A gauge-invariant spectral scale from covariant diffusion in d-dimensional Yang–Mills backgrounds

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We introduce a simple gauge-invariant spectral diagnostic for Euclidean Yang–Mills backgrounds based on covariant diffusion. Using the gauge-covariant Laplacian Δ_A, we consider heat-kernel trace observables built from the semigroup $e^{-t\Delta_A}$ and address the infinite-volume divergence through two complementary normalizations: a trace per unit volume in a finite domain followed by the thermodynamic limit, and a relative trace with respect to the trivial background $A = 0$ that cancels the leading volume contribution. From these trace observables we define an effective spectral scale via a logarithmic derivative in diffusion time, yielding a mass-squared-like parameter that characterizes the large-t decay of covariant diffusion when the limit exists. In parallel, we outline a diffusion-based spectral-dimension diagnostic and discuss basic invariance and positivity properties of the construction. The framework is formulated in general dimension d and is intended as a clean, continuum starting point for connecting gauge-invariant long-time diffusion behaviour with spectral information of the covariant operator, without discretization or lattice input.

Keywords: Yang–Mills theory; covariant Laplacian; heat kernel; gauge invariance; spectral diagnostics.

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1. Introduction

The emergence of macroscopic scales from microscopic gauge dynamics is a central theme in non-Abelian field theory. In Euclidean Yang–Mills settings, one often seeks gauge-invariant diagnostics that translate structural information about a background connection into quantitative long-distance behaviour [1]. Spectral methods provide a natural route: the spectrum of gauge-covariant elliptic operators encodes how fields relax, diffuse, and coarse-grain under heat-type evolution, in a manner stable under smooth deformations and (when defined appropriately) insensitive to gauge choice [2].

Heat-kernel methods and spectral analysis of elliptic operators have a long history in spectral geometry, mathematical physics, and quantum field theory. Classical results relate geometric and analytic properties of differential operators to the asymptotic structure of heat traces, while covariant heat kernels play a central role in effective actions, anomaly calculations, background-field methods, and spectral constructions in gauge theories [2–6]. More recently, diffusion-based observables have been employed to characterize scale-dependent geometric properties through the notion of spectral dimension, particularly in approaches to quantum gravity and random geometries [7–9].

The present work is not intended to introduce a new heat-kernel formalism, a new covariant operator, or a new notion of spectral dimension. Rather, it combines standard ingredients—covariant diffusion, infrared-safe heat-trace normalizations, and logarithmic diffusion-time derivatives—into a gauge-invariant diffusion-based spectral scale associated with the covariant Laplacian. The emphasis is on providing a continuum diagnostic of the infrared spectral behaviour of Yang–Mills backgrounds while remaining independent of gauge choice, discretization schemes, or lattice-specific constructions.

A particularly convenient probe is covariant diffusion generated by a gauge-covariant Laplacian. Given a connection A_μ on $\mathbb{R}^d$ (or on a large finite domain), the semigroup $e^{-t\Delta_A}$ defines a heat kernel that is gauge covariant in its spatial arguments, while trace observables built from it are gauge invariant. The diffusion time t plays the role of a coarse-graining scale: small t probes ultraviolet structure, whereas large t probes infrared spectral information [2].

A technical issue is immediate in infinite volume: $\mathrm{Tr}(e^{-t\Delta_A})$ diverges proportionally to the volume, even for $A = 0$ [2]. We therefore employ two complementary infrared-safe normalizations: (i) an intensive trace per unit volume in a finite domain followed by the thermodynamic limit, and (ii) a relative trace (subtracting the $A = 0$ contribution) which cancels the leading extensive term.

From either normalized trace we extract an *effective spectral scale* via a logarithmic derivative in diffusion time. When a large-t limit exists after taking the thermodynamic limit, the resulting quantity behaves like a mass-squared parameter controlling long-time decay in the diffusion channel under consideration. This is not asserted to be a physical pole

mass; rather, it is a gauge-invariant spectral diagnostic tied to the low end of the spectrum of Δ_A in the chosen representation.

The goals of this paper are sharply defined:

- define a clean gauge-invariant diffusion observable in general dimension d,
- control its infrared normalization and thermodynamic limit,
- define an effective spectral scale from its t-dependence,
- establish basic positivity/monotonicity properties,
- provide benchmarks clarifying what the large-t behaviour can and cannot imply.

The paper is organized as follows. Section 2 introduces covariant diffusion, heat–trace observables, and the two infrared–safe normalizations. Section 3 presents the effective spectral scale (and a complementary spectral–dimension diagnostic), including monotonicity constraints from spectral representations. Section 4 discusses benchmark cases and clarifies finite–volume artifacts versus genuine infrared behaviour. Section 5 summarizes implications and extensions.

2. Covariant diffusion and infrared-safe trace observables

2.1. Set-up and gauge covariance

We work in Euclidean space $\mathbb{R}^d$ (or a bounded domain $\Omega_L \subset \mathbb{R}^d$ of linear size L), with compact gauge group G and Lie algebra $\mathfrak{g}$. Fix a unitary representation ρ acting on a complex vector space V. Let $A_\mu(x)$ be a sufficiently regular background connection, understood as a matrix field in $\mathrm{End}(V)$, and define the covariant derivative

$$D_\mu \equiv \partial_\mu + A_\mu. \tag{1}$$

Under a gauge transformation $g(x) \in G$,

$$\begin{aligned} A_\mu &\mapsto A_\mu^g = gA_\mu g^{-1} - (\partial_\mu g)\, g^{-1}, \\ D_\mu &\mapsto D_\mu^g = gD_\mu g^{-1}. \end{aligned} \tag{2}$$

We consider the gauge-covariant Laplacian acting on V-valued scalar fields,

$$\Delta_A \equiv -D_\mu D_\mu, \tag{3}$$

which is elliptic and nonnegative on appropriate domains and boundary conditions.

For $t > 0$, the semigroup $e^{-t\Delta_A}$ defines covariant diffusion. Formally,

$$\left(e^{-t\Delta_A}\phi\right)(x) = \int_{\Omega_L} \mathrm{d}^d y\, K_A(t;x,y)\, \phi(y), \qquad t > 0, \tag{4}$$

where $K_A(t;x,y)$ is the heat kernel. Gauge covariance implies $K_{A^g}(t;x,y) = g(x)K_A(t;x,y)g(y)^{-1}$, and hence trace observables are gauge invariant.

2.2. Finite-volume heat trace

Let Ω_L be a bounded region of volume $V_L = \mathrm{Vol}(\Omega_L)$, with boundary conditions (periodic/Dirichlet/Neumann) chosen so that Δ_A is self-adjoint and has discrete spectrum $\{\lambda_n(A;L)\}_{n\geq 1}$ with $\lambda_n \geq 0$ [2]. The heat trace is

$$\begin{aligned} \mathrm{Tr}_{\Omega_L}\left(e^{-t\Delta_A}\right) &= \sum_{n\geq 1} \mathrm{e}^{-t\lambda_n(A;L)} \\ &= \int_{\Omega_L} \mathrm{d}^d x\ \mathrm{Tr}_V\ K_A(t;x,x). \end{aligned} \tag{5}$$

Gauge covariance of $e^{-t\Delta_A}$ implies gauge invariance of the trace:

$$\mathrm{Tr}_{\Omega_L}\left(e^{-t\Delta_{A^g}}\right) = \mathrm{Tr}_{\Omega_L}\left(e^{-t\Delta_A}\right), \tag{6}$$

for gauge transformations compatible with the chosen boundary conditions.

2.3. Two infrared-safe normalizations

In infinite volume, $\mathrm{Tr}(e^{-t\Delta_A})$ diverges extensively. We therefore consider:

(i) Intensive trace density. Define the normalized heat-trace density

$$\mathcal{T}_A(t;L) \equiv \frac{1}{V_L}\,\mathrm{Tr}_{\Omega_L}\left(e^{-t\Delta_A}\right). \tag{7}$$

When the thermodynamic limit exists at fixed $t > 0$, we define

$$\mathcal{T}_A(t) \equiv \lim_{L\to\infty} \mathcal{T}_A(t;L). \tag{8}$$

(ii) Relative trace density. Let $\Delta_0 \equiv -\partial^2$ denote the free Laplacian with the *same* boundary conditions on Ω_L. Define

$$\mathcal{R}_A(t;L) \equiv \frac{1}{V_L}\Big[\mathrm{Tr}_{\Omega_L}\left(e^{-t\Delta_A}\right) - \mathrm{Tr}_{\Omega_L}\left(e^{-t\Delta_0}\right)\Big], \tag{9}$$

and, when it exists,

$$\mathcal{R}_A(t) \equiv \lim_{L\to\infty} \mathcal{R}_A(t;L). \tag{10}$$

The relative normalization often improves infrared behaviour by cancelling the leading extensive contribution [2].

2.4. Relation between the two normalizations

The intensive trace density $T_A(t)$ and the relative trace $R_A(t)$ are designed to remove the same extensive volume divergence, but they do not necessarily encode identical information at finite diffusion times.

When the thermodynamic limit exists, the two normalizations provide complementary information about the infrared spectrum, but they need not identify the same asymptotic exponential scale. Their large-t behaviours can differ because $R_A(t) = T_A(t) - T_0(t)$ may be affected by cancellations or by the slower-decaying contribution.

However, the two constructions emphasize different aspects of the spectrum. The intensive trace density $T_A(t)$ remains positive by construction and therefore naturally admits logarithmic derivatives and monotonicity properties. By contrast, the relative trace $R_A(t)$ measures deviations from the free background and may change sign depending on the spectral redistribution induced by the gauge field. As a consequence, logarithmic derivatives based on $R_A(t)$ may become singular or may fail to provide a useful diagnostic in diffusion-time intervals where cancellations occur.

For this reason, $T_A(t)$ will be regarded as the primary spectral observable throughout this work, while $R_A(t)$ is viewed as a complementary quantity that highlights departures from free diffusion. Agreement between the two normalizations in the large-t regime provides evidence that the extracted spectral scale reflects genuine infrared spectral behaviour rather than a normalization artifact.

2.5. Remarks on existence and boundary effects

For fixed $t > 0$, the heat kernel is effectively local on length scale $\ell(t) \sim \sqrt{t}$, and boundary contributions are typically subextensive as $L \to \infty$ [2]. In favorable settings one expects

$$\mathcal{T}_A(t; L) = \mathcal{T}_A(t) + \mathcal{O}\left(\frac{|\partial\Omega_L|}{V_L}\right), \qquad L \to \infty, \tag{11}$$

and similarly for $\mathcal{R}_A(t; L)$. We do not attempt the sharpest minimal hypotheses here; our emphasis is a clean continuum definition and its robust structural consequences.

3. Spectral scale from diffusion: effective mass and spectral dimension

3.1. Effective spectral scale

Assume $\mathcal{T}_A(t; L) > 0$ for $t > 0$ (true at finite volume for standard positive semigroups). We define the diffusion-time dependent effective spectral scale

$$m^2_{\text{eff}}(t; L) \equiv -\frac{\mathrm{d}}{\mathrm{d}t} \ln \mathcal{T}_A(t; L). \tag{12}$$

Using functional calculus,

$$m^2_{\text{eff}}(t; L) = \frac{\frac{1}{V_L}\mathrm{Tr}_{\Omega_L}\left(\Delta_A\, \mathrm{e}^{-t\Delta_A}\right)}{\frac{1}{V_L}\mathrm{Tr}_{\Omega_L}\left(\mathrm{e}^{-t\Delta_A}\right)}, \tag{13}$$

so $m^2_{\text{eff}}(t; L)$ is a heat-kernel weighted average of the spectrum: small t weights higher eigenvalues, while large t concentrates toward the low end of the spectrum.

If the thermodynamic limit exists at fixed t, define

$$m^2_{\text{eff}}(t) \equiv \lim_{L\to\infty} m^2_{\text{eff}}(t; L). \tag{14}$$

When, in addition, the large-t limit exists, we define the macroscopic diffusion spectral scale

$$m^2_{\infty} \equiv \lim_{t\to\infty} m^2_{\text{eff}}(t), \tag{15}$$

provided the order of limits is taken as written (thermodynamic limit first).

Relative version (optional). When $\mathcal{R}_A(t; L)$ has definite sign on a t-window, one may analogously define

$$m^2_{\text{eff,rel}}(t; L) \equiv -\frac{\mathrm{d}}{\mathrm{d}t} \ln|\mathcal{R}_A(t; L)|\,, \tag{16}$$

and its thermodynamic-limit counterpart when it exists. Because $\mathcal{R}_A$ can cross zero, we treat the relative logarithmic derivative as auxiliary.

3.2. Spectral representation and monotonicity (thermodynamic limit)

When the thermodynamic-limit trace density admits a spectral-density representation per unit volume,

$$\mathcal{T}_A(t) = \int_0^\infty \mathrm{e}^{-t\lambda}\, \rho_A(\lambda)\, \mathrm{d}\lambda, \qquad \rho_A(\lambda) \geq 0, \tag{17}$$

in Ref. [2], one can introduce the t-dependent probability measure

$$\mathrm{d}\pi_{A,t}(\lambda) = \frac{\mathrm{e}^{-t\lambda}\rho_A(\lambda)\, \mathrm{d}\lambda}{\mathcal{T}_A(t)}. \tag{18}$$

Then

$$m^2_{\text{eff}}(t) = -\frac{\mathrm{d}}{\mathrm{d}t} \ln \mathcal{T}_A(t) = \int_0^\infty \lambda\, \mathrm{d}\pi_{A,t}(\lambda), \tag{19}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} m^2_{\text{eff}}(t) = -\mathrm{Var}_{A,t}(\lambda) \leq 0. \tag{20}$$

Thus, whenever Eq. refeq:Laplace holds, $m^2_{\text{eff}}(t)$ is nonincreasing in t and can approach a finite limit only from above. This provides a robust constraint and a practical consistency check.

3.3. A diffusion-based spectral dimension (complementary diagnostic)

Define the diffusion-based spectral-dimension diagnostic by

$$d_s^A(t;L) \equiv -2\,\frac{\mathrm{d}}{\mathrm{d}\ln t}\ln \mathcal{T}_A(t;L) = -2t\,\frac{\mathrm{d}}{\mathrm{d}t}\ln \mathcal{T}_A(t;L) = 2t\,m^2_{\rm eff}(t;L), \tag{21}$$

in Ref. [7]. In regimes where $\mathcal{T}_A(t)$ exhibits approximate power-law scaling in t, $d_s^A(t)$ can develop plateaus which diagnose intermediate scaling windows. In contrast, when $\mathcal{T}_A(t)$ contains strong exponential suppression (as in genuinely gapped toy models), $d_s^A(t)$ is typically not the cleanest infrared descriptor; in that case the primary infrared diagnostic is the plateau (or lack thereof) in $m^2_{\rm eff}(t)$.

The quantity $d_s(t)$ is closely related to the spectral dimension commonly employed in spectral geometry, diffusion processes, and quantum-gravity approaches based on scale-dependent effective geometries [7–9]. In those settings, diffusion observables are used to probe how the effective dimensionality perceived by a random walker changes with scale. The role of $d_s(t)$ in the present work is more modest: it serves as a complementary diagnostic accompanying the effective spectral scale $m^2_{\rm eff}(t)$. While both quantities are derived from the same heat-trace observable, $m^2_{\rm eff}(t)$ provides a more direct characterization of infrared spectral suppression, whereas $d_s(t)$ offers additional information regarding the scale dependence of the underlying spectral structure.

3.4. Order of limits and an operational criterion

At fixed finite L, the large-t limit is dominated by the lowest eigenvalue:

$$T_A(t;L) \propto e^{-t\lambda_1(A;L)}, \qquad t\to\infty, \qquad m^2_{\rm eff}(t;L) \xrightarrow[t\to\infty]{} \lambda_1(A;L) \tag{22}$$

Ref. [2]. Therefore a plateau at finite L is *not* evidence of a genuine infrared scale unless it remains nonzero after the thermodynamic limit. The operational criterion advocated here is:

$$m^2_\infty = \lim_{t\to\infty}\Big(\lim_{L\to\infty} m^2_{\rm eff}(t;L)\Big), \tag{23}$$

and $m^2_\infty > 0$ is interpreted as a robust infrared spectral scale in the diffusion channel probed by Δ_A.

4. Benchmarks and interpretation

This section provides calibrating examples clarifying how the diffusion diagnostics behave in simple settings and how finite-volume artifacts appear.

4.1. Pure gauge backgrounds

As a first gauge-theoretic benchmark, consider a pure gauge connection

$$A_\mu = -(\partial_\mu g)g^{-1}, \tag{24}$$

where $g(x)$ is a smooth gauge transformation. The corresponding field strength vanishes identically,

$$F_{\mu\nu} = 0. \tag{25}$$

In this case the covariant derivative is related to the ordinary derivative by conjugation,

$$D_\mu = g\,\partial_\mu\, g^{-1}, \tag{26}$$

and therefore the covariant Laplacian satisfies

$$\Delta_A = g\,\Delta_0\, g^{-1}. \tag{27}$$

Consequently, Δ_A and Δ_0 are unitarily equivalent and possess identical spectra. Whenever boundary conditions are compatible with the gauge transformation, the associated heat traces coincide,

$$\mathrm{Tr}\big(e^{-t\Delta_A}\big) = \mathrm{Tr}\big(e^{-t\Delta_0}\big)\,. \tag{28}$$

It follows that the intensive trace density, the effective spectral scale, and the spectral-dimension diagnostic all reduce to their free-background counterparts. Thus the proposed observables are insensitive to gauge-equivalent descriptions of the trivial connection and respond only to genuine spectral information. This benchmark provides a simple consistency check of the gauge-invariant character of the construction.

4.2. Free background $A = 0$

For $A = 0$ in $\mathbb{R}^d$, the heat kernel is Gaussian and the thermodynamic-limit heat-trace density takes the standard form

$$\mathcal{T}_0(t) = (4\pi t)^{-d/2}\,\dim(V), \tag{29}$$

Ref. [2]. Consequently,

$$m^2_{\rm eff,0}(t) = -\frac{\mathrm{d}}{\mathrm{d}t}\ln \mathcal{T}_0(t) = \frac{d}{2t}, \tag{30}$$

which decays algebraically and has no nonzero infrared plateau:

$$\lim_{t\to\infty} m^2_{\rm eff,0}(t) = 0. \tag{31}$$

The corresponding diagnostic spectral dimension is constant:

$$d_s^0(t) = -2\frac{\mathrm{d}}{\mathrm{d}\ln t}\ln \mathcal{T}_0(t) = d. \tag{32}$$

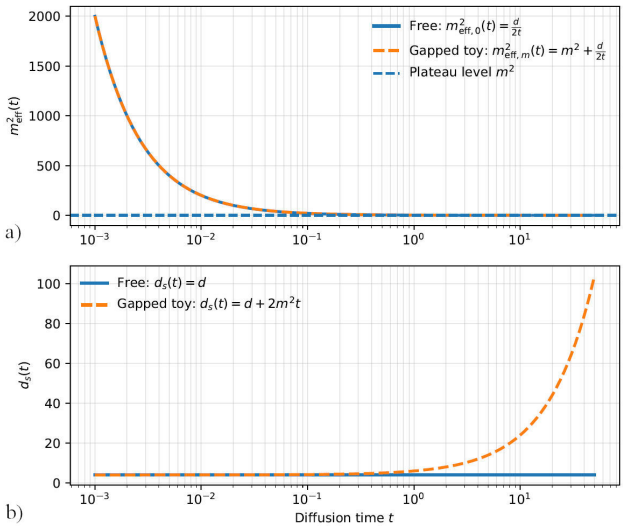


FIGURE 1. Benchmarks for the diffusion-based diagnostics in d dimensions. a) Effective spectral scale $m^2_{\mathrm{eff}}(t)$ for the free background $A = 0$, showing algebraic decay with no infrared plateau, and for the gapped toy operator $-\partial^2 + m^2$, illustrating the emergence of a genuine infrared plateau at m^2. b) Corresponding diffusion-based spectral dimension $d_s(t) = 2t\, m^2_{\mathrm{eff}}(t)$, which remains constant in the free case and is dominated by the exponential regime in the gapped benchmark. The contrast highlights the role of a true spectral gap in producing a stable infrared plateau in the diffusion channel.

4.3. A gapped toy benchmark: $-\partial^2 + m^2$

Consider the scalar operator $-\partial^2 + m^2$ (not necessarily arising as Δ_A, but useful for calibration). One has

$$\mathcal{T}_m(t) = (4\pi t)^{-d/2}\, \dim(V)\, \mathrm{e}^{-m^2 t} \tag{33}$$

Ref. [2], so

$$m^2_{\mathrm{eff},m}(t) = m^2 + \frac{d}{2t},$$
$$\lim_{t\to\infty} m^2_{\mathrm{eff},m}(t) = m^2 > 0. \tag{34}$$

This is the canonical signature of a genuine infrared scale: the effective spectral scale approaches a positive constant. Figure 1 illustrates this canonical behaviour, contrasting the algebraic decay of the free case with the genuine infrared plateau arising in a gapped toy model.

4.4. Finite-volume pseudo-plateaus and stability in L

Equation (22) shows that at fixed L, $m^2_{\mathrm{eff}}(t; L)$ always approaches $\lambda_1(A; L)$ as $t \to \infty$. Therefore, the practical distinction is:

- **Finite-volume artifact:** a plateau in t at fixed L that drifts toward 0 as $L \to \infty$ (typical of gapless spectra).
- **Genuine infrared scale:** a plateau in $m^2_{\mathrm{eff}}(t)$ obtained after $L \to \infty$, converging to $m^2_\infty > 0$ as $t \to \infty$.

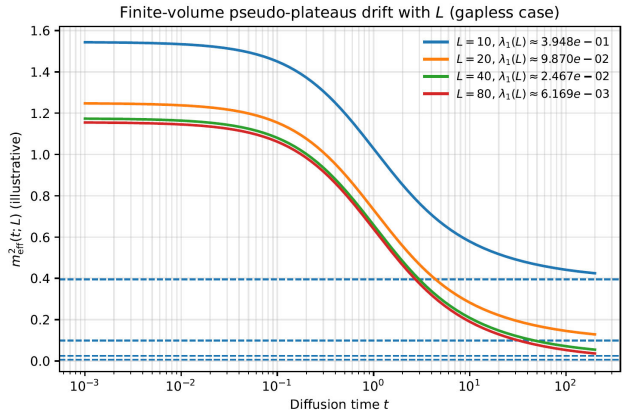


FIGURE 2. Illustration of finite-volume pseudo-plateaus. At fixed L, the effective spectral scale $m^2_{\mathrm{eff}}(t; L)$ approaches the lowest eigenvalue $\lambda_1(A; L)$ as $t \to \infty$, producing an apparent plateau. In a gapless situation, the plateau level drifts toward zero as L increases, motivating the order of limits used to define the macroscopic diffusion scale m^2_∞.

This is precisely why the order of limits in (23) is essential. Figure 2 illustrates this mechanism: at fixed L a plateau in $m^2_{\mathrm{eff}}(t; L)$ is unavoidable, but its level drifts with increasing L, so stability under the limit $L \to \infty$ is the operative criterion for identifying a genuine infrared spectral scale.

4.5. Relative trace as a complementary diagnostic

The relative quantity $\mathcal{R}_A(t) = \mathcal{T}_A(t) - \mathcal{T}_0(t)$ isolates departures from free diffusion and can be useful for identifying t-windows where the background significantly suppresses diffusion. However, since $\mathcal{R}_A(t)$ need not have definite sign for all t, logarithmic derivatives of $\mathcal{R}_A(t)$ can develop spurious singularities near zeros. For this reason, we recommend using $\mathcal{R}_A(t)$ primarily as a qualitative companion to the main diagnostic $m^2_{\mathrm{eff}}(t)$.

It is instructive to emphasize that the appearance of a plateau in $m^2_{\mathrm{eff}}(t)$ is tightly linked to the infrared structure of the spectrum of Δ_A. A genuine plateau arises only when the spectral density exhibits a true infrared gap, leading to an exponential suppression of the heat–trace density at large diffusion times. By contrast, gapless spectra extending down to $\lambda = 0$, such as in the free case, produce an algebraic decay of $T_A(t)$ and therefore prevent the formation of a plateau. Intermediate situations, where the low–lying spectrum is strongly suppressed but not gapped, may lead to a rapid decay of $T_A(t)$ without a well–defined exponential regime. Finally, at finite volume, apparent plateaus are unavoidable but correspond to the lowest discrete eigenvalue and must not be interpreted as genuine infrared scales unless they persist after taking the thermodynamic limit. These possibilities are illustrated in Fig. 3, which compares simple toy spectral densities with and without a true infrared gap and shows how their distinct low–lying spectral structures translate into qualitatively different large–t behaviours of the effective spectral scale $m^2_{\mathrm{eff}}(t)$.

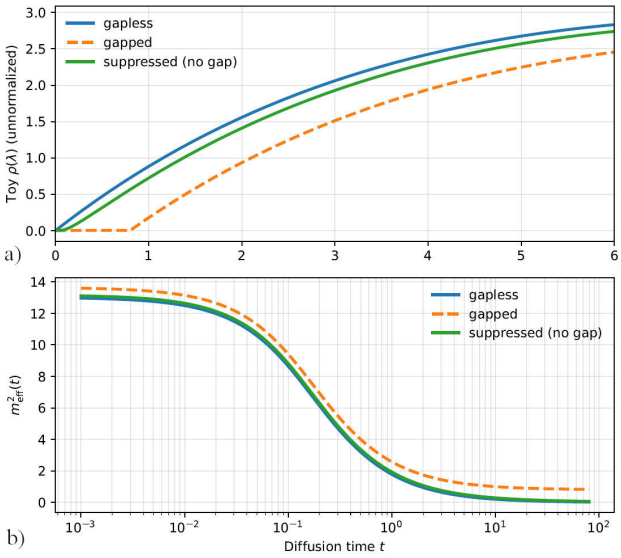


FIGURE 3. Toy spectral densities $\rho(\lambda)$ a) and the resulting effective spectral scale $m^2_{\rm eff}(t)$ obtained from the Laplace representation $\mathcal{T}(t) = \int_0^\infty \mathrm{e}^{-t\lambda}\rho(\lambda)\,\mathrm{d}\lambda$ b). A true infrared gap, corresponding to spectral support starting at $\lambda_0 > 0$, leads to the emergence of a positive infrared plateau in $m^2_{\rm eff}(t)$. By contrast, gapless spectra extending down to $\lambda = 0$ yield algebraic decay and prevent the formation of a plateau. Strong suppression of low-lying modes without a strict gap can produce rapid decay of $\mathcal{T}(t)$ without a clean exponential regime, illustrating intermediate infrared behaviour.

5. Discussion and outlook

We have defined a gauge-invariant, continuum diffusion framework for extracting spectral information from Euclidean Yang–Mills backgrounds in general dimension d. The central observable is the intensive heat-trace density $\mathcal{T}_A(t)$ (or, complementarily, the relative trace $\mathcal{R}_A(t)$), from which we define an effective spectral scale $m^2_{\rm eff}(t)$ by a diffusion-time logarithmic derivative. When the thermodynamic limit exists and is taken before the large-t limit, a nonzero plateau $m^2_\infty > 0$ provides a clean and robust infrared spectral scale in the diffusion channel probed by the covariant Laplacian.

Several structural points are worth emphasizing:

- The construction is manifestly gauge invariant, since it is based on traces of covariant operators.
- When a positive spectral-density representation holds in the thermodynamic limit, $m^2_{\rm eff}(t)$ is monotone non-increasing in t and admits the variance identity (20), providing strong constraints on possible behaviours.
- Finite-volume plateaus are inevitable but can be diagnosed and removed by enforcing the order of limits (23).

Relation to physical mass scales. The diffusion scale m^2_∞ is a spectral diagnostic of Δ_A in a chosen representation and is not, by itself, a statement about a specific gauge-invariant particle mass (*e.g.* a glueball pole). Bridging from diffusion spectra to correlator spectra requires additional input (choice of gauge-invariant operator channels, Osterwalder–Schrader reconstruction, and comparison of decay rates). Nevertheless, the diffusion scale provides a clean continuum quantity that can be compared across ensembles, representations, and deformations, and can serve as a disciplined starting point for such comparisons.

Spectral scales versus physical masses. It is worth emphasizing that the existence of a nonzero diffusion scale m^2_∞ does not necessarily imply the presence of a corresponding physical particle mass. From a purely spectral perspective, an exponential suppression of the heat–trace density at large diffusion time may arise from the collective behaviour of the low–lying spectrum of Δ_A, even in the absence of an isolated pole in any gauge–invariant two–point function. In such situations, the quantity m^2_∞ should be interpreted strictly as a gauge–invariant infrared *spectral scale* associated with covariant diffusion, rather than as the mass of a physical excitation. Establishing a direct relation between diffusion scales and particle masses requires additional dynamical input, such as the analysis of specific gauge–invariant correlator channels and their long–distance decay properties.

Extensions. Natural extensions include: (i) studying other covariant operators (adjoint Laplacian, Dirac-type operators, or Faddeev–Popov operators in suitable formulations), (ii) analyzing background ensembles (rather than fixed backgrounds) and defining disorder-averaged trace densities, (iii) comparing continuum predictions with lattice-measured heat-trace observables as a consistency check, while keeping the definition itself non-discrete.

What would constitute decisive evidence. Within this framework, decisive evidence for a genuine diffusion infrared scale is a stable nonzero plateau in $m^2_{\rm eff}(t)$ after taking $L \to \infty$, together with robustness under changes of boundary conditions and (when available) cross-checks against decay rates of suitable gauge-invariant correlators.

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