,b]$.

It is also based on a modified trapezoidal rule such that

$$\mathcal{T}(f,h) = \frac{h}{2} \sum_{k=1}^{M} f(\xi_{k-1}) + f(\xi_k) \tag{20}$$

To estimate the fractional integral $J^{\alpha} f(x)$ of order $\alpha > 0$, we present an adaptation of the trapezoidal rule (20).

Theorem 1. *Assume that the interval $[0,a]$ is partitioned into k equal-width sub intervals $[x_j, x_{j+1}]$ of equal width $h = a/k$ by using the nodes $x_j = jh$ for $j = 0, 1, ..., k$. The revised trapezoidal rule*

$$\mathcal{T}(f,h,\alpha) = \left((k-1)^{\alpha+1} - (k-\alpha-1)k^{\alpha}\right)\frac{h^{\alpha}f(0)}{\Gamma(\alpha+2)} + \frac{h^{\alpha}f(a)}{\Gamma(\alpha+2)} + \sum_{j=1}^{k-1}\left((k-j+1)^{\alpha+1} - 2(k-j)^{\alpha+1} + (k-j+1)^{\alpha+1}\right)\frac{h^{\alpha}f(x_j)}{\Gamma(\alpha+2)}, \tag{21}$$

is equivalent to the fractional integral.

$$\left(\mathcal{J}^{\alpha}f(x)\right)(a) = \mathcal{T}(f,h,\alpha) - E_T(f,h,\alpha), \qquad a>0; \alpha>0. \tag{22}$$

Additionally, when $f(x) \in C^2[0,a]$, a constant C'_a exists that depends solely on a, leading to the erroneous expression $E_T(f,h,\alpha)$ taking the following form

$$|E_T(f,h,\alpha)| \leq C'_a||f''||_{\infty}a^{\alpha}h^2 = \mathcal{O}(h)^2. \tag{23}$$

Proof. According to the fractional integral

$$\mathcal{J}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-\tau)^{\alpha-1}f(\tau)d\tau, \qquad x>0. \tag{24}$$

We have

$$\left(\mathcal{J}^{\alpha}f(x)\right)(a) = \frac{1}{\Gamma(\alpha)}\int_0^a (a-\tau)^{\alpha-1}f(\tau)d\tau. \tag{25}$$

If $\tilde{f_k}$ is the piecewise linear interpolant for f with nodes picked $x_j; j = 0, 1, 2, \cdots, k$ then the following equations can be used.

$$\int_0^{t_k}(t_k-t)^{\alpha-1}\tilde{f}_k(t))dt = \sum_{j=0}^{k} a_{j,k}f(t_i), \tag{26}$$

where

$$a_{(j.k)} = \frac{h^{\alpha}}{\alpha(\alpha+1)}\begin{cases} (k-1)^{\alpha+1} - (k-1-\alpha)k^{\alpha}, & \text{for} \quad j=0, \\ (k-j+1)^{\alpha+1} + (k-j-1)^{\alpha+1} - 2(k-j)^{\alpha+1}, & \text{for} \quad 1 \leq j \leq k-1, \\ 1, & \text{for} \quad j=k, \end{cases} \tag{27}$$

and

$$\left|\int_0^{(t_k)}(t_k-t)^{\alpha-1}f(t)dt - \sum_{(j=0)^k} a_(j,k)f(t_i)\right| \leq C_{\alpha}||f''||_{\infty}t_k^{\alpha}h^2. \tag{28}$$

for some constant C_α depending only on α. We obtain

$$\int_0^a (a-\tau)^{\alpha-1}\tilde{f}_k(\tau)d\tau = \frac{h^{\alpha}}{\alpha(\alpha+1)}\Big\{(k-1)^{\alpha+1} - (k-\alpha-1)k^{\alpha})f(0) + f(a) + \sum_{j=1}^{k-1}((k-j+1)^{\alpha+1} - 2(k-j)^{\alpha+1} + (k-j-1)^{\alpha+1})f(x_j)\Big\}, \tag{29}$$

and

$$\left|\int_0^a (a-\tau)^{\alpha-1}f(\tau)d\tau - \int_0^a (a-\tau)^{\alpha-1}\tilde{(}f_k)(\tau)d\tau\right| \leq C_{\alpha}||f''||_{\infty}a^{\alpha}h^2. \tag{30}$$

In this case, $C'_a = C_\alpha/\Gamma(\alpha)$. The method's performance is unaffected by the parameter because it behaves quite similarly to the conventional trapezoidal rule. In particular, the modified trapezoidal rule (21) becomes the traditional trapezoidal rule (19) when $\alpha = 1$.

4. Computational details

This study compares the accuracy and efficiency of three major definitions of fractional-order derivatives for calculating fractional-order derivatives. These are Riemann-Liouville, Grünwald-Letnikov, and Caputo definitions. The following pseudo-code describes the computation steps for each method.

4.1. Computation of Riemann-Liouville fractional-order derivatives

The Riemann-Liouville method involves the use of Riemann-Liouville fractional integrals [40] to calculate fractional-order derivatives. The following pseudo-code outlines the computation steps for this method:

% Define the function you want to differentiate
$f = @(x)sin(x)$;
% Define the values of q for which you want to compute derivatives
$q_{values} = [0.5, 0.6, 0.7, 0.8, 0.9, 1]$;
% Define the point at which you want to compute the derivatives
$x = pi/4$; % Adjust as needed
% Define the step size $h = 0.01$; % Adjust for your desired precision
% Function to compute the fractional-order derivative using Riemann-Liouville definition
function derivative = fractional-derivative (q, f, x, h);
$n = length(x)$;
derivative = zeros$(1, n)$;
for $i = 1 : n$
$t = x(i)$;
$sum_{term} = 0$;
for $k = 0 : 100$ % Adjust the range for your desired precision
sum$_t$erm = sum$_t$erm + $((-1)^k)$ * gamma(q + 1) / (gamma(k + 1) * gamma(q - k + 1)) * f(t - k * h);
end
derivative(i) = $(1/h^q)$ * sum$_t$erm;
end
end
% Compute and print the fractional derivatives
for sin(x) for each q, for $q = q_$values
derivative = fractional$_d$erivative(q, f, x, h);
fprintf('$D^{\%}$.1f[sin(x)] at x = %.2f: %.6f', q, x, derivative);
end

In this study, we computed fractional-order derivatives of functions like sine, cosine, and exponential using the Riemann-Liouville definition. MATLAB was employed for implementation. First, the target functions were defined, and fractional-orders (ranging from 0.1 to 1 in increments of 0.1) were selected. The step size $'h'$ was adjusted for precision while avoiding numerical instability. The Riemann-Liouville definition was applied iteratively for each $'q'$ and $'x'$ combination. The resulting fractional-order derivatives were recorded for subsequent analysis. This methodology ensures the accurate computation of fractional derivatives.

4.1.1. Fractional-order derivative simulation comparison using trigonometric and exponential functions based on the Riemann-Liouville definition

Fractional-order derivatives for the sine $(\sin x)$, cosine $(\cos x)$ and (e^x) functions over orders ranging from 0.5 to 1.0 are presented in Fig. 1a), corresponding to Function 1 and Function 2 and Function 3, respectively. The fractional derivative results for the exponential function (e^x) are analyzed separately and are shown in Fig. 1b). Notably, the magnitude values exhibit enhanced sensitivity to fractional differentiation in the region between $x = 5$ and $x = 6$. An in-depth analysis of the exponential function e^x within this interval is therefore discussed exclusively using Fig. 1b), ensuring consistency between the textual description and the graphical representation. The fractional derivatives for orders 0.5, 0.7, 0.8, 0.9, and 1.0 were determined, and the key findings derived from Fig. 1a) and Fig. 1b) are discussed below.

- Values of the magnitude are important sensitivity indicators. The magnitudes of the 0.5 and 0.6 fractional-order derivatives were noticeably bigger than those of the first-order fractional derivative in the region between 5 and 6 on the x-axis. This indicates how keenly they are aware of even the smallest details and variations.
- Significantly responsive to fractional differentiation within this interval was the exponential function (ex), with magnitude values sharply increasing as the order dropped.
- The magnitude values of the fractional derivatives highlight their vulnerability even further. Lower-order fractional derivatives (0.5 and 0.7) yielded magnitude values that were significantly greater than the first-order derivative, indicating their exceptional sensitivity to minute oscillations and fluctuations.
- The 0.5 order derivative was particularly notable due to its unusually high magnitude, which demonstrated how sensitive it was to identify particular features of the exponential function.
- Conversely, higher-order fractional derivatives showed less sensitivity to localized features but continued capture of the function's broad trends, yielding values closer to the first-order derivative (0.9 and 1.0).

The study's findings demonstrate fractional calculus's exceptional sensitivity in identifying minute differences and characteristics inside functions. As shown by their magnitude values, lower-order fractional derivatives stand out as effective

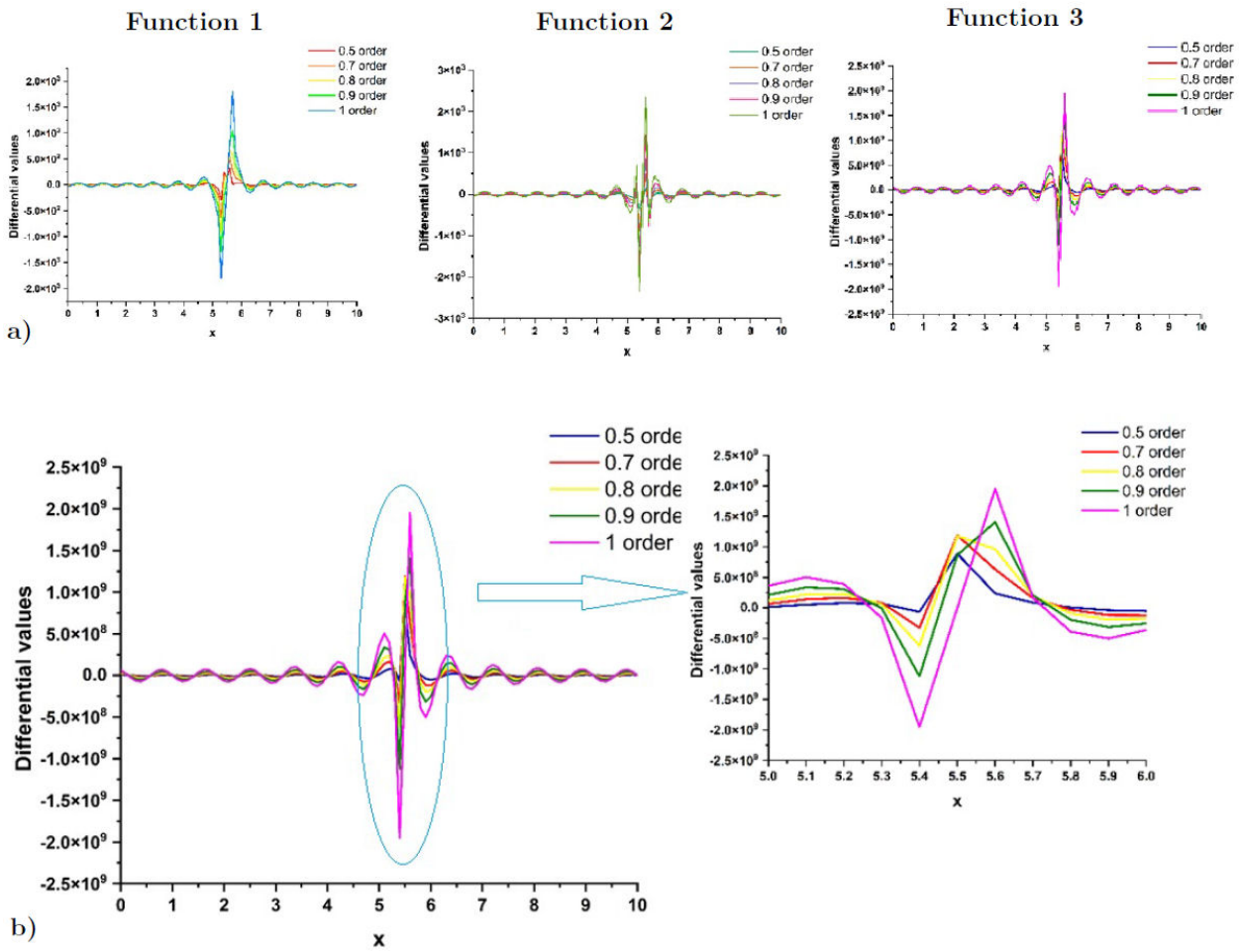


FIGURE 1. Numerical results of fractional-order derivatives computed using the Riemann-Liouville definition. (a) Fractional derivatives of sine (Function 1), cosine (Function 2), and exponential (Function 3) functions for orders $\alpha = 0.5 - 1.0$.(b) Enlarged view of the exponential function (Function 3) illustrating the sensitivity of the fractional derivative magnitude within the interval $x = 5 - 6$.

instruments for spotting fine-grained details, especially in regions with increased sensitivity.

4.2. Computation of Grünwald-Letnikov fractional-order derivatives

Fractional-order derivatives are computed using a recursive formula in the Grünwald-Letnikov method. The following pseudocode illustrates the computation processes for this method:

% Define the function
$f = @(x)exp(x)$;
% Define the order of the fractional derivative
$alpha = 0.5$;
% Define the lower and upper limits of the domain
$a = 0$;
$b = 1$;
% Define the number of points to use in the computation
$N = 100$;
% Define the step size
$h = (b - a)/(N - 1)$; % Define the time vector
$t = a : h : b$; % Compute the Grünwald-Letnikov fractional derivative
D$_a$lpha$_G$L$_f$ = zeros(1,N);
for $k = 1 : N$
D$_a$lpha$_G$L$_f$(k) = (1/gamma(1-alpha)) * sum(((-1)^m * nchoosek(alpha,m) * f(t(k)-m*h)) / (h^{alpha}) for m = 0:k-1);
End

To numerically compute the fractional-order derivatives of the provided function $f(x)$ at various places inside the chosen domain, this code essentially implements the Grünwald-Letnikov approach.

4.2.1. *fractional-order derivative simulation comparison with trigonometric and exponential functions using the Grünwald-Letnikov definition*

We looked at the fractional derivatives of some basic functions, including the sine ($\sin(x)$), cosine ($\cos(x)$), and the exponential function (e^x), as shown in Fig. 2. These results illustrate the flexibility of the fractional derivative approach and provide insight into the complex behavior of the func-

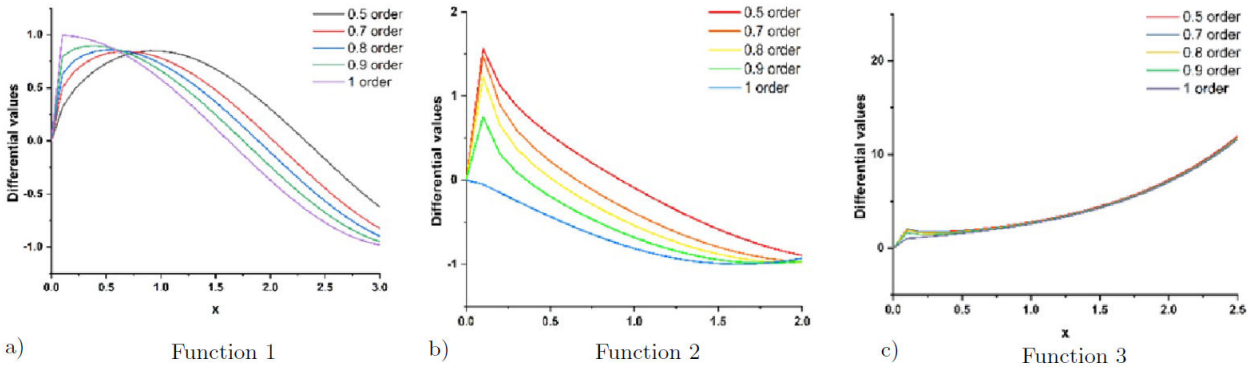


FIGURE 2. Numerical results of fractional-order derivatives computed using the Grünwald-Letnikov definition. a) Fractional derivatives of sine (Function 1), cosine (Function 2), and exponential (Function 3) functions for orders $\alpha = 0.5 - 1.0$.

tions as the fractional-order is varied between 0.5 and 1.0. From the results, it is evident that lower-order derivatives (especially the 0.5 order) exhibit higher magnitudes and increased sensitivity, capturing subtle changes and oscillatory behavior more effectively than higher-order or integer-order derivatives. It is also clear that different types of fractional derivatives behave differently compared to standard first-order derivatives. The 0.5 order derivatives and other lower-order fractional derivatives were the most sensitive, highlighting their ability to detect fine variations in the functions. Their much greater magnitude values provide evidence of this increased sensitivity and demonstrate how well they are capable of capturing nuances and detailed differences in complex environments. In particular, the comparison across different functions shows that the sine and cosine functions are more responsive to fractional differentiation than the exponential function, reflecting the intrinsic periodicity of trigonometric functions. This is testified to the first-order derivative's magnitude values, which were the lowest here, showing smoother reactions to changes in the functions. Fractional derivatives are seen to be very sensitive to the behavior of functions, and of all the presented orders, 0.5^{th}-order seemed to be the most sensitive. Such results have important consequences: fractional derivatives, which are more sensitive than integer-order ones, are valuable tools for extracting patterns and latent data; this is particularly relevant for data analysis and signal processing, where it is important to detect even the slightest changes.

4.3. Computation of caputo fractional-order derivatives

The Caputo technique makes use of a modified derivative to adjust for the function's original circumstances. The following pseudo-code explains the computation operations for this method.

% Define the function
$f = @(x)exp(x)$;
% Define the order of the fractional derivative
$alpha = 0.5$;
% Define the lower and upper limits of the domain
$a = 0$;
$b = 1$;
% Define the number of points to use in the computation
$N = 100$;
% Define the step size
$h = (b - a)/(N - 1)$;
% Define the time vector
$t = a : h : b$;
% Compute the Caputo fractional derivative
$D_a lpha_{C\ f}$ = zeros(1,N);
for $k = 1 : N$;
% $D_a lpha_{C\ f}$(k) = (1/gamma(1-alpha)) * integral
(@(s) (f(s)-f(a))./((s-a)$^{(1+alpha)}$),a,t(k)',
ArrayValued', true);
End

This MATLAB code computes the Caputo fractional derivative of a given function numerically across a specified domain, which is useful for analyzing and understanding the behavior of functions having fractional-order differentiation.

4.3.1. *Simulation comparison of fractional-order derivatives using the caputo formulation with trigonometric and exponential functions*

Our investigation of fractional-order derivatives for the sine, cosine, and exponential functions ($\sin x$, $\cos x$, and e^x) reveals important insights into the sensitivity of these functions with respect to different fractional-orders. The numerical results demonstrate that the sine and cosine functions exhibit pronounced sensitivity in the region $0 \leq x \leq 0.5$, where even small variations in the fractional-order lead to noticeable changes in the computed derivative values. This behavior indicates that fractional differentiation in this interval amplifies subtle variations in the underlying functions. As the order of differentiation is reduced from the classical first-order derivative ($\alpha = 1.0$) toward lower fractional-orders, particularly $\alpha = 0.5$, significant changes in the derivative response are observed. Among all tested orders, the 0.5-order derivative consistently exhibits the highest sensitivity, highlighting its effectiveness in capturing fine-scale variations that are not

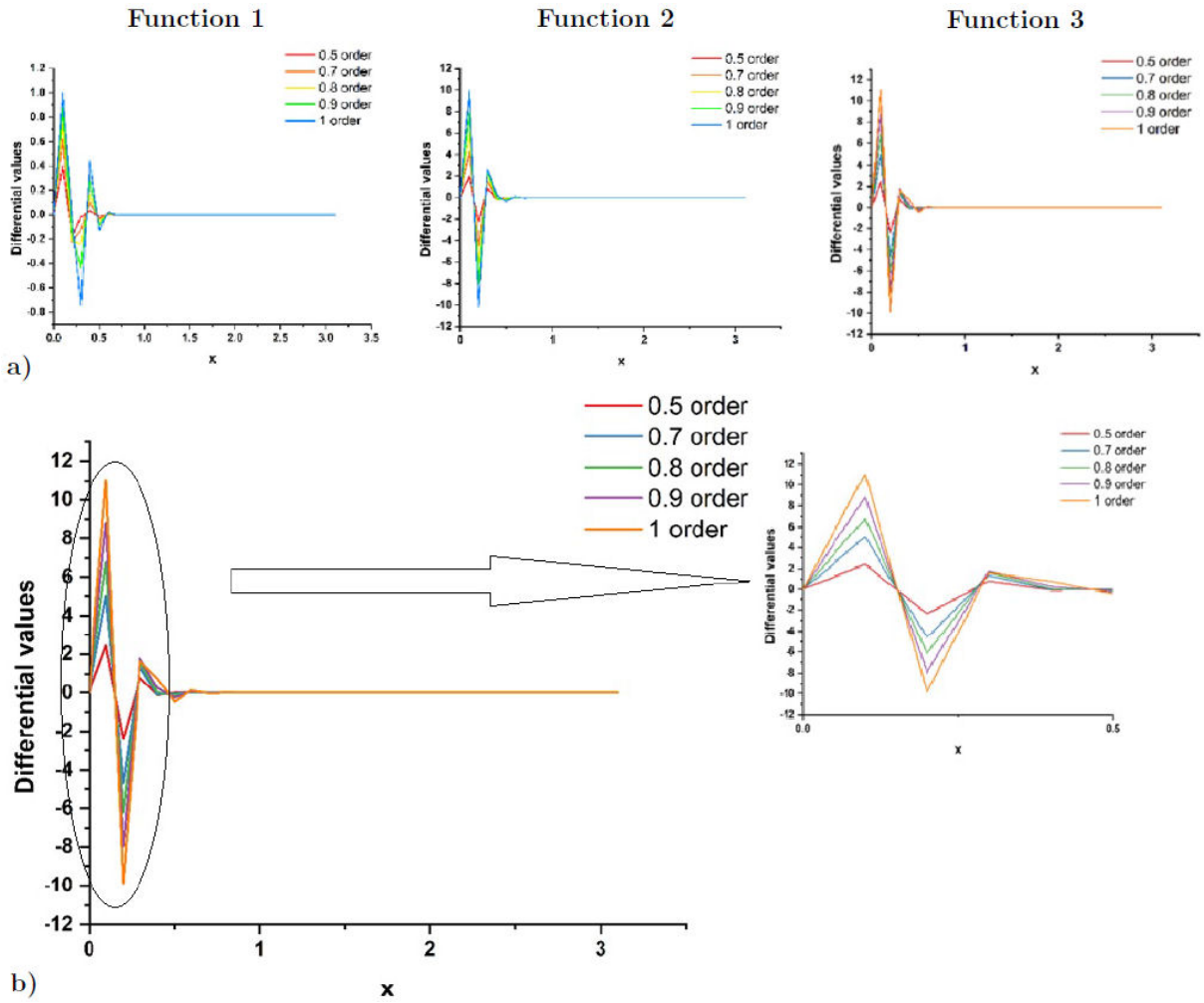


FIGURE 3. Fractional-order derivative (FD) simulation analysis using the Caputo definition. a) Fractional derivatives of sine (Function 1), cosine (Function 2), and exponential (Function 3) functions for orders $\alpha = 0.5 - 1.0$. b) Enlarged view of the exponential function (Function 3) illustrating the sensitivity of the fractional derivative magnitude within the interval $x = 0 - 0.5$.

detectable using integer-order differentiation. The exponential function (e^x) is found to be even more sensitive to changes in the fractional-order compared to the trigonometric functions. As the fractional-order decreases, the magnitude of the computed derivatives changes markedly, with the strongest sensitivity again occurring at $\alpha = 0.5$. This enhanced response becomes especially evident as the analysis moves away from the classical first-order derivative, confirming the superior resolving capability of lower-order fractional derivatives. These behaviors are clearly illustrated in Fig. 3a), which presents the fractional derivatives of the sine, cosine, and exponential functions over the range of fractional-orders from 0 to 1.0. To further highlight the sensitivity of the exponential function, Figure 3(b) provides an enlarged view within the interval $x = 0 - 0.5$, where the effect of fractional differentiation is significantly magnified. The graphical results confirm that lower-order derivatives, particularly at $\alpha = 0.5$, produce the most pronounced variations in derivative magnitude. In the context of the RLC circuit model, similar trends are observed, where the fractional-order $\alpha = 0.5$ yields the highest sensitivity to changes in the system response, even for small variations in the applied derivative order. This reinforces the conclusion that lower-order fractional derivatives are highly effective for modeling systems with memory and hereditary properties. Overall, the results presented in Fig. 3 emphasize the critical role of fractional-order selection in fractional calculus applications. The strong sensitivity associated with lower-order derivatives, especially $\alpha = 0.5$, underscores the necessity for precise control of the fractional-order in practical applications such as signal processing, engineering systems, and physical modeling.

5. Absolute value error analysis

In this detailed study, we examined the sensitivity of several Caputo derivatives to different functions and numerical parameters. Using various orders (0, 0.5, 0.7, 0.8, 0.9, and 1) with a step size (h) of 0.1 and 10 steps, we estimated the Caputo derivatives for the sine function (sin(x)), as shown in Table I. The absolute value errors, $Ec(f, h, \text{order})$, were

TABLE I. Caputo fractional derivatives for Function 1

Order of Derivative (α)	k	h	C (f,h,order)	E_C (f,h,order)
0^{th}	10.0000	0.1000	0.8408	0.0053
0.5^{th}	10.0000	0.1000	0.8454	0.0007
0.7^{th}	10.0000	0.1000	0.7665	0.0795
0.8^{th}	10.0000	0.1000	0.7059	0.1401
0.9^{th}	10.0000	0.1000	0.6305	0.2155
1^{st}	10.0000	0.1000	0.5403	0.3058

TABLE II. Caputo fractional derivatives for Function 2.

Order of Derivative (α)	k	h	C (f,h,order)	E_C (f,h,order)
0^{th}	2.0000	0.5000	0.8239	0.022
0.5^{th}	2.0000	0.5000	0.8299	0.00161
0.7^{th}	2.0000	0.5000	0.7548	0.0912
0.8^{th}	2.0000	0.5000	0.6971	0.1489
0.9^{th}	2.0000	0.5000	0.6256	0.2204
1^{st}	2.0000	0.5000	0.5403	0.3058

TABLE III. Caputo fractional derivatives for Function 3

Order of Derivative (α)	k	h	C (f,h,order)	E_C (f,h,order)
0^{th}	1.0000	1.0000	0.7702	0.0759
0.5^{th}	1.0000	1.0000	0.7826	0.0635
0.7^{th}	1.0000	1.0000	0.7202	0.1258
0.8^{th}	1.0000	1.0000	0.6719	0.1742
0.9^{th}	1.0000	1.0000	0.6119	0.2342
1^{st}	1.0000	1.0000	0.5403	0.3058

recorded after comparing the computed values, $C(f,h,order)$, with the genuine values. Notably, within this numerical framework, the most sensitive and accurate alternative for approximating the sine function was the 0.5^{th}-order Caputo derivative, which showed the minimum absolute value error (0.0007). However, the absolute value errors grew over time as we increased the Caputo derivative order beyond 0.5, showing a reduction in. A similar analysis is shown in Table II for the cosine function (cos(x)). In this instance, a greater step size of 0.5 was used with just 2 steps. Like Table I, the 0.5^{th}-order Caputo derivative showed the maximum sensitivity and the lowest absolute value error (0.0161), whereas higher-order derivatives showed decreasing accuracy. This inquiry is expanded to include the exponential function in Table III. Using several orders (0, 0.5, 0.7, 0.8, 0.9, and 1) with a step size of 1 and only one step, the Caputo derivatives for the exponential function were determined. The 0.5^{th}-order Caputo derivative once more demonstrated to be the most sensitive, with the lowest absolute value error (0.0635), while higher-order derivatives showed decreasing accuracy. In summary, while approximating distinct functions under varied numerical conditions, the 0.5^{th}-order Caputo derivative consistently appeared as the most sensitive and accurate option across all three tables. In these particular numerical conditions, higher-order derivatives showed progressively bigger absolute value mistakes, illustrating their decreased sensitivity and precision. This shows that using a 0.5^{th}-order Caputo derivative for related numerical problems may produce the most accurate outcomes while minimizing computational errors.

6. Sensitivity and stability analysis of fractional-order derivatives

To quantify the sensitivity of fractional-order derivatives, the correlation index (CI) method is employed. The correlation index measures the degree of linear relationship between two datasets, here between experimental functions and their simulated fractional derivatives. Mathematically, for two vectors X and Y of length n, the correlation coefficient (CC) is defined as:

$$CC(X,Y)=\frac{\sum_{i=1}^{n}(X_i-\bar{X})(Y_i-\bar{Y})}{\sqrt{\sum_{i=1}^{n}(X_i-\bar{X})^2}\sqrt{\sum_{i=1}^{n}(Y_i-\bar{Y})^2}}. \quad (31)$$

Here, $\bar{X}$ and $\bar{Y}$ are the mean values of X and Y, respectively. A CC value close to 1 indicates a strong positive correlation, whereas a value near 0 indicates a weak correlation. The correlation index provides a quantitative metric for evaluating how sensitive a function is to changes in the fractional derivative order. Lower-order derivatives are expected to capture subtle variations in the data more effectively, which can be confirmed through higher CC values.

6.1. Correlation index-based sensitivity analysis

However, the strength of correlation coefficients (CCs) is higher when evaluated with experimental functions $\sin(x)$, $\cos(x)$, and $\exp(x)$. To further analyze the effect of different fractional derivatives, the CCs were calculated separately for Riemann-Liouville, Grünwald-Letnikov, and Caputo derivatives. Specifically, CCs data of the 0th order spectrum, the primary objective of this study is to observe the behavior of CCs for functions $\sin(x)$, $\cos(x)$, and $\exp(x)$ in the application of differential filters. It is observed that the Caputo derivative produces slightly higher CCs in comparison to the Riemann-Liouville and Grünwald-Letnikov derivatives, indicating its better sensitivity for the current functions. To achieve a more robust and quantitative examination of functions $\sin(x)$, $\cos(x)$, and $\exp(x)$, it is essential to pinpoint the sensitive order of the differential transformation of functions responsible for refining the accuracy of the model. The depicted CCs values correspond to the functions $\sin(x)$, $\cos(x)$, and $\exp(x)$ are calculated. In Figs. 4a) - 4c), the strength of CCs for each axis, belonging to each order, is portrayed as a

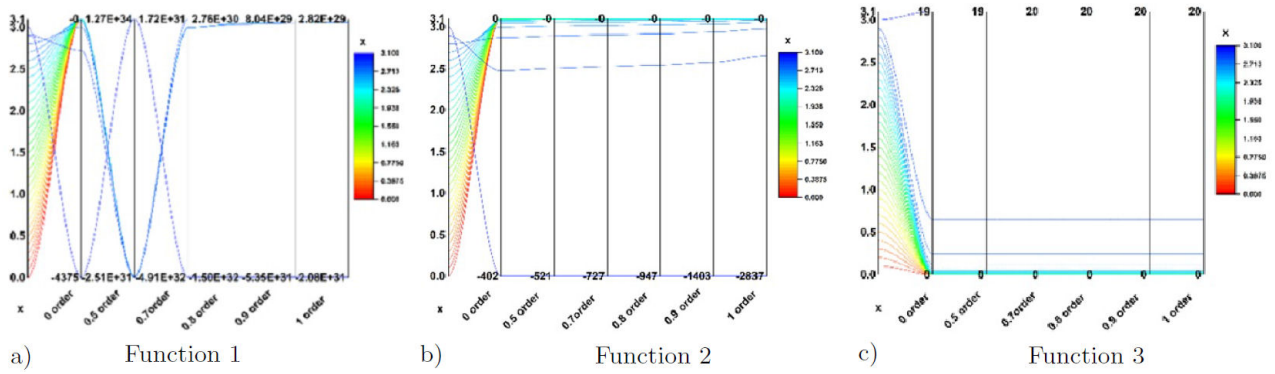


FIGURE 4. Principal Component Analysis of Functions (Function1, Function2, Function3) and simulated fractional derivatives (FD).

parallel plot, enhancing the visualization of point-to-point correlation between experimental and simulated fractional data, with annotations indicating which derivative type is represented in each subplot. In the data representations showcased in Figs. 4a) - 4c), a more convergent and robust distribution of CCs indicates a negligible contribution of quasi-static and background noise, with minimal loss of information during the transformation of experimental functions, and vice versa. Notably, Figs. 4a) - 4c) highlights that the strength of CCs for the differential transformation of the 0.5^{th} order is higher, with a more convergent distribution over the spectral range compared to other differential orders. This trend holds for all three sine $(\sin x)$, cosine $(\cos x)$, and exponential $(\exp(x))$ functions and the effect of the derivative type is also evident, with Caputo showing superior convergence. Furthermore, the strength and convergence of CCs exhibit a decreasing trend when moving towards higher orders of differentiation for all regions of interest (refer to Figs. 1 - 3). These findings affirm that the application of differential filters of lower order (0.5^{th} in the current study) is more appropriate due to the strength and point-to-point distribution of CCs across the functions $\sin(x)$, $\cos(x)$, and $\exp(x)$ of interest, while also highlighting the influence of the choice of fractional derivative type on sensitivity.

6.2. Statistical analysis using principal component analysis

Insightful conclusions have been drawn from our investigation utilizing Principal Component Analysis (PCA) on fractional derivatives of sine, cosine, and exponential functions computed using the Riemann-Liouville (RL) concept. Principal Component Analysis is a well-established multivariate statistical technique that transforms a set of possibly correlated variables into a new set of uncorrelated variables, called principal components, by projecting the data onto orthogonal directions that maximize variance. The first principal component captures the maximum variance in the dataset, while subsequent components represent decreasing amounts of variance and are mutually orthogonal. PCA is widely used for dimensionality reduction, feature extraction, and identification of dominant patterns in complex datasets. We found various patterns and behaviors in the fractional derivatives of these fundamental functions across a range of fractional-orders, from integer derivatives $0^{th} - order)$ to extremely non-integer derivatives $0.5^{th}, 0.7^{th}, 0.9^{th}$, and $1st$-order).

In the present study, PCA is employed to quantitatively assess the sensitivity of different fractional-orders by comparing their variance structure against the 0th-order (raw data) baseline. In Fig. 5 of our study, we gave a detailed examination of fractional-order derivatives obtained from the Riemann-Liouville (RL) definition applied to various functions. In particular, "Function 1" showed the sine function, "Function 2" the cosine function, and "Function 3" an exponential function. Surprisingly, "Function 3", which represents an exponential function, showed a distinct parabolic trajectory after being subjected to Principal Component Analysis (PCA) analysis. Notably, "Function 3," corresponding to the exponential function, exhibits a distinct parabolic trajectory in the PCA space. This behavior arises from the inherently non-linear and rapidly increasing nature of the exponential function, which leads to stronger variance contributions along higher principal components when compared to trigonometric functions. Under fractional differentiation, the exponential function preserves its monotonic growth while introducing scale-dependent curvature, causing the PCA projection to assume a parabolic structure. This contrasts with the oscillatory and bounded nature of sine and cosine functions, whose PCA representations remain more linear and symmetric. This trait made our analysis more sophisticated and interesting. Such parabolic trajectories in PCA space indicate nonlinear correlations and complex variance redistribution introduced by fractional differentiation. Our primary focus here was to test these fractional-orders computed using RL for sensitivity and relevance with the 0th order, which is often used as a baseline of the raw data. The parabolic behavior observed for "Function 3" therefore reflects the combined effect of exponential growth and fractional-order memory inherent in the RL operator, rather than numerical instability or anomaly. By examining the distribution of data points along the principal component axes, the deviation of each fractional-order from the original signal can be quantitatively

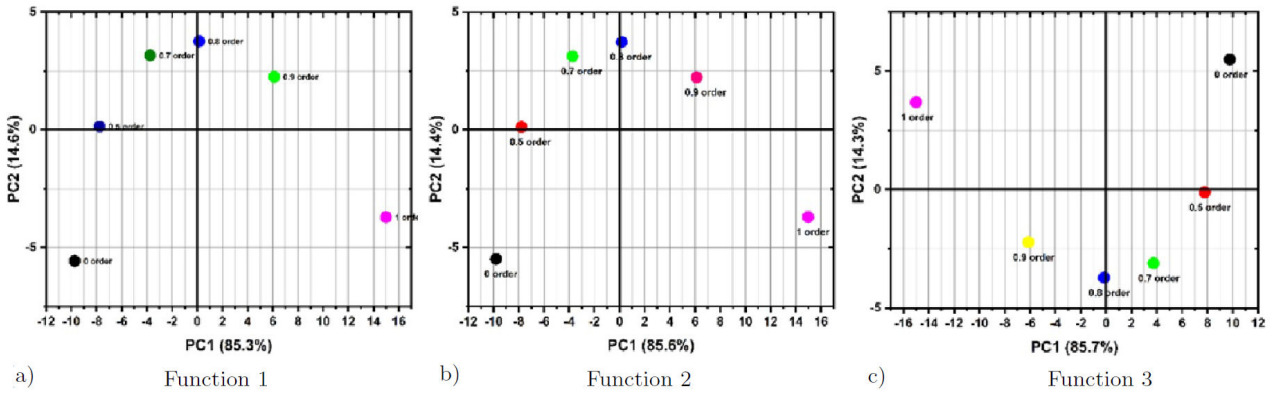


FIGURE 5. Coefficients of point-to-point correlations of experimental and simulated FD spectral data for a) $\cos(x)$, b) $\sin(x)$, c) $\exp(x)$.

evaluated. It was at this point that we understood that more validation was needed to increase the reliability of the given results. We then started using Principal Component analysis (PCA) to validate the findings of the first query and paid special attention to "Function 3" and the parabolic nature of its curve. A more thorough investigation of ratio values obtained for each RL-derived fractional-order at the x and y axes was made, and a shocking revelation was made. These ratios provided useful insights into the nature of each fractional-order derived by RL analysis for numerically quantifying the deviation from the 0th order. These ratios represent the relative contribution of each principal component and serve as quantitative indicators of similarity or deviation from the 0th-order reference signal. For the "Function 1" ($\sin x$), the fractional derivative, which has been obtained from the 0.5^{th}-order RL ratio, was 0.80. This near alignment showed that fractional-order effects can be introduced while maintaining a close mirror image of the underlying sine function, as stated by the RL approach. The 0.5^{th}-order RL-derived fractional derivative for "Function 2" (cosx) showed ratios of 0.79 on the x-axis and -0.02 on the y-axis, which are quite similar to the 0th order values of -9.79 and -5.48, respectively. Surprisingly, these results were extended to "Function 3," which, when PCA analysis was performed, showed an exponential function with a parabolic trajectory. The 0.5^{th}-order RL-derived fractional derivative showed ratios of 0.79 on the x-axis and -0.02 on the y-axis even in this complicated scenario, demonstrating its adaptability in capturing the underlying data patterns while accommodating the intriguing parabolic trajectory, all within the RL definition. The 1st order RL-derived fractional derivative, on the other hand, consistently showed higher absolute ratios on both axes, indicating more dramatic departures from the original data. Overall, the parabolic trajectory observed in the exponential function is a direct consequence of its nonlinear growth amplified by fractional-order differentiation, and its consistent behavior under PCA further validates the sensitivity and stability of the 0.5^{th}-order RL fractional derivative. This behavior reflects the stronger smoothing and loss of fine-scale information typically associated with higher-order differentiation.

In light of this, our thorough analysis, strengthened by the use of PCA, and highlighted by the parabolic trajectory in "Function 3," supports the finding that the 0.5^{th}-order RL-derived fractional derivative consistently demonstrates to be the most sensitive and significant choice for capturing variations within the data across different functions. Researchers now have a powerful tool for extracting nuanced information from complex datasets while preserving the key elements of the original data patterns, thanks to this fractional-order, which was derived using the RL definition and validated through statistical methods.

7. Application: modeling and control of viscoelastic materials using fractional calculus

7.1. Workflow: modeling and control of viscoelastic materials using fractional calculus

The following workflow outlines the step-by-step process for modeling and controlling viscoelastic materials using numerical fractional calculus:

1. **Data Collection:** Obtain experimental stress-strain data from viscoelastic material tests (*e.g.*, creep or relaxation experiments).
2. **Model Selection:** Choose an appropriate fractional viscoelastic model, such as the fractional Maxwell or Kelvin-Voigt model.
3. **Numerical Discretizations:**
   - Select a numerical method for the fractional derivative (Riemann-Liouville, Grünwald-Letnikov, or Caputo).
   - Discretized the model equations using the chosen method.
4. **Parameter Estimation:** Fit model parameters (*e.g.*, E, λ, η, α) to the experimental data using optimization techniques (*e.g.*, least squares).
5. **Model Validation:**

- Compare the model output with experimental results using error metrics (RMSE, R^2).
- Apply Principal Component Analysis (PCA) for statistical validation of model reliability.

6. **Controller Design (Optional):** Design and simulate a fractional-order controller (*e.g.*, fractional PID) using the validated model.

7. **Visualization and Reporting:** Present results with plots (stress/strain curves, PCA scree plots) and summarize findings.

7.2. Technical details and example: fractional modeling of viscoelastic materials

Viscoelastic materials are inherently characterized by memory effects, meaning that their present stress response depends not only on the instantaneous strain but also on the entire deformation history. Fractional calculus provides a natural and physically meaningful framework to model such hereditary behavior through nonlocal fractional derivatives, whose kernels explicitly account for long-range temporal memory.

7.2.1. Fractional viscoelastic model

Viscoelastic materials exhibit both elastic and viscous behavior, characterized by a history-dependent (memory) response, meaning that the current stress state depends not only on the instantaneous strain but also on its entire deformation history. This intrinsic memory effect is a defining feature of viscoelastic materials and cannot be adequately captured using classical integer-order differential models. Fractional calculus naturally incorporates memory through convolution-type operators with power-law kernels, making it particularly suitable for modeling viscoelastic phenomena. The fractional Maxwell model is given by:

$$\sigma(t) + \lambda^{\alpha} D_t^{\alpha} \sigma(t) = E D_t^{\alpha} \epsilon(t), \tag{32}$$

where $\sigma(t)$ is the stress, $\epsilon(t)$ is the strain, E is the elastic modulus, λ is a material parameter, and D_t^{α} denotes the fractional derivative of order $0 < \alpha < 1$. The fractional derivative D_t^{α} introduces a memory kernel that governs how past strain and stress values contribute to the present response. Lower values of α correspond to stronger long-term memory effects, while values approaching unity reduce the model toward classical, memory-less viscoelastic behavior.

7.2.2. Grünwald-Letnikov Fractional Derivative

The Grünwald-Letnikov (GL) definition is well-suited for numerical computation, especially with discrete experimental data:

$$D_t^{\alpha} f(t_n) \approx \frac{1}{h^{\alpha}} \sum_{k=0}^{n} (-1)^k \binom{\alpha}{k} f(t_{n-k}), \tag{33}$$

where h is the time step and $\binom{\alpha}{k}$ is the generalized binomial coefficient:

$$\binom{\alpha}{k} = \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)}. \tag{34}$$

```python
import numpy as np
from scipy.Special import gamma

def grunwald_letnikov(f, alpha, h):
    N = len(f)
    coeffs = [((-1)**k) * gamma(alpha+1)
    / (gamma(k+1) * gamma(alpha-k+1))
    for k in range(N)]
    D_alpha_f = np.zeros(N)
    for n in range(N):
        s = 0
        for k in range(n+1):
            s += coeffs[k] * f[n-k]
        D_alpha_f[n] = s / h**alpha
    return D_alpha_f

# Example usage:
# f = np.array([...])
# your sampled function values
# alpha = 0.5
# fractional order
# h = 0.01
# time step
# result = grunwald_letnikov
          (f, alpha, h)
```

LISTING 1. `Python` implementation of the Grünwald-Letnikov fractional derivative.

To ensure reproducibility and to clarify the numerical implementation used in this study, a Python-based algorithm for computing the Grünwald-Letnikov (GL) fractional derivative is presented in this subsection. The listed code evaluates the fractional derivative of a discrete signal f of length N for a given fractional-order α and step size h. The algorithm is based on the classical Grünwald-Letnikov definition, where the fractional derivative is approximated through a weighted summation of past signal values using generalized binomial coefficients involving the Gamma function. Specifically, the function `Grünwald_letnikov(`f`,`α`,`h`)` returns the numerical approximation of the fractional derivative $D^{\alpha} f$, which is subsequently used in this work to generate simulated fractional-order responses of sine, cosine, and exponential functions. These simulated fractional derivatives form the basis for the comparative analysis, sensitivity evaluation, and statistical investigations (correlation index and principal component analysis) discussed in Secs. 4-6. By varying the fractional-order α between 0 and 1, the code enables a controlled analysis of how fractional differentiation enhances sensitivity to subtle signal variations while preserv-

ing the essential structural characteristics of the original functions.

7.2.3. Numerical example

Suppose we have strain data $\epsilon(t_n)$ sampled at uniform intervals h, and we wish to compute the fractional derivative of order $\alpha = 0.5$.

1. **Compute the GL coefficients:**

$$c_k = (-1)^k \binom{0.5}{k}, \tag{35}$$

for $k = 0, 1, \ldots, n$.

2. **Compute the fractional derivative:**

$$D_t^{0.5}\epsilon(t_n) \approx \frac{1}{h^{0.5}} \sum_{k=0}^{n} c_k \epsilon(t_{n-k}). \tag{36}$$

3. **Insert into the fractional Maxwell model:**

$$\sigma(t_n) + \lambda^{0.5} D_t^{0.5}\sigma(t_n) = E D_t^{0.5}\epsilon(t_n). \tag{37}$$

The numerical implementation explicitly demonstrates how memory effects are incorporated through weighted contributions of past strain values, with the decay rate governed by the fractional-order α and the chosen derivative definition. This equation can be solved numerically for $\sigma(t_n)$ given $\epsilon(t_n)$ and fitted to experimental data to estimate E, λ, and α.

The numerical implementation explicitly demonstrates how memory effects are incorporated through weighted contributions of past strain values, with the decay rate governed by the fractional-order α and the chosen derivative definition.

7.2.4. Model validation using principal component analysis

To statistically validate the model:

- Construct a data matrix with columns representing experimental stress data and model-predicted stress for various parameter sets.
- Apply Principal Component Analysis (PCA) to this matrix.
- If the model output aligns with the dominant principal component(s), the model reliably captures the essential features of the viscoelastic response.

By comparing PCA projections obtained using RL, Caputo, and GL-based formulations, the influence of different memory representations on model fidelity and sensitivity can be quantitatively assessed.

7.3. Summary

This workflow demonstrates how the Grünwald-Letnikov numerical technique can be integrated into fractional viscoelastic modeling and validated using PCA. Moreover, the comparative inclusion of Riemann-Liouville, Caputo, and Grünwald-Letnikov derivatives establishes a unified framework for analyzing memory effects in viscoelastic materials, directly aligning the application with the manuscript's stated focus on fractional-order memory behavior.

8. Conclusion

Using three main definitions (Riemann-Liouville, Grünwald-Letnikov, and Caputo derivatives), our thorough investigation of fractional calculus and the computation of fractional-order derivatives has shown their critical importance in modern scientific research. We have demonstrated the exceptional sensitivity and accuracy of lower-order derivatives, particularly the 0.5^{th} order, in capturing subtle fluctuations within functions through a comprehensive analysis of fractional derivatives for fundamental trigonometric and exponential functions across several orders. A comparative analysis reveals distinct advantages and limitations among the three definitions: Riemann-Liouville derivatives exhibited the highest sensitivity for detecting minor variations in the function behavior, particularly in lower-order derivatives, and were statistically validated using Principal Component Analysis (PCA), showing consistent alignment with the original data patterns. Grünwald-Letnikov derivatives provided a flexible and computationally straightforward approach suitable for numerical simulations, capturing memory effects effectively while maintaining moderate sensitivity across orders. Caputo derivatives were advantageous for practical modeling of physical systems, as they naturally incorporate initial conditions, although their sensitivity for subtle function changes was slightly lower compared to Riemann-Liouville derivatives. Examining the strength and dispersion of point-to-point correlation coefficients (CCs) across each order of transformation further highlighted that lower-order derivatives lead to higher information precision and reduced loss of data integrity, as observed in functions $\sin(x)$, $\cos(x)$, and $\exp(x)$. Notably, the 0.5^{th} order Riemann-Liouville-derived derivative consistently emerged as the most potent tool for capturing nuanced variations while maintaining a reliable statistical representation. These findings underscore the complementary roles of the three definitions in fractional calculus: Riemann-Liouville for sensitivity, Grünwald-Letnikov for computational versatility, and Caputo for practical modeling of physical phenomena. Such comparative insights provide researchers with a framework to select the most appropriate fractional derivative for a given application, depending on whether sensitivity, numerical implementation, or boundary-condition management is prioritized. Future research could explore more complex applications, investigate additional numerical techniques, and extend the scope of

fractional calculus into new interdisciplinary areas, thereby enhancing our understanding of this revolutionary mathematical tool and its real-world applications across scientific and engineering domains.

Conflicts of interest

All the authors declared that they do not have any competing interests.

Data availability statement

All data generated or analyzed during this study are included in this article.

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Contributions of authors

All authors contributed equally to writing and reviewing this manuscript.

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