NUCLEAR MATTER PROPERTIES*

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RESUMEN

The Moszkowski and Scott approximation of Brueckner's many body theory is put in a suitable form for calculation, when the two body model has tensor spin orbit and quadratic spin orbit coupling.

I INTRODUCTION

In 1960 Moszkowski and Scott[1] proposed an approximation to the Brueckner method, when it is applied to nuclear matter. This approximation which was proposed as a simplification of the Brueckner method, is based on two observations:

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1. At short distances (less than 1 Fermi) the two nucleons do not feel the presence of nuclear matter (i. e., the rest of nucleons), since the average distance between nucleons is about two Fermis.

2. For the long range part the nucleon is strongly affected by the presence of the other nucleons because of the Pauli principle.

Then the scattering of two nucleons is not like the scattering of two free nucleons. The Pauli principle does not allow the energies of the scattered nucleons to be different from the energies that they had before the scattering. Thus, the physical presence of the other nucleons restricts elastic scattering to the forward direction. Therefore, the nuclear matter two particle wave function (i. e., the wave which describes the mutual behavior of two nucleons in the presence of nuclear matter) at long distances becomes a plane wave.

II. THEORY

A. Summary of the Brueckner Method

Before discussing the method of Moszkowski and Scott in more detail, some of the mathematical features of the Brueckner theory of nuclear matter[2] will be reviewed. According to Brueckner, one generates the nuclear matter ground state by perturbing the ground state of a nucleon Fermi gas. The unperturbed Hamiltonian is

$$H_o = \sum_{i=0}^{P_F} P_i^2/2m,$$

where m is the average mass of a neutron and P_F the Fermi momentum, related to the density of nucleons by[3]

$$\rho = 4/h^3 \frac{4\pi r}{3} P_F^3,$$

where the first factor 4 is due to the fact that for each momentum state in the ground state of a Fermi gas there are four combinations of spin and isotopic spin. The energy E_o of the ground state of H_o is

$$E_o = \frac{3}{5} \frac{P_F^2}{2m} N,$$

where N is the number of nucleons.

The perturbation is assumed equal to the sum of all the two nucleon interactions, namely

$$W = \frac{1}{2} \sum_{\substack{i,j \\ i \neq j}} V_{ij},$$

where V_{ij} is the potential between the nucleons i and j. The possibility of three or more body interactions (i. e., potentials like V_{ijk}, etc.) is ignored.

The total Hamiltonian is then,

$$H = H_o + W.$$

If the antisymmetric state vector $| \psi_o >$ and $| \psi >$ denote the ground state of H_o and H respectively, then

$$(H_o + W) | \psi > = (E_o + E) | \psi >,$$

where E is the energy shift due to the perturbation. Multiplying on the left by $< \psi_o |$,

$$< \psi_o | H_o | \psi > + < \psi_o | W | \psi > = (E_o + E) < \psi_o | \psi >.$$

Since one supposes that $| \psi_o >$ and $| \psi >$ have the same boundary conditions,

$$< \psi_o | H_o | \psi > = E_o < \psi_o | \psi >.$$

Taking the normalization of $|\psi>$ and $|\psi_o>$ such that

$$<\psi_o|\psi_o> = 1 \text{ and } <\psi_o|\psi> = 1,$$

$$E = <\psi_o|W|\psi>. \tag{II-1}$$

One can think of E as the sum of all one particle potentials in nuclear matter. Specifically,

$$E(\vec{k}_F) = \sum_{\vec{k}_i = 0}^{\vec{k}_F} u(\vec{k}_i), \tag{II-2}$$

where $u(\vec{k}_i)$ is the one particle potential such that the total energy of the nucleon i with momentum k_i in nuclear matter is given by

$$\epsilon(\vec{k}_i) = \frac{1}{m} k_i^2 + u(\vec{k}_i) \tag{II-3}$$

That is, $u(\vec{k}_i)$ represents the total interaction of the nucleon i with momentum $\vec{k}_i$ when it moves through nuclear matter.

Brueckner, rearranging terms in the explicit perturbation expansion, found the expression for $u(\vec{k}_i)$ as

$$u(\vec{k}_i) = \frac{1}{2}\sum_j <\psi_o|K_{ij}|\psi_o>, \tag{II-4}$$

where K_{ij} is a two particle operator, given by the implicit equation,

$$K_{ij} = V_{ij} + V_{ij}\frac{Q}{e}K_{ij}. \tag{II-5}$$

The subscripts identify the two particles i and j, which collide with momenta $\vec{k}_i$ and $\vec{k}_j$, inside nuclear matter. $\underset{e}{Q}$ is the propagator that characterizes the way that particles i and j are excited when they are dispersed in nuclear matter. Specifically,

$$e = \epsilon(\vec{k}_i) + \epsilon(\vec{k}_j) - \epsilon(\vec{k}_i') - \epsilon(\vec{k}_j'),$$

and

$$Q = 0, \text{ if } k_i', k_j', \leq k_F$$

$$= 1, \text{ if } k_i', k_j', > k_F .$$

That is, Q is the projection operator in momentum space for the region outside the Fermi Sea.

$\epsilon(\vec{k}_i')$ represents a virtual excitation of the particle i from momentum k_i to k_i' and is given by an equation similar to (II-3).

Since all of the energies appearing in the expression for e depend on the K matrix from equations (II-3) and (II-4), the solution of equation (II-5) requires a self-consistent procedure. That is, one must assume an $\epsilon(\vec{k})$ to obtain K from equation (II-5), which is later used to obtain a new, generally different $\epsilon(\vec{k})$ from equation (II-4). The procedure is continued until $\epsilon(\vec{k})$ is the same as the one calculated.

Finally Brueckner obtained the average binding energy per nucleon as

$$E_{av} = 3/P_F^3 \int_0^{P_F} dP \ [\ P^2/2m + u(p)\ ] .$$

Besides equation (II-5), there exists another important relationship between V_{ij} and K_{ij}, which will be used in the next section. From equations (II-1), (II-3) and (II-4), one easily sees that the energy shift E of the ensemble of nucleons has

the two expressions,

$$E = \frac{1}{2} \sum_{i,j} \langle \psi_o | V_{ij} | \psi \rangle$$

and

$$E = \frac{1}{2} \sum_{i,j} \langle \psi_o | K_{ij} | \psi_o \rangle ,$$

It is assumed that ψ_o can be expressed as a product of single-particle wave functions. Then calling φ^N the two-particle-wave function, which represents the mutual behavior of i and j in the presence of nuclear matter and φ the corresponding two-free-particle wave function (plane wave).

$$E = \frac{1}{2} \sum_{\substack{i,j \\ i \neq j}} \langle \varphi | V_{ij} | \varphi^N \rangle = \frac{1}{2} \sum_{\substack{i,j \\ i \neq j}} \langle \varphi | K_{ij} | \varphi \rangle ,$$

and

$$\langle \varphi | V_{ij} | \varphi^N \rangle = \langle \varphi | K_{ij} | \varphi \rangle .$$

Since φ is any eigenfunction of the free-two-particle Hamiltonian,

$$V_{ij} | \varphi^N \rangle = K_{ij} | \varphi \rangle .$$

B. Separation Method of Moszkowski and Scott.

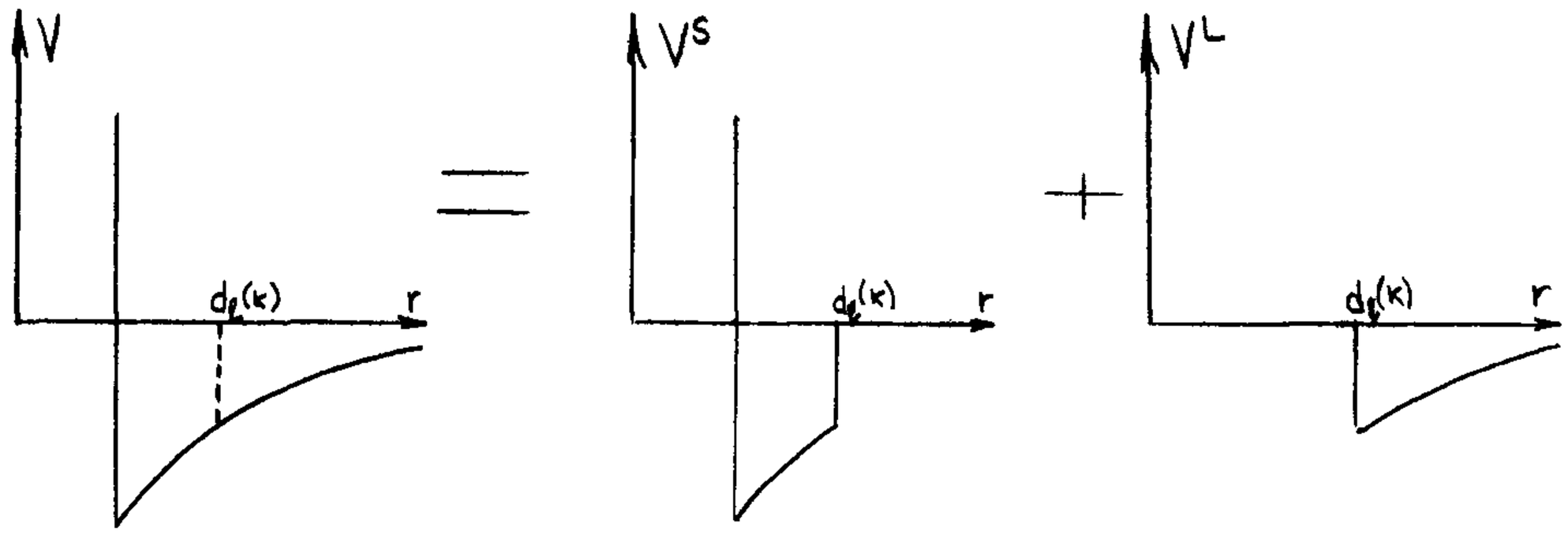
This method consists of two successive approximations to the Brueckner method. These successive approximations henceforth will be called AP1 and AP2. As a result of numerical calculations*, AP1 gives roughly 90% of the average of one particle potential in nuclear matter and AP2 accounts for almost all of the remaining 10%.

The advantage of this procedure is in the simple physical foundation of AP1. This foundation was explained in the introduction and will be rewieved and formally expressed in the following paragraphs.

The two nucleon potential V is divided into short and long range parts, namely

$$V = V^S + V^L \,, \tag{II-7}$$

or graphically:



The separation distance $d_l\,(k)$ is determined so that V^S produces zero phase shift for nucleon-nucleon scattering in the particular angular momentum state l and for a given relative momentum k.

First Approximation

Let φ^N be the true two-particle wave function in nuclear matter when the two-body potential is V.

Let φ^S and φ^L be the corresponding φ^N when the two-body potential is V^S and V^L respectively.

φ^F will denote the two-particle wave function when no nuclear matter is present, that is to say, φ^F is simply the two-particle-Schrodinger-equation solution with potential V.

For the first approximation of Moszkowski and Scott, the following assumption is made: φ^N is equal to φ^F for distances less than d and equal to the unperturbed wave function (plane wave) for distances greater than d. This means that

the behavior of the two nucleons in nuclear matter for distances less than d is the same as for two free colliding nucleons. In other words, the presence of the other nucleons is not felt by the two nucleons when their mutual separation distance r is less than d.

From equation (II-6a) the energy shift of the particles i and j is given by

$$\Delta E_{ij} = \langle \varphi | K_{ij} | \varphi \rangle , \qquad \text{(II-8)}$$

where $| \varphi \rangle$ is the unperturbed state vector (plane wave). This rather formal expression does not show the statistical factors which arise from the different spin orientations and the Pauli principle. Now by equations (II-6c) and (II-7) the preceding equation becomes

$$\Delta E_{ij} = \langle \varphi | V^S_{ij} | \varphi^N \rangle_0^d + \langle \varphi | V^L_{ij} | \varphi^N \rangle_d^\infty . \qquad \text{(II-9)}$$

The limits o, d and ∞ are those of the radial integrals.

In AP1, φ^N becomes φ^F in the first matrix element of equation (II-9) and equal to φ in the second one. Therefore, in AP1

$$\Delta E_{ij} = \langle \varphi | V_{ij} | \varphi^F \rangle_0^d + \langle \varphi | V^L_{ij} | \varphi \rangle_d^\infty . \qquad \text{(II-10)}$$

The boundary conditions of φ and φ^F at zero and d are:

$$\varphi(0) = \varphi(0) = \varphi^F(0) = 0 ,$$

and, by definition of d,

$$\varphi(d) = \varphi^F(d) \text{ and } \left(\frac{d\varphi}{dr}\right)_d = \left(\frac{d\varphi^F}{dr}\right)_d .$$

These three equations make the first matrix element of equation (II-10) vanish, and therefore,

$$\Delta E_{ij} = \langle \varphi \,|\, V^L_{ij} \,|\, \varphi^L \rangle^{\infty}_{d}.$$

If the phase shift $\delta_l(k)$ of V^S does not vanish for any distance d, one chooses d such that $\delta_l(k)$ be a minimum and then the first matrix element of the expression (II-9) contributes $-4\pi\hbar^2/m \ \dfrac{\delta_l(k)}{k}$ to the energy shift.

Second Approximation

The second approximation (AP2) of Moszkowski and Scott[1] is obtained by substituting AP1 into the correct expression for K (equation (II-5)) in the following way (the subscripts i and j have been dropped),

$$K = V + V \frac{Q}{e} K. \tag{II-11}$$

If $V = V^S$, one has what could be called the short distance reaction matrix K^S,

$$K^S = V^S + V^S \frac{1}{e_o} K^S ,$$

where e_o is the free propagator since at short distances the presence of nuclear matter is neglected. Thus,

$$e_o = E_o - H_o ,$$

where H_o is the free Hamiltonian and E_o its value in the ground state. The corresponding long distance reaction matrix K^L in AP1 is just V^L since in the exact expression

$$V^L \mid \varphi^N> = K^L \mid \varphi>$$

one assumes that $\mid \varphi^N> = \mid \varphi>$ for r greater than d. For r less than d, V^L equals zero by definition of V^L. Consequently in AP1 K_1 is

$$K_1 = K^S + V^L .$$

From

$$V = V^L + V^S \quad \text{and} \quad V^S = (1 + K^S \frac{1}{e_o})^{-1} K^S$$

one has

$$V = V^L + (1 + K^S \frac{1}{e_o})^{-1} K^S .$$

Substituting this expression in equation (II-11),

$$K = K^S + K^S (\frac{1}{e} - \frac{1}{e_o}) K + (1 + K^S \frac{1}{e_o}) V^L (1 + \frac{1}{e} K) .$$

Making K in the second member equal K_1, one obtains K in AP2; that is

$$K_2 = K_1 + \text{P.T.} + \text{D.T.} + \text{L.D.T.} + \text{C.T.},$$

where

$$\text{P.T.} = K^S \frac{Q-1}{e} K^S \qquad \text{(Pauli Term)},$$

$$\text{D.T.} = K^S \left(\frac{1}{e} - \frac{1}{e_o}\right) K^S \qquad \text{(Dispersion Term)},$$

$$\text{L.D.T.} = V^L \underline{\underline{Q}}_{e} V^L \qquad \text{(Long Distance Term)},$$

and

$$\text{C.T.} = V^L \underline{\underline{Q}}_{e} K^S + K^S \underline{\underline{Q}}_{e} V^L \qquad \text{(Cross Term)},$$

C. First and Second Approximations in Suitable Form for Computation

It is now convenient to rewrite the formal relations, which appear in AP1 and AP2, in a form suitable for calculation. This has to be done according to the mathematical structure of the two-body potential.

The Hamada-Johnston potential, which is the most complex potential considered in this work, has the following terms:

1) Central $= V_C(r)\ (\sigma_1 \cdot \sigma_2)\ (\tau_1 \cdot \tau_2)$,

2) Tensor $= V_T(r)\ (\tau_1 \cdot \tau_2)\ S_{12}$,

3) Spin orbit coupling $= V_{L \cdot S}(r)\ (L \cdot S)$.

4) Quadratic spin orbit coupling $= V_{LL}(r)\ L_{12}$,

where

$$S_{12} = 3 - \frac{(\sigma_1 \cdot r)\ (\sigma_2 \cdot r)}{r^2} - (\sigma_1 \cdot \sigma_2),$$

$$L \cdot S = \frac{1}{2}(r \times p)\ (\sigma_1 + \sigma_2),$$

$$L_{12} = \frac{1}{2}\left[\ (\sigma_1 \cdot L)\ (\sigma_2 \cdot L) + (\sigma_2 \cdot L)\ (\sigma_1 \cdot L)\ \right],$$

$S_1 = \frac{\sigma_1}{2}$ = spin of nucleon 1,

$S_2 = \frac{\sigma_2}{2}$ = spin of nucleon 2,

$\vec{r}$ = relative distance between the two nucleons.

$T_1 = \frac{1}{2}\tau_1$ = isotopic spin of nucleon 1,

and $T_2 = \frac{1}{2}\tau_2$ = isotopic spin of nucleon 2.

The good quantum numbers for a system with this kind of potential are:

1) the total angular momentum $J = L + S$ and its component J_z along the z axis,
2) the parity since the potential is invariant under spatial reflection,
and 3) the total spin S, since the potential is not altered under the exchange of σ_1 and σ_2.

Since the nucleon is a fermion, any state of the system must be antisymmetric with respect to the interchange of all coordinates of the two nucleons. Therefore the following relation holds:

(spin exchange) (space exchange) (isotopic spin exchange) $= -1$.

The state of the system will be a linear combination of the total angular momentum L by the formula

$$F_l^{jm_j s}(\hat{k},\hat{r}) = \sum_{m_s=-1}^{1} (jm_j, lm_j - m_s, sm_s)\, Y_l^{m_j - m_s}(\hat{k},\hat{r})\, X_s^{m_s},$$

where $(jm_j, lm_j - m_s, sm_s)$ are the Clebsch-Gordan coefficients for $m_l = m_j - m_s$.

The plane wave of momentum k, $\langle r \mid X_s^{m_s} K\rangle = e^{i\vec{k}\cdot\vec{r}} X_s^{m_s}$, with the spin eigenfunction explicitly indicated, has the partial wave expansion:

$$\langle r | X_s^{m_s} k \rangle = \sum_{l=0}^{\infty} (2l+1)\, i^l \left(\frac{4\pi}{2l+1}\right)^{1/2} Y_l^0(\hat{k},\hat{r})\, j_l(kr)\, X_s^{m_s},$$

where $j_l(kr)$ is the regular spherical Bessel function of angular momentum l.

Since

$$X_s^{m_s}\, Y_l^0(\hat{k},\hat{r}) = \sum_{j=l-1}^{l+1} (jm_s, lo, sm_s)\, F_l^{jm_j s}(\hat{k},\hat{r}),$$

the plane wave expansion becomes

$$\langle r | X_s^{m_s} k \rangle = \sum_{j=0}^{\infty} \sum_{l=j-1}^{l=j+1} (2l+1)\, i^l \left(\frac{4\pi}{2l+1}\right)^{1/2} j_l(kr)\, (jm_s, lo, sm_s)\, F_l^{jm_s s}(\hat{k},\hat{r}). \tag{II-12}$$

The two nucleon state, φ^F, which is the solution of the Schrödinger equation with a tensor potential, has an expansion similar to that of the plane wave. The difference lies in the coupled state: $l = j+1$ and $l = j-1$. The unperturbed solutions $F_{j+1}^{jm_j s}(\hat{k},\hat{r})\, j_{j+1}(kr)$, under the tensor action, goes over into

$$F_{j+1}^{jm_j s}(\hat{k},\hat{r})\, U_{j+1,\,j+1}(r) + F_{j-1}^{jm_j s}(\hat{k},\hat{r})\, U_{j+1,\,j-1}(r)$$

where $U_{j+1,\,j+1}(r)$ and $U_{j+1,\,j-1}(r)$ are solutions of the coupled radial Schrödinger equations. If the tensor force is switched off, then $U_{j+1,\,j-1}(r)$ vanishes.

In an analogous way one finds that $F_{j-1}^{jm_j s}(\hat{k},\hat{r})$ goes over into $F_{j-1}^{jm_j s}(\hat{k},\hat{r})\, U_{j-1,\,j-1}(r) + F_{j+1}^{jm_j s}(\hat{k},\hat{r})\, U_{j-1,\,j+1}(r)$. Consequently, the expression for φ^F, $^0P_1$ written explicity as the sum of the two coupled and the uncoupled term, has the following form:

$$\varphi_k^F(r) = \sum_{j=0}^{\infty} [\alpha_{j-1,\, ms,\, j}(F_{j-1}^{jm_s s}(\hat{k},\hat{r})\ U_{j-1,\, j-1}(r) + F_{j+1}^{jm_s s}(\hat{k},\hat{r})\ U_{j-1,\, j+1}(r))$$

$$+ \alpha_{j+1,\, m_s,\, j}(F_{j+1}^{jm_s s}(\hat{k},\hat{r})\ U_{j+1,\, j+1}(r) + F_{j-1}^{jm_s s}(\hat{k},\hat{r})\ U_{j+1,\, j-1}(r))$$

$$+ \alpha_{j,\, m_s,\, j} F_j^{jm_s s}(\hat{k},\hat{r})\ U_{j,\, j}(r)\ ] , \qquad \text{(II-13)}$$

where

$$\alpha_{l,\, m_s,\, j} = (2l+1)\ i^l \left(\frac{4\pi}{2l+1}\right)^{1/2} (jm_s,\, lo,\, m_s s) .$$

Since in AP2 one has to deal with matrix elements taken between functions corresponding to different momenta k and k', the plane wave and φ^F must be re-expanded in other forms more convenient for future developments.

The usual plane wave expansion is

$$\langle r \,|\, K_s^{m_s} k\rangle = e^{i\vec{k}\cdot\vec{r}}\, X_s^{m_s} = \sum_{l=0}^{\infty} (2l+1)\ i^l \left(\frac{4\pi}{2l+1}\right)^{1/2} j_l(kr)\ Y_l^0(\hat{k},\hat{r})\ X_s^{m_s}$$

From the addition theorem,

$$\frac{(2l+1)}{4\pi}^{1/2} Y_l^0(\hat{k},\hat{r}) = \sum_{m_l=l}^{l} Y_l^{m_l^*}(\hat{k},\hat{k}')\, Y_l^{m_l}(\hat{k},\hat{r}) \quad .$$

Using this equation in the plane wave expression,

$$\langle r \,|\, X_s^{m_s} k\rangle = 4\pi \sum_{l=0}^{\infty} \sum_{m_l=-l}^{l} j_l(kr)\ i^l\, Y_l^{m_l^*}(\hat{k},\hat{k}')\ Y_l^{m_l}(\hat{k},\hat{r})\ X_s^{m_s} .$$

Introducing now the eigenfunction of j given by the equation

$$F_{l\ m_l}^{jm_l + m_s,\ s}(\hat{k},\hat{r}) = \sum_{j=l-1}^{l+1} Y_l^{m_l}(\hat{k},\hat{r})\ X_s^{m_s}(jm_s + m_l, lm_l, sm_s)\ ,$$

one finally obtains

$$\langle r \mid X_s^{m_s} k\rangle = 4\pi \sum_{j=0}^{\infty} \sum_{l=j-1}^{j+1} \sum_{m_l} j_l(kr)\ i^l\ Y_l^{m_l *}(\hat{k},\hat{k}')\ F_l^{jm_l + m_{s'},\ s}(\hat{k},\hat{r})\ . \quad \text{(II-14)}$$

Similar treatment for φ^F leads to

$$\varphi_k^F(r) = 4\pi \sum_{j=0}^{\infty} \sum_{l=j-1}^{j+1} \sum_{m_l=-l}^{l} Y_l^{m_l}(\hat{k},\hat{k}')\ (j_{m_l + m_{s'}}, l_{m_l}, s_{m_s})\ i^l$$
$$\left(\sum_{l'=j-1}^{j+1} U_{l,l'}^{k}(r)\ F_l^{jm_l + m_{s'},\ s}(\hat{k},\hat{r}) \right)\ . \quad \text{(II-15)}$$

The subscripts (l, l') can only have values of the same parity which are:

$$(j,j),\ (j-1,j-1),\ (j+1,j+1),\ (j-1,j+1) \text{ and } (j+1,j-1)\ .$$

First approximation - AP1. The B.E. of the nucleon Fermi gas is given by

$$\text{B.E.} = \frac{1}{2}\hbar^{2/m}\left(\frac{3}{5}k_F^2\right) + \bar{U}(k_F)\ ,$$

where $\frac{3}{5}k_F^2$ is the average kinetic energy in the Fermi sea and $\bar{U}(k_F)$ is the average one nucleon potential inside nuclear matter. That is,

$$\bar{U}(k_F) = \int_0^{k_F} u(k)\ P(k)\ dk, \qquad \text{(II-16)}$$

where $P(k)$ is the probability of finding two nucleons in the Fermi sea with a relative momentum k.

It can be shown that

$$P(k) = \frac{8k}{\pi^2}\left(1 - \frac{3}{2}\, k/k_F + \frac{1}{2}\, k^3/k_F^3\right).$$

$u(k)$ is the interaction of two nucleons in nuclear matter with relative momentum k. Including the statistical factors, the expression for $u(k)$ is

$$u(k) = \frac{1}{4} \sum_{T=0}^{1} \frac{2T+1}{2} \sum_{S=0}^{1} \sum_{m_s=-1}^{1} u_{s,m_s,T}(k).$$

Here T is the total isotopic spin and $u_{s,m_s,T}(k)$ is the energy due to the interaction of two nucleons with relative momentum k in a state with total spin s, total isotopic spin T and spin component m_s. If $\vec{k}_1$ and $\vec{k}_2$ are the momenta of the two particles before the interaction, then

$$u_{s,m_s,T}(k) = \frac{1}{\Omega} \langle X_s^{m_s}(k_1 + k_2) \,|\, V^L \,|\, X_s^{m_s}(k_1 + k_2) \rangle .$$

Going over to the relative and center of mass coordinates, this equation becomes

$$u_{s,m_s,T}(k) = \langle X_s^{m_s}\ k \,|\, V^L \,|\, X_s^{m_s}\ k \rangle .$$

According to equation (II-12)

$$u_{s,m_s,T}(k) = \sum_{j=0}^{\infty} \sum_{l'=j-1}^{j+1} \sum_{l=j-1}^{j+1} (2l+1)\, i^l \left(\frac{4\pi}{2l+1}\right)^{1/2} (2l'+1) \left(\frac{4\pi}{2l'+1}\right)^{1/2} i^{l'}$$

$$\int_0^{\infty} dr\, j_l(kr)\, V_{ll'}^{js}\, j_{l'}(kr)\, r^2, (jm_s, l'\, o, s m_s)\, (jm_s, lo.\, s m_s)$$

where

$$V_{ll'}^{js} = \int F_l^{jm_s s^*}(\hat{k},\hat{r})\, V\, F_{l'}^{jm_s, s}(\hat{k},\hat{r})\, d(\hat{k},\hat{r}) .$$

Since $u_{s,m_s,T}(k)$ depends on m_s only in the Clebsch-Gordan coefficients, which have the property that

$$\sum_{m_s=-1}^{1} (jm_s, l'\, o, m_s s)\, (jm_s, l'\, o, m_s s) = \frac{2j+1}{2j+1}\, \delta_{ll'} ,$$

one concludes that

$$u_{sT}(k) = \sum_{m_s=-1}^{1} \mu_{s,m_s T}(k) = 4\pi \sum_{j=0}^{\infty} \sum_{l=j-1}^{j+1} (2j+1) \int_d^{\infty} r^2\, j_l(kr)\, V_{ll}^{js}\, j_l(kr)\, dr .$$

Finally,

$$u(k) = \frac{1}{4} \sum \frac{2T+1}{2} \sum_s u_{s,T}(k) .$$

The coefficient $\frac{1}{4} \sum_{T=0}^{1} \frac{2T+1}{2} \sum_{s=0}^{1}$ accounts for the statistical properties of the nucleon Fermi gas.

Second approximation - AP2. As in AP1 the average one-particle-potential (O.P.P.) is given by

$$\overline{U}(k_F) = \int_0^{k_F} u(k)\ P(k)\ dk .$$

In AP2 $u(k)$ will be the sum of the contributions from the Pauli term, the dispersion term, the cross term and the long range term; that is,

$$u(k) = u^P(k) + u^D(k) + u^C(k) + u^L(k) .$$

a.- Pauli Term.

$$u^P(k) = \frac{1}{4} \sum \frac{2T+1}{2} \sum_s \sum_{m_s} u^P_{s, m_s, T}(k) ,$$

where

$$u^P_{s, m_s, T}(k) = \langle X_s^{m_s}\ k \,|\, K^s\ \frac{Q-1}{e}\ K^s \,|\, X_s^{m_s}\ k \rangle .$$

We rewrite this equation as

$$u^P_{s, m_s, T}(k) = \langle X_s^{m_s}\ k \,|\, K^s\ \frac{1}{e_o}\ (Q-1)\ \frac{e_o^2}{e}\ \frac{1}{e_o}\ K^s \,|\, X_s^{m_s}\ k \rangle . \quad \text{(II-17)}$$

Since

$$V^S \,|\, \varphi^F \rangle = k^s \,|\, X_s^{m_s}\ k \rangle ,$$

or

$$V^S \,|\, \varphi^F\!> \;=\; V^S \,|\, X_s^{m_s}\!> \;+ V^S \;\frac{1}{e_o}\; K^S \,|\, X_s^{m_s}\!> \,,$$

we have

$$|\, \varphi^F - X_s^{m_s}\; k\!> \;=\; \frac{1}{e_o}\; K^S \,|\, X_s^{m_s}\; k\!> \,.$$

Substituting this expression in equation (II-17), one obtains

$$u^P_{s,\, m_s,\, T}(k) \;=\; <\varphi^F - X_s^{m_s}\; k\,|\; (Q-1)\;\frac{e_o^2}{e}\,|\; \varphi^F - X_s^{m_s}\; k\!> \,,$$

or, in momentum space,

$$u^P_{s,m_s,\,T}(k) \;=\; \sum_{k'}\; <\varphi^F - X_s^{m_s}\; k\,|\, X_s^{m_s}\; k'\!><X_s^{m_s}\; k'\;|\; (Q-1)\;\frac{e_o^2}{e}\,|\,\varphi^F - X_s^{m_s}\; k\!> \,.$$

Using the equivalence,

$$\sum_k \;\rightarrow\; \frac{\Omega}{(2\pi)^3}\int d_3 k \,,$$

$$u^P_{s\,,\,m_s,\,T}(k) \;=\; \frac{1}{(2\pi)^3}\;\left(\frac{\hbar^2}{m}\right)^2 \int_o^\infty dk'\; k'^2\,(k^2 - k'^2)^2\;\frac{Q\,(P,\,k')\;-\;1}{e\,(P,\,k,k')}\int d\,(\hat{k},\,\hat{k}')\;.$$

$$<X_s^{m_s}\ k' \mid \varphi^P - X_s^{m_s}\ k> ^2 ,$$

where P is the center of mass (C.M.) momentum and $|X_s^{m_s}\ k'>$ is the state vector of two non-interacting nucleons in relative and C.M. coordinates. From equations (II-14) and (II-15) one obtains

$$<X_s^{m_s}\ k' \mid \varphi^P - X_s^{m_s}\ k> = (4\pi)^2 \sum_j \sum_l \ [\ (\sum_{l'}\ A_{ll'}^{kk'}\ i^{(l-l')}\ I_{ll'}^{kk'}) - A_{ll}^{kk'}\ H_{ll}^{kk'}\ ]\ . \tag{II-18}$$

The integration over the solid angle $(\hat{k}, \hat{r})$ has already been done.

In the above equation,

$$A_{ll'}^{kk'} = \sum_{m_l} Y_l^{m_l *}\ (\hat{k}, \hat{k})\ Y_l^{m_l}(\hat{k}, \hat{k}')\ (jm_l + m_s, lm_{l'} sm_s)\ (jm_{l'} + m_s, l'm_{l'} sm_s),$$

$$A_{ll}^{kk'} = \sum_{m_l} Y_l^{m *}\ (\hat{k}, \hat{k})\ Y_l^{m_l}(\hat{k}, \hat{k}')\ (jm_l + m_s, lm_l, sm_s)^2 ,$$

$$I_{ll'}^{kk'} = \int_o^d U_{ll'}^{k}\ (r)\ j_{l'}\ (kr)\ r^2\ dr$$

and

$$H_{ll}^{kk'} = \int_o^d j_l\,(kr)\ j_l\,(k'r)\ r^2\ dr ,$$

where $d(k)$ is the cutoff distance which separates the long from the short range parts of the potential.

When $\langle X_s^{m_s}\, k' \mid \varphi^F - X_s^{m_s}\, k\rangle$ is squared and integrated over the solid angle $(\hat{k}, \hat{k}')$ one obtains

$$\int d(\hat{k}, \hat{k}')\, \langle I\rangle^2 = (4\pi)^4 \Big\{ \sum_j \sum_{l_1} \sum_{l_2} \Big[\Big(\sum_{l_1'} (AA)_{l_1 l_2 l_1'}\, I^{kk}_{l_1 l_1'}\, I^{kk}_{l_2 l_1'} \Big)$$

$$- (Aa)_{l_1 l_2}\, I^{kk}_{l_1 l_2}\, H^{kk}_{l_2 l_2} - (Aa)_{l_2 l_1}\, H^{kk}_{l_1 l_1}\, I^{kk}_{l_2 l_1} + a_{l_1 l_1}\, H^{kk}_{l_1 l_1}\, H^{kk}_{l_1 l_1} \Big] \Big\},$$

(II-19)

where

$$(AA)_{l_1 l_2 l_1'} = i^{l_1 - l_2} \sum_m Y^{m_1 *}_{l_1}(\hat{k}, \hat{k})\; Y^{m_1}_{l_2}(\hat{k}, \hat{k})\; (jm_1 + m_s, l_1 m_1, s m_s)$$

$$(jm_1 + sm, l_2 m_1, s m_s)\; (jm_1 + m_s, l_1' m_s, s m_s)^2,$$

$$(Aa)_{l_1 l_2} = i^{(l_1 - l_2)} \sum_{m_1} Y^{m_1}_{l_{11}}(\hat{k}, \hat{k})\; Y^{m_1}_{l_2}(\hat{k}, \hat{k})\; (jm_1 + m_s, l_1 m_1, s m_s)\; (jm_1 + m_s,$$

$$l_2 m_1, s m_s)^3,$$

$$(Aa)_{l_2 l_1} = i^{(l_1 - l_2)} \sum_{m_1} Y^{m_1}_{l_1}(\hat{k}, \hat{k})\; Y^{m_1}_{l_2}(\hat{k}, \hat{k})\; (jm_1 + m_s, l_1 m_1, s m_s)^3$$

$$(jm_1 + m_s, l_2, m_1 s m_s)$$

and

$$(aa)_{l_1 l_1} = \sum_{m_1} Y_{l_1}^{m_1}(\hat{k}, \hat{k})\, Y_{l_1}^{m_1}(\hat{k}, \hat{k})\, (jm_1 + s_m, l_1 m_1, s m_s)^4 .$$

Since (l_1, l_2) and (l_1, l_1') are coupled with the same parity, namely: (j, j), $(j-1, j-1)$, $(j+1, j+1)$, $(j+1, j-1)$ and $(j-1, j+1)$, the preceding expressions become quite simple. (See Table I.)

For the singlet state, (AA), (Aa), (aa) are all equal to

$$\sum_{m_l=-l}^{l} Y_l^{m^*}(o)\, Y_l^m(o) = \frac{2l+1}{4\pi}\, Y_l^o(o) = \frac{2l+1}{4\pi} .$$

In this case the Pauli term takes the form

$$u^P(k) = 8\left(\frac{\hbar^2}{m}\right)^2 \int_o^\infty dk' \left(k^2 - k'^2\right)^2 k'^2 \frac{\left(Q(P, k') - 1\right)}{e(P, k', k)} \sum c_l (2l+1)$$

$$\left[\left(I_{ll}^{kk'} - H_{ll}^{kk'} \right)^2 \right],$$

where c_l is the statistical factor equaling 3/8 for the singlet even and 1/8 for singlet odd state.

b.—Dispersion Term.

$$u^D(k) = \frac{1}{4} \sum_{T=0} \frac{2T+1}{2} \sum_{S=0}^{1} \sum_{m_s} u^D_{S, m_S, T}(k)$$

where

$$u^D_{S, m_S, T}(k) = \langle X_s^{m_s}\, k \,|\, K^S \left(\frac{1}{e} - \frac{1}{e_o}\right) K^S \,|\, X_s^{m_s}\, k \rangle ,$$

or, explicitly,

Table I

Coefficients for AP2 Calculation

$AA_{j(jj)} = (2j+1)/8\pi$

$AA_{j-1(j-1,j-1)} = (j^2 + (j+1)^2/2)/8\pi(2j-1)$

$AA_{j-1(j-1,j+1)} = (j(j+1))^{1/2}(1-j)/8\pi(2j-1)$

$AA_{j-1(j+1,j-1)} = AA_{j-1(j-1,j+1)}$

$AA_{j+1(j-1,j-1)} = 3(j+1)j/8\pi(2j+3)$

$AA_{j+1(j-1,j+1)} = (j(j+1))^{1/2}(1+j/2)/4\pi(2j+3)$

$AA_{j+1(j+1,j-1)} = A_{j+1}(j-1,j+1)$

$AA_{j+1(j+1,j+1)} = ((j+1)^2 + j^2/2)/4\pi(2j+3)$

$AA_{j-1(j+1,j+1)} = 3j(j+1)/8\pi(2j-1)$

The table gives the coefficients $AA_{l_3(l_1,l_2)}$ which are defined by $AA_{l_3(l_1,l_2)} = (2l_1+1)^{1/2}(2l_2+1)^{1/2}/4\pi \sum_{m_s=-1}^{1} (jm_s, l_1 0, sm_s)(jm_s, l_2 0, sm_s)(jm_s, l_3 0, sm_s)^2$, where (jm_j, lm_l, sm_s) is the Clebsch-Gordan coefficient.

According to the notation used in Section II, one has:

$$\sum_{m_s=-1}^{1} (AA)_{l_1 l_2 l_1'} = AA_{l_1'(l_1,l_2)}, \quad \sum_{m_s=-1}^{1} (Aa)_{l_1 l_2} = AA_{l_2(l_1,l_2)},$$

$$\sum_{m_s=-1}^{1} (Aa)_{l_2 l_1} = AA_{l_1(l_1,l_2)}, \text{ and } \sum_{m_s=-1}^{1} (aa)_{l_1 l_1} = AA_{l_1(l_1,l_1)}.$$

$$u^{D}_{s,\,m_s,\,T}(k) = \frac{1}{(2\pi)^3}\left(\frac{\hbar^2}{m}\right)^2 \int_0^{\infty} dk'\, k'^2 \left(k^2 - k'^2\right)^2 \left(\frac{1}{e(P,k,k')} - \frac{1}{\hbar^2/m\,(k^2 - k'^2)}\right)$$

$$\int d(\hat{k},\hat{k}')\, \left\langle \varphi^F - X_s^{m_s}\, k \middle| X_s^{m_s}\, k' \right\rangle^2 ,$$

where the last integral has been found in equation (II-19) .

For singlet state this formula is simplified as

$$u^{D}_{s,\,T}(k) = \sum_{m_s} u^{D}_{s,\,m_s,\,T}(k) = 8\,\hbar^2/m \int_0^{\infty} dk' k'^2 \left(k^2 - k'^2\right)$$

$$\left[\hbar^2/m\, \frac{(k^2 - k'^2) - e(P,k,k')}{e(P,k,k')}\right] \sum (2l+1)\left(I_{ll}^{kk'} - H_{ll}^{kk'}\right)^2$$

c.- *Cros term*

$$u^{c}(k) = \frac{1}{4} \sum_{T=0}^{1} \frac{(2T+1)}{2} \sum_{S=0}^{1} \sum_{m_s=-1}^{1} u^{c}_{s,\,m_s,\,T}(k) ,$$

where

$$u^{c}_{s,\,m_s,\,T}(k) = 2 \left\langle X_s^{m_s}\, k \middle| V^L\, \underline{\underline{Q}}_{e}\, K^S \middle| X_s^{m_s}\, k \right\rangle$$

since

$$\left| \varphi^F - X_s^{m_s}\, k \right\rangle = \frac{1}{e_o}\, K^s \left| X_s^{m_s}\, k \right\rangle .$$

$u^c_{s,\,m_s,\,T}(k)$ can be rewritten as

$$u^c_{s,\,m_s,\,T} = 2\,\langle X_s^{m_s}\,k\,|\,V^L\,Q\,\frac{e_o}{e}\,|\,\varphi^F - X_s^{m_s}\,k\rangle$$

or, in momentum space,

$$u^c_{s,\,m_s,\,T}(k) = 2\,\frac{I}{(2\pi)^3}\,\hbar^2/m\,\int_o^\infty dk'\,k'^2\,(k^2 - k'^2)\,\frac{Q(P,k')}{e(P,k,k')}\,\int d(\hat{k},\hat{k}')$$

$$\langle kX_s^{m_s}\,|\,V^L\,|\,k'\,X_s^{m_s}\rangle\,\langle k'\,X_s^{m}\,|\,\varphi^F - kX_s^{m_s}\rangle\,.$$

By equation (II-14)

$$\langle kX_s^{m_s}\,|\,V^L\,|\,k'\,X_s^{m_s}\rangle = (4\pi)^2\,\sum_j\,\sum_l\,\sum_{l'}\,A^{kk'}_{ll'}(V)\,i^{(l-l')}\,I(V)^{kk'}_{ll'}$$

where

$$A^{kk'}_{ll'}(V) = \sum_m\,\sum_{m'}\,Y_l^m(\hat{k},\hat{k})\,Y_{l'}^{m'*}(\hat{k},\hat{k}')\,(jm_s + m_l, lm_l, sm_s)\,(jm_s + m_l, l'm_{l'}, sm_s)$$

and

$$I^{kk'}_{ll'}(V) = \int_o^\infty j_l(kr)\,V^L_{ll'}\,j_{l'}(k'r)\,r^2\,dr\,.$$

Using this result as well as equation (II-18),

$$\int d(\hat{k},\hat{k}')\,\langle k\,X_s^{m_s}\,|\,V^L\,|\,k'\,X_s^{m_s}\rangle\,\langle k'\,X_s^{m_s}\,|\,\varphi^F - kX_s^{m_s}\rangle =$$

$$(4\pi)^4 \sum_j \sum_{l_1'} \sum_l \left[(AA)_{l_1 l l'} \; I^{kk'}_{l_1 l'} \; I^{kk'}_{l l'} - (AA)_{l_1 l l'} \; I^{kk'}_{l l_1} \; (V) \; H^{kk'}_{l_1 l_1} \right].$$

The notation is consistent with that used previously.

For the singlet state

$$u^c_{s,T}(k) = \sum_{m_s} u^c_{s,m_s,T}(k) = 2 \left(\frac{\hbar^2}{m}\right) 8 \sum_l (2l+1) \; I^{kk}_{ll} \; (V) \; (I^{kk'}_{ll} - H^{kk'}_{ll})$$

d.- Long Range Term

$$u^L_{s,m_s,T}(k) = \langle X^{m_s}_s \; k \,|\, V^L \, \underline{\underline{Q}}_e \, V^L \,|\, X^{m_s}_s \; k \rangle$$

$$= \int_o^\infty dk' \; k^{12} \; \frac{Q(P,k')}{e(P,k,k')} \int d(\hat{k},\hat{k}') \; \langle X^{m_s}_s \; k \,|\, V^L \,|\, k \; X^{m_s}_s \rangle^2 .$$

Using equation (II-3),

$$\langle X^{m_s}_s \; k \,|\, V^L \,|\, k \; X^{m_s}_s \rangle^2 = (4\pi)^4 \sum_j \sum_{l_1} \sum_{l_2} \sum_{l_1'} (AA)_{l_1 l_2 l_1'} \; I^{kk'}_{l_1 l_1'} \; (V) \; I^{kk'}_{l_2 l_1'} \; (V) .$$

In the singlet state,

$$u^L_{s,T}(k) = \sum_{m_s} u^L_{s,m_s,T}(k) = 8 \int_o^\infty dk' \; k'^2 \; \frac{Q(P,k')}{e(P,k,k')} \sum (2l+1) \left(I^{kk'}_{ll'}(V)\right)^2 .$$

e.- *Simplifying Approximations AP2*

Let $\vec{k}_1'$ and $\vec{k}_2'$ be the momenta of the two nucleons after the interaction.

They are related to the total momentum $\vec{P}$ and the relative momentum $\vec{k}'$ by

$$\vec{P} = \vec{k}'_1 + \vec{k}'_2 \text{ and } \vec{k}' = \frac{\vec{k}'_1 - \vec{k}'_2}{2}$$

By definition of Q,

$$Q(P, k') = 0 \quad \text{if} \quad k'_1 \text{ or } k'_2 \leqslant k_F$$

$$= 1 \quad \text{if} \quad k'_1 \text{ and } k'_2 > k_F .$$

It can be shown that these equations are equivalent to

$$Q(P, k') = 0 \quad \text{if} \quad k'^2 + P^2/4 < k_F^2 ,$$

$$= 1 \quad \text{if} \quad k' - P/2 > k_F ,$$

$$= \frac{k'^2 + P^2/4 - k_F^2}{k' P} \qquad \text{otherwise .}$$

The value of $\vec{P}$ will depend on $\vec{k}_1$ and $\vec{k}_2$, the momenta of the particles before the collision

The problem is simplified if $\vec{P}$ is taken equal to the average center of mass momentum over the Fermi sea of two nucleons having relative momentum $\vec{k} = \frac{\vec{k}_1 - \vec{k}_2}{2}$. It can be shown that this average is such that

$$P^2/4 = \frac{3}{5} k_F (k_F - k) \left[1 + \frac{k^2}{3 k_F (2k_F + k)} \right] , \; k < k_F$$

$$= 0 \qquad , \; k > k_F .$$

The propagator e can also be simplified if it is calculated in AP1 instead of using the self-consistent procedure. Thus

$$e = \hbar^2/m\,(k^2 - l'^2) + U\,(\vec{P}/2 + \vec{k}) + U\,(\vec{P}/2 - \vec{k}) - U\,(\vec{P}/2 + \vec{k}') - U\,(\vec{P}/2 -$$

where U is calculated in AP1.

The simplification can be facilitated if one supposes that $U\,(\vec{m})$ is calculated at an average momentum. Moszkowski and Scott chose this momentum as $(k_F^2/8 + m^2/4)^{1/2}$, which corresponds to the average value of the relative momentum of a particle with momentum $\vec{m}$ with respect to a particle of momentum $k_F/\sqrt{2}$.

The same authors assumed that

$$|\vec{P}/2 - \vec{k}| = |\vec{P}/2 + \vec{k}|\,,\ |\vec{P}/2 + \vec{k}'| = |\vec{P}/2 - \vec{k}'| \text{ and that } P = k_F\,.$$

Consequently, the propagator under these simplifications becomes

$$e\,(P, k, k') = \hbar^2/m\,(k^2 - k'^2) + 2k_F^3/3\pi^2\,(2\,U\,(k) - 2\,U\,(k'),$$

where $U(k)$ and $U(k')$ are calculated in AP1 at $(k_F^3/8 + k^2/4)^{1/2}$ and $(k_F^2/8 + k'^2/4)^{1/2}$, respectively.

D. Determination of the Cutoff Distance d

The distance, d, which separates the short from the long range potential, was determined by setting equal to zero those nuclear parameters, $x's$ and $y's$ used by J. L. McHale and R. L. Thaler[5], which are proportional to the matrix element of the potential in the Born approximation. In a coupled state they are defined by the asymptotic form of two independent solutions to the coupled equations in the form

$$U_{j-1,j}(r) \simeq F_{j-1} + x_{j-1,j}\, G_{j-1}(r)\,,$$

$$U_{j+1,j}(r) \simeq Y_j\, G_{j+1}(r)\,,$$

$$U_{j-1,j}(r) \simeq Y_j\, G_{j-1}(r)$$

and

$$U_{j+1,j}(r) \simeq F_{j+1}(r) + x_{j+1}\, G_{j+1}(r)\,.$$

Here $r^{-1} F_l(r)$ and $r^{-1} G_l(r)$ are the regular and irregular solutions of the Bessel radial wave functions in the l, j channel.

These nuclear parameters are related to the Blatt and Biedenharn phase shifts by the equations:

$$x_{j+1} = \cos \epsilon \tan \delta_{j\mp1,j} + \sin^2 \epsilon \tan \delta_{j\mp1,j}$$

and

$$y_j = \frac{1}{2} \sin 2\epsilon \,(\tan \delta_{j-1} - \tan \delta_{j+1})\,.$$

The $x's$, for a given j are called the tangent matrix elements because when there is no coupling (i.e., $\epsilon = 0$) x becomes the tangent of the ordinary phase shift.

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