Multiplicative calculus in optical fiber analysis: an alternative frame perspective

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Received 30 July 2025; accepted 13 August 2025

This work uses the techniques of a non-Newtonian calculus (or multiplicative calculus) in the 3D Riemannian manifold to investigate the geometric features of linearly polarized light waves along optical fibers using the alternative moving frame. The evolution of a linearly polarized light wave is linked to a geometric phase since the optical fiber is thought to be a one-dimensional object embedded in a 3D Riemannian manifold. Thus, we produce a novel kind of multiplicative derivative geometric phase model. Furthermore, we present magnetic curves that are produced by the electric field E, defined by the electromagnetic curve. Then we define the Rytov curve, which consists of the combination of the space curve and the electromagnetic curve. In conclusion, we gave examples that match the theory and visualized them using the MATLAB program and analyzed the results using multiplicative calculus, which allows us to interpret the results proportionally.

Keywords: Optical fiber; non-Newtonian calculus; electromagnetic curves; polarized light wave; alternative moving frame.

DOI: https://doi.org/10.31349/RevMexFis.72.011301

1. Introduction

Calculus, one of the most fundamental and practical mathematics tools, enables solving complex problems encountered in science, engineering, economics, and many other fields through differential and integral calculus methods. The independent studies conducted by I. Newton and G. W. Leibniz in the 17th century marked a significant milestone in the development of calculus. Newton used calculus to formulate the laws of motion and the theory of universal gravitation; Leibniz significantly contributed to this field by developing modern integral and derivative notation. M. Grossman and R. Katz gave the multiplicative calculus that emerged as an alternative to traditional additive calculus and treated the rates of change multiplicatively rather than additively [1]. Although operations are performed additively in classical derivative and integral calculus, these operations are expressed in multiplicative terms in multiplicative calculus. Especially in fields such as finance, engineering, and physics, multiplicative rates of change can provide more accurate results. For example, the growth of an investment with compound interest can be modeled more accurately using multiplicative derivatives and integrals. This approach is more efficient, especially in systems that work logarithmically, and provides a practical and new perspective on problems. In addition, Bashirov et al. gave basic definitions and theorems about multiplicative calculus [2]. We see the application of multiplicative calculus to geometry in Georgiev's books [3, 4]. In these books, she developed the basic concepts of differential and analytic geometry using multiplicative calculus. Moreover, non-Newtonian calculus, while offering novel insights into phenomena characterized by proportional growth or exponential behavior, presents certain limitations when applied to real-world fluid dynamics problems. For instance, Ma *et al*. [5] introduced a framework based on multiplicative Euclidean space to analyze non-Newtonian wavefronts, demonstrating theoretical potential; however, its practical implementation remains constrained due to strict function positivity and differentiability requirements. In contrast, classical analysis remains more adaptable to diverse physical systems. Wilson and Thomas [6] provided foundational insights into turbulent flow behavior in non-Newtonian fluids using conventional differential methods, a versatility echoed in subsequent empirical and computational studies. Li *et al*. [7], for example, effectively employed classical rheological tools, such as the Marsh funnel and CFD modeling, to distinguish between Newtonian and non-Newtonian flow behaviors under varying field conditions. Similarly, Rashidi *et al*. [8] leveraged the homotopy analysis method within classical frameworks to capture complex thermal and velocity fields in wedge-driven flows. Even finite element methods, as applied by Böhme and Rubart [9], have shown sustained efficacy in classical contexts where the multiplicative approach might falter due to numerical instability or limited boundary condition compatibility. Consequently, while multiplicative calculus introduces a fresh perspective, classical analysis remains dominant due to its broader applicability, computational robustness, and deeper integration into engineering practices.

Electromagnetic curves are mathematical structures that describe the interactions of electromagnetic waves and magnetic fields. These curves are essential in advanced technologies, especially in the propagation of electromagnetic waves, optical fibers, telecommunications, and wireless communications. This concept of electromagnetic theory and differential geometry offers us many applications in physics and engineering. Since it is a current topic, it has attracted the attention of many mathematicians and physicists. Studies have been carried out using different spaces and frames with respect to electromagnetic curves in optical fibers [10, 11, 13, 15, 15–17, 26]. Moreover, it is used in the expression of the Rytov curve that describes phase perturbations and diffraction effects on the wavefront. Moreover, the curve is particularly used in the Rytov approximation (Rytov method), which analyzes wave propagation in turbulent media.

This study aims to examine electromagnetic curves in optical fibers with multiplicative calculus using an alternative moving frame. Hence, the study is organized as follows. Section 1 is an introduction providing an overview of the research objectives and scope. Section 2 presents foundational background information on the theory of non-Newtonian analysis and its relevance to the study. Section 3 explores the geometric phase of the polarization plane of a light wave propagating through an optical fiber within the $\{e_1, e_2, e_3\}$ framework, incorporating the concept of multiplicative derivatives. Section 4 is about the electromagnetic (EM) trajectories that utilize the polarization plane of a light wave in an optical fiber and emphasizes the role of multiplicative derivatives. Section 5 illustrates practical examples and visualizations using the Matlab program to demonstrate the application of multiplicative derivatives in this context. In Section 6, the study is summarized. Moreover, the aspects of the results that differ from the literature are interpreted.

2. Preliminaries

In this section, some fundamental definitions and theorems are represented for the multiplicative space obtained by choosing the exponential function of the generator $(\exp)$. The generator function α is chosen as the $(\exp)$ function. Then the multiplicative Frenet frame of the multiplicative curve γ is given. The foundational material will be based on the books by S. Georgiev; see [3, 18]:

$$\alpha : \mathbb{R} \to \mathbb{R}^+,$$
$$p \to \alpha(p) = e^p$$

and

$$\alpha^{-1} : \mathbb{R}^+ \to \mathbb{R},$$
$$q \to \alpha^{-1}(q) = \log q.$$

The exponential function defines a mapping from $\mathbb{R}$ to $\mathbb{R}^+$, assigning each real input a strictly positive output. Hence, the real number set in the multiplicative space is defined as below:

$$\mathbb{R}_\star = \{\exp(p) : p \in \mathbb{R}\} = \mathbb{R}^+.$$

Likewise, we define positive and negative multiplicative numbers as follows:

$$\mathbb{R}_\star^+ = \{\exp(p) : p \in \mathbb{R}^+\} = (1, \infty),$$

and

$$\mathbb{R}_\star^- = \{\exp(p) : p \in \mathbb{R}^-\} = (0, 1).$$

Table demonstrates the basic operations in the multiplicative space using the function exp. $\forall p, q \in \mathbb{R}_\star, q \neq 1$.

Multiplicative Operation	Exponential Form	Equivalent Form
$p +_* q$	$e^{\log p + \log q}$	pq
$p -_* q$	$e^{\log p - \log q}$	$\frac{p}{q}$
$p \cdot_* q$	$e^{\log p \log q}$	$p^{\log q}$
$p /_* q$	$e^{\log p / \log q}$	$p^{\frac{1}{\log q}}$

The field $(\mathbb{R}_\star, +_\star, -_\star)$ gives a multiplicative structure as given by the table. Every element of the space $\mathbb{R}\star$ is called a multiplicative number and is denoted by $p\star \in \mathbb{R}\star$, where $p\star = \exp(p)$ for some $p \in \mathbb{R}$. To simplify notation, we shall refer to multiplicative numbers as $p \in \mathbb{R}_\star$ instead of $p_\star$ throughout the research. In this framework, the identity element for multiplicative addition is $0_\star = 1$, while for multiplicative multiplication it is $1_\star = e$.

We now proceed to discuss several useful operations within the multiplicative space. Absolute value multiplication defines multiplicative space. Absolute value in Newtonian space is defined additively, reflecting the additive nature of distance. Since distance represents a multiplicative change in space, its absolute value is defined as follows:

$$|p|_\star = \begin{cases} p, & p \geq 0_\star \\ -_\star p, & p < 0_\star, \end{cases}$$

where $-_\star p = 1/p$. In the multiplicative space, we have

$$p^{k*} = p \cdot_\star p \cdot_\star ... p = e^{(\log p)^k}$$

for $p \in \mathbb{R}_\star$ and $k \in \mathbb{R}$. Furthermore, we have

$$p^{\frac{1}{2}*} = e^{(\log p)^{\frac{1}{2}}} = \star\sqrt{p}.$$

For $p, q \in \mathbb{R}_\star$, the following formulas are denoted:

$$(p +_\star q)^{2_\star} = p^{2_\star} +_\star e^2 \cdot_\star p \cdot_\star q +_\star q^{2_\star},$$
$$p^{2_\star} -_\star q^{2_\star} = (p -_\star q) \cdot_\star (p +_\star q).$$

The vector definition in $n-$ dimensional multiplicative space $\mathbb{R}_\star^n$ is represented by

$$\mathbb{R}_\star^n = \{(x_1, x_2, ..., x_n) : x_i \in \mathbb{R}_\star, i \in 1, 2, ..., n\}.$$

$\mathbb{R}^n\star$ represents a vector space over $\mathbb{R}\star$, equipped with the following pair of operations:

$$u +_\star v = (u_1 +_\star v_1, u_2 +_\star v_2, ..., u_n +_\star v_n)$$
$$= (u_1 v_1, u_2 v_2, ..., u_n v_n),$$

and

$$\begin{aligned} a \cdot_\star v &= (a \cdot_\star u_1, a \cdot_\star u_2, ..., a \cdot_\star u_n) \\ &= (u_1 v_1, u_2 v_2, ..., u_n v_n) \\ &= (e^{\log a \log u_1}, e^{\log a \log u_2}, ..., e^{\log a \log u_n}) \\ &= e^{\log a \log u}, \end{aligned}$$

where $u, v \in \mathbb{R}^n_\star$. Let u and v be arbitrary multiplicative vectors in the multiplicative vector space $\mathbb{R}^n_\star$. The corresponding multiplicative inner product is defined as follows:

$$\langle u, v \rangle_\star = e^{\langle \log u, \log v \rangle}.$$

Furthermore, if the multiplicative vectors u and v are orthogonal, the following relation holds:

$$\langle u, v \rangle_\star = 0_\star.$$

The multiplicative norm associated with the vector u is defined as follows:

$$||u||_\star = e^{\langle \log u, \log u \rangle^{\frac{1}{2}}}.$$

The multiplicative cross product has typical algebraic and geometrical characteristics. Let u and v be two unit multiplicative vectors in the multiplicative vector space. Let θ denote the multiplicative angle between the multiplicative unit vectors, defined as follows:

$$\theta = \arccos_\star (e^{\langle \log u, \log v \rangle}).$$

The multiplicative cosine of the multiplicative angle between two multiplicative unit vectors is

$$\cos_\star \theta = \langle u, v \rangle_\star.$$

For $\theta \in \mathbb{R}_\star$, the definitions of multiplicative trigonometric functions are given as follows:

$$\begin{aligned} \sin_\star \theta &= e^{\sin \log \theta}, \quad \cos_\star \theta = e^{\cos \log \theta}, \\ \tan_\star \theta &= e^{\tan \log \theta}, \quad \cot_\star \theta = e^{\cot \log \theta}. \end{aligned}$$

Let f be a function defined on the multiplicative space $\mathbb{R}\star$, where x belongs to an interval $I \subset \mathbb{R}\star$. The multiplicative derivative of the function f is defined as follows:

$$\begin{aligned} f^\star(x) &= \lim_{h \to 0_\star} (f(x +_\star h) -_\star f(x)) /_* h \\ &= \lim_{h \to 1} \left(\frac{f(xh)}{f(x)}\right)^{\frac{1}{\log h}} = \lim_{h \to 1} e^{\frac{\log \frac{f(xh)}{f(x)}}{\log h}}. \end{aligned}$$

Applying L'Hospital's rule in this context, we obtain

$$f^\star(x) = e^{\frac{x f'(x)}{f(x)}}.$$

Moreover, if f is both multiplicatively differentiable and continuous, it is called a $*$-differentiable (multiplicative differentiable) function. It also satisfies the multiplicative derivative of Leibniz and the chain rule.

The multiplicative integral serves as the inverse operation of the multiplicative derivative. The function $f(x)$ has its indefinite multiplicative integral defined as follows:

$$\int_\star f(x) \cdot_\star d_\star x = e^{\int \frac{1}{x} \log f(x) dx}, \quad x \in \mathbb{R}_\star.$$

A curve γ in $\mathbb{R}^3_\star$, which is in a multiplicative parametrization of class $C^k_\star (k \geq 1_\star)$, is defined as a multiplicative vector-valued function $\gamma : I \subset \mathbb{R}_\star$, where $\gamma(s) = (\gamma_1(s), \gamma_2(s), \gamma_3(s))$. The multiplicative curve $\gamma(s)$ is regular if and only if $||\gamma(s)||_\star \neq 0_\star$. Moreover, if the velocity vector of $\gamma(s)$ equals $1_\star$, then $\gamma(s)$ is called a multiplicative unit speed curve. Given $s_0 \in I$, the multiplicative arc length corresponding to the multiplicatively regular curve $\gamma(s)$ can be expressed as:

$$h(s) = \int_{\star s_0}^{s} ||\gamma(s)||_\star \cdot_\star d_\star s. \tag{1}$$

The multiplicative Frenet trihedron of $\gamma(s)$ is denoted by

$$\begin{aligned} t(s) &= \gamma^\star(s), \quad n(s) = \frac{\gamma^{\star\star}(s)}{||\gamma^{\star\star}(s)||_\star}, \\ b(s) &= t(s) \times_\star n(s), \end{aligned} \tag{2}$$

where $t(s), n(s)$ and $b(s)$ are called multiplicative tangent, multiplicative principal normal, and multiplicative binormal vectors, respectively. Also, the multiplicative curvature and multiplicative torsion are given by

$$\begin{aligned} \kappa(s) &= ||\gamma^{\star\star}(s)||_\star, \\ \tau(s) &= \langle n(s), b(s) \rangle_\star. \end{aligned}$$

Moreover, the multiplicative Frenet formulae of γ are represented by

$$\begin{aligned} t^\star(s) &= \kappa(s) \cdot_\star n(s), \\ n(s) &= -_\star \kappa(s) \cdot_\star t(s) +_\star \tau(s) \cdot_\star b(s), \\ b^\star(s) &= -_\star \tau(s) \cdot_\star n(s). \end{aligned}$$

Let $\gamma(s)$ be a unit speed curve in the Euclidean 3-space, and

$$e_1(s) = \frac{t'(s)}{\|t'(s)\|}, \quad e_2(s) = \frac{n'(s)}{\|n'(s)\|}$$

and

$$e_3(s) = \frac{\tau(s) t(s) + \kappa(s) b(s)}{\sqrt{\kappa^2(s) + \tau^2(s)}}$$

be the unit principal normal vector, the derivative of the principal normal vector, and the Darboux vector, respectively. Then $\{e_1(s), e_2(s), e_3(s)\}$ is the alternative moving frame of the curve $\gamma(s)$. Derivatives of the alternative moving frame can be written as

$$\begin{pmatrix} e_1'(s) \\ e_2'(s) \\ e_3'(s) \end{pmatrix} = \begin{pmatrix} 0 & \alpha(s) & 0 \\ -\alpha(s) & 0 & \beta(s) \\ 0 & -\beta(s) & 0 \end{pmatrix} \begin{pmatrix} e_1(s) \\ e_2(s) \\ e_3(s) \end{pmatrix}, \tag{3}$$

where $\alpha(s) = \sqrt{\kappa^2(s) + \tau^2(s)}$, $\beta(s) = \sigma \cdot \alpha(s)$ and

$$\sigma = \frac{\kappa^2(s)}{(\kappa^2(s) + \tau^2(s))^{3/2}} \cdot \left(\frac{\tau}{\kappa}\right)'$$

are curvatures of the curve $\gamma(s)$ with respect to alternative moving frame.

3. Linearly polarized light wave in optical fiber in a geometric phase with a multiplicative alternative moving frame

In this section, an optical fiber is examined through a space curve considering the multiplicative alternative moving frame.

Definition 1. *Let $\gamma(s)$ be a unit speed multiplicative curve, and $e_1(s) = t^\star(s)/_\star ||t^\star(s)||_\star$, $e_2(s) = e_1(s)/_\star \|e_1(s)\|_\star$ and $e_3(s) = (\tau \cdot_\star t +_\star \kappa \cdot_\star b)/_\star \star \sqrt{\kappa^{2_\star} +_\star \tau^{2_\star}}$ be the unit principal normal vector, the multiplicative derivative of the principal normal vector, and the Darboux vector, respectively. Then $\{e_1(s), e_2(s), e_3(s)\}$ is the alternative moving frame of the curve $\gamma(s)$. Derivatives of the alternative moving frame are given as follows:*

$$e_1(s) = \alpha(s) \cdot_\star e_2(s), \tag{4}$$

$$e_2(s) = -_\star \alpha(s) \cdot_\star e_1(s) +_\star \beta(s) \cdot_\star e_3(s), \tag{5}$$

$$e_3(s) = -_\star \beta(s) \cdot_\star e_2(s), \tag{6}$$

where $\alpha(s) = \star\sqrt{\kappa^{2_\star} +_\star \tau^{2_\star}}$, $\beta(s) = \sigma \cdot_\star \alpha(s)$ and $\sigma = \kappa^{2_\star} /_\star (\kappa^{2_\star} +_\star \tau^{2_\star})^{\frac{3}{2}} * \cdot_\star (\tau /_\star \kappa)^\star$ are curvatures of the curve γ with respect to alternative moving frame.

Assume that γ is a space curve in a multiplicative alternative moving frame. We demonstrate the relationship between a geometric phase and the evolution of a linearly polarized light wave, as the optical fiber is a one-dimensional object embedded in the 3D Riemann manifold. The electric field $E(s)$, on the other hand, defines the direction of the linearly polarized light wave's state. Thus, the multiplicative alternative moving frame along an optical fiber can be used to write the direction of $E(s)$ as follows:

$$E^\star(s) = \mu_1 \cdot_\star e_1(s) +_\star \mu_2 \cdot_\star e_2(s) +_\star \mu_3 \cdot_\star e_3(s), \tag{7}$$

where $\mu_i (i = 1, 2, 3)$ denote differentiable functions. Three different situations are examined in the following analysis of the polarized light's state direction.

Case 1. Let the electric field $E(s)$ lie on a plane orthogonal to $e_1(s)$. Hence, we have

$$\langle E(s), e_1(s) \rangle_\star = 0_\star. \tag{8}$$

Considering the multiplicative derivative of Eq. (8), we obtain

$$\langle E^\star(s), e_1(s) \rangle_\star +_\star \langle E(s), e_1(s) \rangle_\star = 0_\star.$$

So, we have

$$\langle \mu_1 \cdot_\star e_1(s) +_\star \mu_2 \cdot_\star e_2(s) +_\star \mu_3 \cdot_\star e_3(s), e_1(s) \rangle_\star = 0_\star.$$

If all the necessary arrangements are done, the first coefficient is calculated as

$$\mu_1 = -_\star \alpha(s) \cdot_\star \langle E(s), e_2(s) \rangle_\star. \tag{9}$$

We state that $\langle E(s), E(s) \rangle_\star$ is constant if we assume that there is no mechanical loss in the optical fiber as a result of absorption. Next, if we calculate the multiplicative derivative of this align, we obtain

$$\langle E^\star(s), E(s) \rangle_\star = 0_\star.$$

After some algebraic calculations, we acquire

$$\mu_2 \cdot_\star \langle E(s), e_2(s) \rangle_\star = -_\star \mu_3 \cdot_\star \langle E(s), e_3(s) \rangle_\star.$$

Since $\langle E(s), e_2(s) \rangle_\star \neq 0_\star$ and $\langle E(s), e_3(s) \rangle_\star \neq 0_\star$, μ_2 and μ_3 are represented by

$$\begin{aligned} \mu_2 &= \mu \cdot_\star \langle E(s), e_3(s) \rangle_\star, \\ \mu_3 &= -_\star \mu \cdot_\star \langle E(s), e_2(s) \rangle_\star. \end{aligned} \tag{10}$$

All the coefficients are written in Eq. (7), we obtain

$$\begin{aligned} E^\star(s) = &-_\star \alpha(s) \cdot_\star \langle E(s), e_2(s) \rangle_\star \cdot_\star e_1(s) \\ &+_\star \mu \cdot_\star \langle E(s), e_3(s) \rangle_\star \cdot_\star e_2(s) \\ &-_\star \mu \cdot_\star \langle E(s), e_2(s) \rangle_\star \cdot_\star e_3(s), \end{aligned}$$

and then

$$\begin{aligned} E^\star(s) = &-_\star \alpha(s) \cdot_\star \langle E(s), e_2(s) \rangle_\star \cdot_\star e_1(s) \\ &+_\star \mu \cdot_\star (\langle E(s), e_3(s) \rangle_\star \cdot_\star e_2(s) \\ &-_\star \langle E(s), e_2(s) \rangle_\star \cdot_\star e_3(s)). \end{aligned}$$

Hence, we write

$$\begin{aligned} E^\star(s) = &-_\star \alpha(s) \cdot_\star \langle E(s), e_2(s) \rangle_\star \cdot_\star e_1(s) \\ &+_\star \mu \cdot_\star (E(s) \times_\star e_1(s)). \end{aligned} \tag{11}$$

The second term on the right side of the align above provides rotation around the n. Assuming that n is transmitted in parallel, we observe that

$$E^\star(s) = -_\star \alpha(s) \cdot_\star \langle E(s), e_2(s) \rangle_\star \cdot_\star e_1(s) \cdot_\star e_1(s). \tag{12}$$

Moreover, the polarization vector is given by

$$\begin{aligned} E(s) = &\langle E(s), e_2(s) \rangle_\star \cdot_\star e_2(s) \\ &+_\star \langle E(s), e_3(s) \rangle_\star \cdot_\star e_3(s). \end{aligned} \tag{13}$$

The multiplicative derivative of Eq. (13) is obtained by

$$\begin{aligned}E^{\star}(s) &= (-_{\star}\alpha(s)\cdot_{\star}\langle E(s),e_2(s)\rangle_{\star})\cdot_{\star} e_1(s)\\ &+_{\star}((\langle E(s),e_2(s)\rangle_{\star})^{\star}\\ &-_{\star}\beta(s)\cdot_{\star}\langle E(s),e_3(s)\rangle_{\star})\cdot_{\star} e_2(s)\\ &+_{\star}(\langle E(s),e_3(s)\rangle_{\star}+_{\star}\beta(s)\cdot_{\star}\langle E(s),e_2(s)\rangle_{\star})\cdot_{\star} e_3(s).\end{aligned}$$

So, we have

$$\begin{bmatrix}(\langle E(s),e_2(s)\rangle_{\star})^{\star}\\(\langle E(s),e_3(s)\rangle_{\star})^{\star}\end{bmatrix}=\begin{bmatrix}0_{\star} & \beta(s)\\ \beta(s) & 0_{\star}\end{bmatrix}\cdot_{\star}\begin{bmatrix}\langle E(s),e_2(s)\rangle_{\star}\\ \langle E(s),e_3(s)\rangle_{\star}.\end{bmatrix}$$

Since $\langle E(s),E(s)\rangle_{\star}$ is constant, the following align can be represented by

$$E(s)=\sin_{\star}\theta(s)\cdot_{\star} e_2(s)+_{\star}\cos_{\star}\theta(s)\cdot_{\star} e_3(s). \tag{14}$$

Taking the multiplicative derivative of Eq. (14), we get

$$\begin{aligned}E^{\star}(s) &= -_{\star}\alpha(s)\cdot_{\star}\langle E(s),e_2(s)\rangle_{\star}\cdot_{\star} e_1(s)\\ &+_{\star}(\theta^{\star}(s)._{\star}\langle E(s),e_3(s)\rangle_{\star}\\ &-_{\star}\beta(s)\cdot_{\star}\langle E(s),e_3(s)\rangle_{\star}).{}_{\star}e_2(s)\\ &+_{\star}(\langle E(s),e_2(s)\rangle_{\star}\cdot_{\star}\beta(s)\\ &-_{\star}\theta^{\star}(s)\cdot_{\star}\langle E(s),e_3(s)\rangle_{\star})\cdot_{\star} e_3(s),\end{aligned}$$

and then

$$\begin{aligned}E^{\star}(s) &= -_{\star}\alpha(s)\cdot_{\star}\langle E(s),e_2(s)\rangle_{\star}\cdot_{\star} e_1(s)\\ &+_{\star}(\theta^{\star}(s)-_{\star}\beta(s))\cdot_{\star}(E(s)\times_{\star} e_1(s)).\end{aligned}$$

Under conditions exhibiting optical activity, the field rotates to the left if the fiber does not favor it. Therefore, it is appropriate to assume that the coefficient of the second term on the right side of the align above equals zero as

$$\theta^{\star}(s)-_{\star}\beta(s)=0_{\star}.$$

Integrating the above align, we obtain that

$$\theta^{\star}(s)=\int_{\star}\beta(s)._{\star}d_{\star}s.$$

Thus, for the condition $\langle E(s),e_1(s)\rangle_{\star}=0_{\star}$, we obtain the $E_{e_1}-$ Rytov curve. Moreover, the curves corresponding to the polarization vector and the optical polarization vector are traced, respectively:

$$E_{e_1(s)}(s)=\gamma(s)+_{\star}E(s), \tag{15}$$

and

$$\begin{aligned}E(s) &= \sin_{\star}\left(\int_{\star}\beta(s)._{\star}d_{\star}s\right)._{\star}e_2(s)\\ &+_{\star}\cos_{\star}\left(\int_{\star}\beta(s)._{\star}d_{\star}s\right)._{\star}e_3(s).\end{aligned} \tag{16}$$

Case 2. Let the electric field $E(s)$ lie on a plane orthogonal to $e_2(s)$ in the second particular case. Presumably, we write

$$\langle E(s),e_2(s)\rangle_{\star}=0_{\star}. \tag{17}$$

Taking the multiplicative derivative of Eq. (17), we acquire

$$\langle E^{\star}(s),e_2(s)\rangle_{\star}+_{\star}\langle E(s),e_2(s)\rangle_{\star}=0_{\star}.$$

So, we have

$$\langle\mu_1._{\star}e_1(s)+_{\star}\mu_2._{\star}e_2(s)+_{\star}\mu_3._{\star}e_3(s),e_2(s)\rangle_{\star}=0_{\star}.$$

After some calculations, the second coefficient is computed by

$$\mu_2=\alpha(s)._{\star}\langle E(s),e_1(s)\rangle_{\star}-_{\star}\beta(s)._{\star}\langle E(s),e_3(s)\rangle_{\star}. \tag{18}$$

Since $\langle E(s),E(s)\rangle_{\star}$ is constant and the relevant calculations have been carried out, we obtain

$$\mu_1._{\star}\langle E(s),e_1(s)\rangle_{\star}=-_{\star}\mu_3._{\star}\langle E(s),e_3(s)\rangle_{\star}$$

and then μ_2 and μ_3 are expressed by

$$\begin{aligned}\mu_1 &= \mu._{\star}\langle E(s),e_3(s)\rangle_{\star},\\ \mu_3 &= -_{\star}\mu._{\star}\langle E(s),e_1(s)\rangle_{\star}.\end{aligned} \tag{19}$$

All the coefficients are written in Eq. (7), we have

$$\begin{aligned}E^{\star}(s) &= \mu._{\star}\langle E(s),e_3(s)\rangle_{\star}._{\star}e_1(s)+_{\star}(\alpha(s)._{\star}\langle E(s),e_1(s)\rangle_{\star}\\ &-_{\star}\beta(s)._{\star}\langle E(s),e_3(s)\rangle_{\star})._{\star}e_2(s)\\ &-_{\star}\mu._{\star}\langle E(s),e_1(s)\rangle_{\star}._{\star}e_3(s).\end{aligned}$$

Therefore, we write

$$E^{\star}(s)=(\alpha(s)._{\star}\langle E(s),t(s)\rangle_{\star} \tag{20}$$

$$-_{\star}\beta(s)._{\star}\langle E(s),b(s)\rangle_{\star})._{\star}e_1(s) \tag{21}$$

$$-_{\star}\mu._{\star}(E(s)\times_{\star}e_1(s)). \tag{22}$$

Rotation around $e_2(s)$ is represented by the second part of the align above, just like in the first case. Assuming that $e_2(s)$ is carried in parallel, we can write it as

$$\begin{aligned}E^{\star}(s) &= \alpha(s)._{\star}\langle E(s),e_1(s)\rangle_{\star}\\ &-_{\star}\beta(s)._{\star}\langle E(s),e_3(s)\rangle_{\star})._{\star}e_2(s).\end{aligned} \tag{23}$$

Additionally, the polarization vector is defined by

$$\begin{aligned}E(s) &= \langle E(s),e_1(s)\rangle_{\star}._{\star}e_1(s)\\ &+_{\star}\langle E(s),e_3(s)\rangle_{\star}._{\star}e_3(s).\end{aligned} \tag{24}$$

The multiplicative derivative of Eq. (24) is acquired by

$$\begin{aligned}E^{\star}(s) &= (\langle E(s),e_1(s)\rangle_{\star})^{\star}._{\star}e_1(s)+_{\star}(\alpha(s)._{\star}\langle E(s),e_1(s)\rangle_{\star}\\ &-_{\star}\beta(s)._{\star}\langle E(s),e_3(s)\rangle_{\star})._{\star}e_2(s)\\ &+_{\star}(\langle E(s),e_3(s)\rangle_{\star})^{\star}._{\star}e_3(s).\end{aligned}$$

Therefore, the following matrix representation can be written as follows:

$$\begin{bmatrix}(\langle E(s),e_1(s)\rangle_\star)^\star\\(\langle E(s),e_3(s)\rangle_\star)^\star\end{bmatrix}=\begin{bmatrix}0_\star & 0_\star\\0_\star & 0_\star\end{bmatrix}\cdot_\star\begin{bmatrix}\langle E(s),e_1(s)\rangle_\star\\\langle E(s),e_3(s)\rangle_\star.\end{bmatrix}$$

The polarization vector is written in terms of spherical coordinates as follows:

$$E(s)=\cos_\star\theta(s).{}_\star e_1(s)+_\star\sin_\star\theta(s).{}_\star e_3(s). \tag{25}$$

Taking the multiplicative derivative of Eq. (25), we obtain

$$\begin{aligned}E^\star(s)=&(\alpha(s).{}_\star\langle E(s),e_1(s)\rangle_\star\\&-_\star\beta(s).{}_\star\langle E(s),e_3(s)\rangle_\star).{}_\star e_2(s)\\&+_\star\theta^\star(s).{}_\star(E(s)\times_\star e_1(s)).\end{aligned}$$

We have $\theta^\star(s)=0_\star$ from the above align. The parallel transport is thus moved along $e_2(s)$ by E. Thus, in the second scenario, we can state that the optical fiber is $E_{e_2}-$ Rytov curve. $e_2(s)$ determines a change in the direction of the polarized light state. Therefore, using the conditions $\langle E(s),e_2(s)\rangle_\star=0_\star$, we obtain the $E_{e_2}-$ Rytov curve and polarization vector in the optical as

$$E_{e_2}(s)=\gamma(s)+_\star E(s), \tag{26}$$

and then

$$\begin{aligned}E(s)&=\cos_\star\theta.{}_\star e_1(s)+_\star\sin_\star\theta.{}_\star e_3(s),\\\theta&=constant.\end{aligned} \tag{27}$$

Case 3. Assume that the electric field $E(s)$ lies on a plane orthogonal to $e_3(s)$. Therefore, we have

$$\langle E(s),e_3(s)\rangle_\star=0_\star. \tag{28}$$

Taking the multiplicative derivative of Eq. (28), we have

$$\langle E^\star(s),e_3(s)\rangle_\star+_\star\langle E(s),e_3(s)\rangle_\star=0_\star.$$

Hence, we have

$$\langle\mu_1.{}_\star e_1(s)+_\star\mu_2.{}_\star e_2(s)+_\star\mu_3.{}_\star e_3(s),e_3(s)\rangle_\star=0_\star.$$

Exploiting the multiplicative Frenet frame, the third coefficient is computed by

$$\mu_3=\beta(s).{}_\star\langle E(s),e_2(s)\rangle_\star. \tag{29}$$

Since $\langle E(s),E(s)\rangle_\star$ is constant, we obtain

$$\langle E^\star(s),E(s)\rangle_\star=0_\star.$$

After some algebraic calculations, we acquire

$$\mu_1\cdot_\star\langle E(s),e_1(s)\rangle_\star=-_\star\mu_2\cdot_\star\langle E(s),e_2(s)\rangle_\star.$$

Since $\langle E(s),e_1(s)\rangle_\star\neq0_\star$ and $\langle E(s),e_2(s)\rangle_\star\neq0_\star$, μ_1 and μ_2 are, respectively,

$$\begin{aligned}\mu_1&=\mu\cdot_\star\langle E(s),e_2(s)\rangle_\star,\\\mu_2&=-_\star\mu\cdot_\star\langle E(s),e_1(s)\rangle_\star.\end{aligned} \tag{30}$$

All the coefficients are written in Eq. (7), we acquire

$$\begin{aligned}E^\star(s)=&\beta(s).{}_\star\langle E(s),e_2(s)\rangle_\star\cdot_\star e_3(s)\\&+_\star\mu\cdot_\star(E(s)\times_\star e_3(s)).\end{aligned}$$

and then

$$E^\star(s)=\beta(s)\cdot_\star\langle E(s),e_2(s)\rangle_\star\cdot_\star e_3(s),$$

where $e_3(s)$ is parallel transported, and the second term on the right side of the above align provides a rotation around $e_3(s)$. Furthermore, the polarization vector is given by

$$\begin{aligned}E(s)=&\langle E(s),e_1(s)\rangle_\star\cdot_\star e_1(s)\\&+_\star\langle E(s),e_2(s)\rangle_\star\cdot_\star e_2(s).\end{aligned} \tag{31}$$

The multiplicative derivative of Eq. (31) is computed by

$$\begin{aligned}E^\star(s)=&(((\langle E(s),e_1(s)\rangle_\star)^\star\\&-_\star\alpha(s)\langle E(s),e_2(s)\rangle_\star)\cdot_\star e_1(s)\\&+_\star(\alpha(s)\cdot_\star\langle E(s),e_1(s)\rangle_\star\\&+_\star(\langle E(s),e_2(s)\rangle_\star)^\star)\cdot_\star e_2(s)\\&+_\star(\beta(s)\cdot_\star\langle E(s),e_2(s)\rangle_\star)\cdot_\star e_3(s).\end{aligned}$$

The matrix representation is given by

$$\begin{bmatrix}(\langle E(s),e_1(s)\rangle_\star)^\star\\(\langle E(s),e_2(s)\rangle_\star)^\star\end{bmatrix}=\begin{bmatrix}0_\star & \kappa(s)\\-_\star\kappa(s) & 0_\star\end{bmatrix}\cdot_\star\begin{bmatrix}\langle E(s),e_1(s)\rangle_\star\\\langle E(s),e_2(s)\rangle_\star.\end{bmatrix}$$

Also, the polarization vector is given as follows:

$$E(s)=\cos_\star\theta(s).{}_\star e_1(s)+_\star\sin_\star\theta(s)\cdot_\star e_2(s). \tag{32}$$

Taking the multiplicative derivative of Eq. (32), we obtain

$$\begin{aligned}E^\star(s)=&-_\star\beta(s)\cdot_\star\langle E(s),e_2(s)\rangle_\star\cdot_\star e_3(s)\\&-_\star(\alpha(s)+_\star\theta^\star(s))\cdot_\star(E(s)\times_\star e_3(s)).\end{aligned}$$

Since it denotes an optical fiber, we have $\alpha(s)+_\star\theta^\star(s)=0_\star$. Integrating this align, we obtain the following:

$$\theta(s)=-_\star\int_\star\alpha(s).{}_\star d_\star s.$$

Therefore, for the condition $\langle E(s),e_3(s)\rangle_\star=0_\star$, we obtain the $E_{e_3}-$ Rytov curve. Hence, the Rytov curve represents the traced curve of the polarization vector and the optical polarization vector, respectively.

$$E_{e_3}(s)=\gamma(s)+_\star E(s), \tag{33}$$

and

$$\begin{aligned}E(s)=&\cos_\star\left(\int_\star\alpha(s).{}_\star d_\star s\right).{}_\star e_1(s)\\&-_\star\sin_\star\left(\int_\star\alpha(s).{}_\star d_\star s\right).{}_\star e_2(s).\end{aligned} \tag{34}$$

4. Multiplicative calculus of electromagnetic curves on the optical fiber along the polarization plane

In this section, we examine the electromagnetic curves on the optical fiber according to the orthogonality situations of the alternative moving frame.

As a charged particle interacts with an electromagnetic field, it creates a force known as the Lorentz force. This Lorentz force caused by the electromagnetic field affects the motion of the particle, which follows a path along the optical fiber. The paths followed by this charged particle are called electromagnetic curves [25]. Let F be an optical fiber defined by the curve γ in 3D Riemannian space. Then, the electromagnetic curve with respect to the multiplicative calculus holds the following relation:

$$\Phi(E) = V \times_\star E, \tag{35}$$

where V is a Killing magnetic vector field. This section analyzes the orthogonality of E with the elements of the alternating frame $\{e_1, e_2, e_3\}$.

4.1. Electromagnetic curves on the optical fiber for the case $E \perp e_1$

From Sec. 3, if $\langle E(s), e_1(s)\rangle_\star = 0_\star$, we find the align of $E^\star$ as follows:

$$E^\star(s) = -f \cdot_\star \langle E^\star(s), e_2(s)\rangle_\star \cdot_\star e_1(s) \tag{36}$$
$$+_\star \mu\langle E(s), e_3(s)\rangle_\star \cdot_\star e_2(s) \tag{37}$$
$$-_\star \mu \cdot_\star \langle E(s), e_2(s)\rangle_\star \cdot_\star e_3(s). \tag{38}$$

According to the Lorentz force align, we know that

$$\langle \Phi(E(s)), e_1(s)\rangle_\star = -\langle \Phi(e_1(s)), E(s)\rangle_\star\ , \tag{39}$$
$$\langle \Phi(E(s)), e_2(s)\rangle_\star = -\langle \Phi(e_2(s)), E(s)\rangle_\star\ , \tag{40}$$
$$\langle \Phi(E(s)), e_3(s)\rangle_\star = -\langle \Phi(e_3(s)), E(s)\rangle_\star\ . \tag{41}$$

Using the $\Phi(s)$ on multiplicative Frenet frame, we get

$$\Phi(e_1(s)) = \alpha(s) \cdot_\star e_2(s)\ , \tag{42}$$
$$\Phi(e_2(s)) = \alpha(s) \cdot_\star e_1(s) -_\star \mu \cdot_\star e_3(s)\ , \tag{43}$$
$$\Phi(e_3(s)) = \mu \cdot_\star e_2(s)\ . \tag{44}$$

If we write the Killing vector field according to the multiplicative Frenet frame, we obtain

$$V = a_1 \cdot_\star e_1(s) +_\star a_2 \cdot_\star e_2(s) +_\star a_3 \cdot_\star e_3(s). \tag{45}$$

And also, we know that

$$\Phi(e_1(s)) = V \times_\star e_1(s)\ , \tag{46}$$
$$\Phi(e_2(s)) = V \times_\star e_2(s)\ , \tag{47}$$
$$\Phi(e_3(s)) = V \times_\star e_3(s)\ . \tag{48}$$

Thus, we have

$$V(s) = -_\star\mu \cdot_\star e_1(s) +_\star \alpha(s) \cdot_\star e_3(s). \tag{49}$$

Since we consider n as perpendicular to E, the Lorentz force aligns are obtained according to the multiplicative Frenet frame as follows:

$$\Phi(e_1(s)) = \alpha(s) \cdot_\star e_2(s)\ , \tag{50}$$
$$\Phi(e_2(s)) = -_\star\alpha(s) \cdot_\star e_1(s)\ , \tag{51}$$
$$\Phi(e_3(s)) = 0_\star \tag{52}$$

and

$$V(s) = \alpha(s) \cdot_\star e_3(s). \tag{53}$$

4.2. Electromagnetic curves on the optical fiber for the case $E \perp e_2$

For the case $\langle E(s), e_2(s)\rangle_\star = 0_\star$, we have

$$E^\star(s) = \mu \cdot_\star \langle E(s), e_3(s)\rangle_\star \cdot_\star e_1(s) \tag{54}$$
$$+_\star (\kappa \cdot_\star \langle E(s), e_1(s)\rangle_\star \tag{55}$$
$$-_\star \beta(s) \cdot_\star \langle E(s), e_3(s)\rangle_\star) \cdot_\star e_2(s) \tag{56}$$
$$-_\star \mu \cdot_\star \langle E(s), e_1(s)\rangle_\star e_3(s). \tag{57}$$

From the Lorentz force, we get

$$\Phi(e_1(s)) = \alpha(s) \cdot_\star e_2(s) -_\star \mu \cdot_\star e_3(s)\ , \tag{58}$$
$$\Phi(e_2(s)) = -_\star\alpha(s) \cdot_\star e_1(s) +_\star \beta(s) \cdot_\star e_3(s)\ , \tag{59}$$
$$\Phi(e_3(s)) = \mu \cdot_\star e_1(s) -_\star \beta(s) \cdot_\star e_2(s)\ . \tag{60}$$

We can compute the magnetic vector field according to the multiplicative Frenet frame as

$$V(s)=\beta(s) \cdot_\star e_1(s) +_\star \mu \cdot_\star e_2(s) +_\star \alpha(s) \cdot_\star e_3(s). \tag{61}$$

Since $e_2(s)$ is perpendicular to $E(s)$, we can write the Lorentz force aligns as follows:

$$\Phi(e_1(s)) = \alpha(s) \cdot_\star e_2(s)\ , \tag{62}$$
$$\Phi(e_2(s)) = -_\star\alpha(s) \cdot_\star e_1(s) +_\star \beta(s) \cdot_\star e_3(s)\ , \tag{63}$$
$$\Phi(e_3(s)) = -_\star\beta(s) \cdot_\star e_2(s) \tag{64}$$

and

$$V(s) = \alpha(s) \cdot_\star e_3(s). \tag{65}$$

4.3. Electromagnetic curves on the optical fiber for the case $E \perp e_3$

Assume that $\langle E(s), e_3(s)\rangle_\star = 0_\star$. From Section 3, we know that

$$E^\star(s) = \mu \cdot_\star \langle E(s), e_2(s)\rangle_\star \cdot_\star e_1(s) \tag{66}$$

$$+_\star -_\star \beta(s) \cdot_\star \langle E(s), e_1(s)\rangle_\star \cdot_\star e_2(s) \tag{67}$$

$$-_\star \langle E(s), e_2(s)\rangle_\star e_3(s). \tag{68}$$

Using the Φ on the multiplicative Frenet frame, we obtain

$$\Phi(e_1(s)) = -_\star \mu \cdot_\star e_2(s)\ , \tag{69}$$

$$\Phi(e_2(s)) = \mu \cdot_\star e_1(s) +_\star \beta(s) \cdot_\star e_3(s)\ , \tag{70}$$

$$\Phi(e_3(s)) = -_\star \mu \cdot_\star e_2(s)\ . \tag{71}$$

Thus, we have

$$V(s) = \beta(s) \cdot_\star e_1(s) -_\star \mu \cdot_\star e_3(s). \tag{72}$$

Since $e_3(s)$ is perpendicular to the polarization vector $E(s)$, we get the Lorentz force as

$$\Phi(e_1(s)) = 0_\star\ , \tag{73}$$

$$\Phi(e_2(s)) = g \cdot_\star e_3(s)\ , \tag{74}$$

$$\Phi(e_3(s)) = -_\star \beta(s) \cdot_\star e_2(s) \tag{75}$$

and

$$V(s) = \beta(s) \cdot_\star e_1(s). \tag{76}$$

5. Example

In this section, we give some illustrative examples to verify the results. Moreover, in Example 5.1, we discuss the difference between multiplicative E-Rytov curves and E-Rytov curves for multiplicative helix curve and the general helix curve with some figures.

Example 5.1 Let $\gamma(s) = ((e^4/_\star e^5) \cdot_\star \sin_\star s, (e^4/_\star e^5) \cdot_\star \cos_\star s, (e^3/_\star e^5) \cdot_\star e^s)$ be a multiplicative helix curve. Then, the multiplicative alternative moving apparatus of $\gamma(s)$ is given as

$$e_1(s) = \gamma^{\star\star}(s) /_{\star} \|\gamma^{\star\star}(s)\|_\star = (-\star \sin_\star s, -\star \cos_\star s, 0_\star),$$

$$e_2(s) = e_1(s) /_{\star} \|e_1(s)\|_\star = (-\star \cos_\star s, \sin_\star s, 0_\star),$$

$$e_3(s) = (\tau \cdot \star\, t +_\star \kappa \cdot \star\, b) /_{\star} \star\sqrt{\kappa^{2\star} +_\star \tau^{2\star}} = (0_\star,\ 0_\star,\ -_\star 1_\star),$$

$$\alpha(s) = \star\sqrt{\kappa^{2_\star} +_\star \tau^{2_\star}} = 1_\star,$$

$$\beta(s) = \sigma \cdot_\star \alpha(s) = 0_\star.$$

In the graphs below, we have examined the E-Rytov curve according to the three cases, which are examined in Sec. 3. In these cases, we draw the curves E_{e_1}, E_{e_2}, and E_{e_3}, as well as the electric field E on the sphere in Fig. 1-3, respectively.

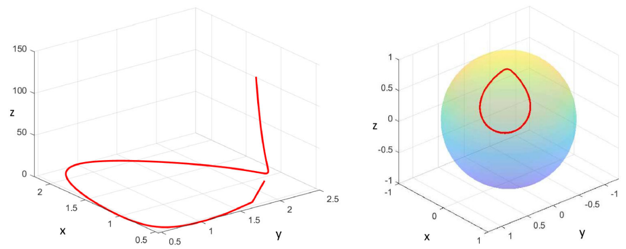

FIGURE 1. E_{e_1} - Rytov curve and the electric field E for $\langle E, e_1\rangle_* = 0_*$, respectively.

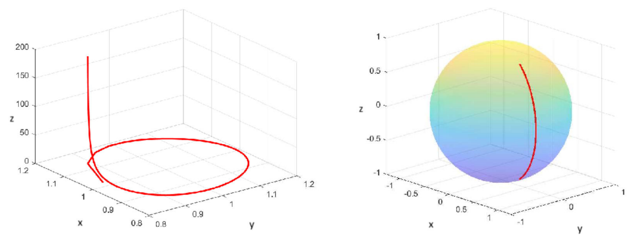

FIGURE 2. E_{e_2} - Rytov curve and the electric field E for $\langle E, e_2\rangle_* = 0_*$, respectively.

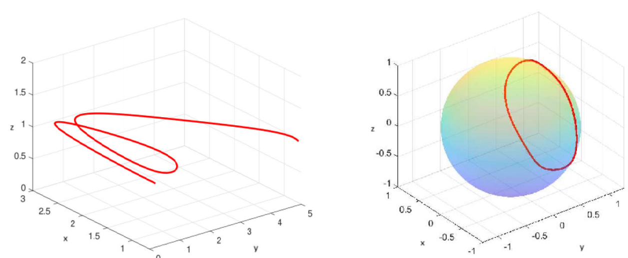

FIGURE 3. E_{e_3} - Rytov curve and the electric field E for $\langle E, e_3\rangle_* = 0_*$, respectively.

Let us reconsider the example in the classical analysis: Assume that $\gamma(s) = [(4/5)\sin s, (4/5)\cos s, (3/5)s)$ is the helix curve. Then the alternative moving operators of $\gamma(s)$ are computed by

$$e_1(s) = (-\sin s, -\cos s, 0),$$

$$e_2(s) = (-\cos s, \sin s, 0),$$

$$e_3(s) = (0, 0, -1),$$

$$\alpha(s) = 1,$$

$$\beta(s) = 0.$$

In the following figures, we show the E-Rytov curves in terms of three cases. Additionally, we present the electric fields on the sphere, respectively.

The study [26] is about the characterization of electromagnetic curves with the help of the alternative frame by means of the classical derivative. Now let's give Example 5.1 in Ref. [26] using multiplicative calculus and compare it with the results in classical derivation.

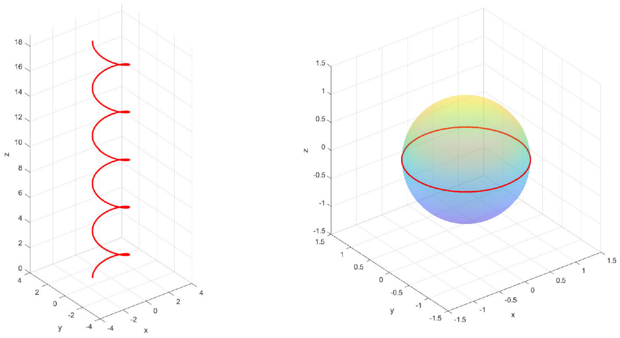

FIGURE 4. E_{e_1} - Rytov curve and the electric field E for $\langle E, e_1 \rangle = 0$, ($\theta = \pi/2$) respectively.

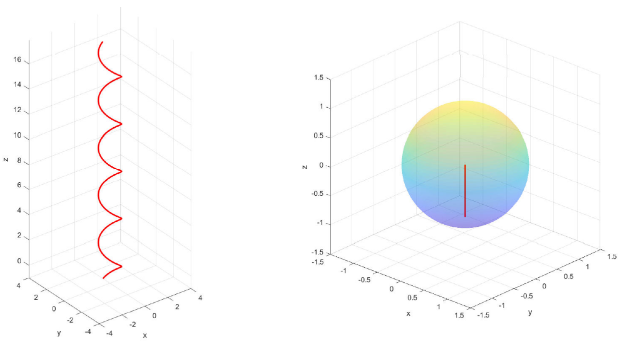

FIGURE 5. E_{e_2} - Rytov curve and the electric field E for $\langle E, e_2 \rangle = 0$, ($\theta = \pi/2$) respectively.

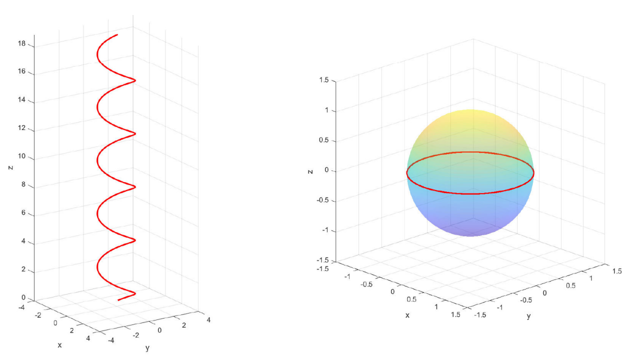

FIGURE 6. E_{e_3} - Rytov curve and the electric field E for $\langle E, e_3 \rangle = 0$, ($\theta = \pi/2$) respectively.

Example 5.2 Let $\gamma(s) = ((e^3/_\star e^4) \cdot_\star \sin_\star s -_\star (e/_\star e^{12}) \cdot_\star \sin_\star s, -_\star(e^3/_\star e^4) \cdot_\star \cos_\star s +_\star (e/_\star e^{12}) \cdot_\star \cos_\star 3s, (e^{\sqrt{3}}/_\star e^2) \cdot_\star \sin_\star s)$ be a unit speed curve. Then we have the Frenet apparatus of $\gamma(s)$ to the alternative moving frame:

$$
\begin{aligned}
e_1(s) &= \gamma^{\star\star}(s)/_\star \|\gamma^{\star\star}(s)\|_\star \\
&= ((e^{\sqrt{3}}/_\star e^2) \cdot \star \cos_\star 2s, \\
&\quad (e^{\sqrt{3}}/_\star e^2) \cdot \star \sin_\star 2s, -_\star(e/_\star e^2)), \\
e_2(s) &= e_1(s)/_\star \|e_1(s)\|_\star = (-_\star \sin_\star 2s, \cos_\star 2s, 0_\star), \\
e_3(s) &= (\tau \cdot \star t +_\star \kappa \cdot \star b)/_\star \star\sqrt{\kappa^{2\star} +_\star \tau^{2\star}} \\
&= ((e/_\star e^2) \cdot \star \cos_\star 2s, (e/_\star e^2) \cdot \star \sin_\star 2s, (e^{\sqrt{3}}/_\star e^2)), \\
\alpha(s) &= \star\sqrt{\kappa^{2\star} +_\star \tau^{2\star}} = e^{\sqrt{3}}, \\
\beta(s) &= \sigma \cdot_\star \alpha(s) = -_\star 1_\star.
\end{aligned}
$$

In the figures below, we present the E-Rytov curves for three different cases. Specifically, the electric fields are shown on the sphere in Figs. 7 to 9, respectively:

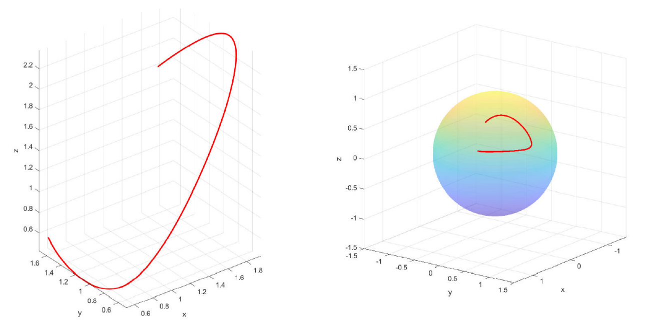

FIGURE 7. E_{e_1} - Rytov curve and the electric field E for $\langle E, e_1 \rangle = 0$, respectively.

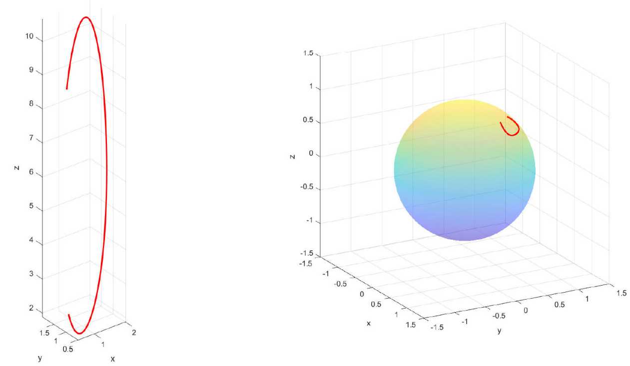

FIGURE 8. E_{e_2} - Rytov curve and the electric field E for $\langle E, e_2 \rangle = 0$, respectively.

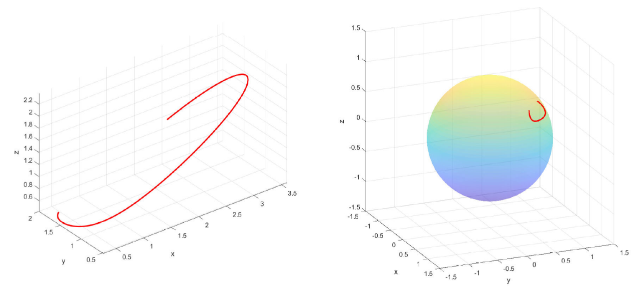

FIGURE 9. E_{e_3} - Rytov curve and the electric field E for $\langle E, e_3 \rangle = 0$, respectively.

6. Conclusion

The relationship between multiplicative and classical (additive) derivatives is elegantly captured by the expression $f^{\star}(x) = e^{xf^{'}(x)/f(x)}$, where $f^{\star}(x)$ denotes the multiplicative derivative and $f^{'}(x)$ the classical derivative. This formula highlights how the multiplicative derivative encodes relative (percentage-like) changes in a function, in contrast to the absolute rate of change measured by the classical derivative. A key advantage of the multiplicative derivative lies in its natural applicability to problems involving exponential growth, scaling behaviors, or multiplicative processes, as seen in fields such as economics, biology, and information theory. It allows for more intuitive modeling in systems where changes are better understood in proportional rather than additive terms. However, a notable drawback is that it requires the function to be strictly positive and differentiable, limiting its applicability compared to the more broadly defined classical derivative. The comparison between multiplicative helix and general helix is given for E-Rytov curves in Example 5.1. Geometrically, the multiplicative derivative can be interpreted as a proportion.

Electromagnetic waves consist of electric and magnetic fields that oscillate perpendicular to each other and propagate synchronously. The paths of these waves can be bent or curved due to different environmental conditions and boundary effects. In addition, they are applied in different areas such as optics, fiber systems, radio waves, communication, astronomy, medical imaging, etc. Moreover, it is used in the definition of the Rytov curve, where phase perturbations and diffraction effects on the wavefront are described by the curve. The wavefront is distorted by diffraction and scattering when waves travel over a randomly changing medium, such as the atmosphere. The small-scale diffraction effects of optical and electromagnetic waves are examined using the Rytov approximation. Furthermore, there are several significant application areas, such as optics, laser beams, radio waves, acoustic waves, etc. Considering these fundamental application areas, in this paper, we reinterpreted these curves using an alternative moving frame in non-Newtonian analysis. Unlike studies in the literature, these curves were expressed by exponential functions using non-Newtonian analysis techniques. Therefore, the results helped to understand the growth, scaling, and ratio-based systems for these curves. Moreover, we gave an example and visualized their images using the MATLAB program in multiplicative calculus.

Acknowledgements

The authors would like to thank the editor and the anonymous reviewers for their valuable comments and constructive suggestions, which have greatly improved the quality of this paper.

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