Dimensional consistency in fractional differential equations with non singular kernels

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The purpose of this article is to address the issues of dimensional consistency that arise in the process of replacing the ordinary time derivative operator by a fractional derivative operator in order to write a fractional differential equation. We show that by performing a simple change of variables fulfilling certain conditions ensures the consistency in physical dimensions for fractional differential equations with non singular kernels. An example of the proposed method is given.

Keywords: Fractional calculus; Caputo-Fabrizio fractional derivative; RC circuit.

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1. Introduction

From a mathematical point of view, fractional calculus was introduced as a generalization of classical calculus and has become a powerful tool for modeling complex dissipative phenomena, attracting the interest of scientists and being currently the subject of many theoretical and experimental investigations [1-3]. Among the various definitions of fractional derivatives, Riemann-Liouville and Caputo operators, based on power-law kernels, are widely used to capture memory effects with algebraic decay [4]. However, these kernels are singular at the origin, which can lead to difficulties in handling initial conditions and interpreting physical processes. To overcome these limitations, Caputo and Fabrizio proposed a fractional derivative with smooth non-singular exponential kernels, which have the ability to describe material heterogeneities and multi-scale fluctuations [5]. Despite these kernel design development, incorporating fractional derivatives into physical models introduces additional challenges related to dimensional consistency.

Fractional derivatives, whether defined with singular or non-singular kernels, are incorporated into differential equations to model physical phenomena through fractional differential equations (FDEs). Through these formulations, non-integer models have been applied to systems such as the parallel RLC circuit or the damped harmonic oscillator [6]. However, the straightforward replacement of an integer order derivative by a fractional order derivative in equations involving physical variables leads to dimensional imbalance that raises questions about the interpretability of the models [7].

To address this issue, some strategies are implemented, such as introducing a scaling parameter or characteristic constant that restores the dimensional homogeneity in the equations. For example, Gómez-Aguilar *et al*. [8,9], and Bachuin [10] introduced an auxiliary parameter σ that represents the fractional time components of the system, naming the non-local time as the *cosmic time*. This parameter allows the fractional derivative to be dimensionally consistent with its classical counterpart, even though its physical interpretation remains challenging, as it typically carries units such as $\sec^{-\alpha}$ that are not directly measurable in practice.

Alternatively, Vaz and Capelas de Oliveira [7] describe a dimensional regularization technique for the fractional derivative of Caputo, where a multiplicative term involving the independent variable is introduced to maintain the original physical dimension of the problem. This approach avoids the arbitrariness of auxiliary parameters by systematically balancing the dimensions through characteristic scales.

Other methods include normalization procedures that transform variables into dimensionless forms to avoid dimensional balancing, but when reintroducing physical dimensions into fractional form, some problems are encountered when interpreting the terms of the FDE [7].

The main goal of this article is to answer the question on how physical dimensions in a fractional derivative operator has to be taken into account to model real problems and to have dimensional consistency in the fractional differential system.

2. Dimensional consistency in fractional derivative operators

The simplest procedure for constructing a fractional differential equation consists of simply replacing all conventional derivative operators of a given system by the fractional derivative operator. The replacement of an integer order derivative operator by a fractional derivative operator in a differential equation leads to a dimensional inconsistency. For example, let us consider the definition of the classical

Caputo fractional derivative, which is given by

$$ {}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^1 \frac{df(s)}{ds} \frac{1}{(t-s)^{\alpha}} ds, \quad (1) $$

with $\alpha \in (0,1]$ and $a \in [-\infty, t]$ and $f(t) \in H^1(a,b)$ for $b > a$, where $H^1(a,b)$ denotes a Sobolev space. If we do such replacements we have a dimensional inconsistency since the time dimension of ${}^{C}D_t^{\alpha}$, i.e. the Caputo fractional derivative, is $\sec^{-\alpha}$ while that of d/dt is $\sec^{-1}$. For achieving dimensional consistency, Gómez *et al*. introduce the following simple rule which consists of replacing the ordinary time derivative operator by the fractional one in the following way

$$ \frac{d}{dt} \rightarrow \frac{1}{\sigma^{1-\alpha}} {}^{C}D_t^{\alpha} \quad \text{where} \quad 0 < \alpha \leq 1, \quad (2) $$

where $\sigma > 0$ is a new parameter which has the dimension of time, *i.e.* $[\sigma] = s$. Using this procedure, it is easy to write down the fractional differential equation of any given system. Note that the dimensions of d/dt and $(1/\sigma^{1-\alpha})\, {}^{C}D_t^{\alpha}$ are now consistent.

However, there are many different definitions of fractional derivatives in the literature. Traditional definitions of fractional derivatives, such as the Caputo operator, employ power-law kernels with singularities at the origin. Although these kernels effectively represent long-range memory, their singular behavior often leads to analytical difficulties, numerical instabilities, and ambiguity in the physical interpretation of initial conditions. In order to overcome these limitations, Caputo and Fabrizio proposed a new fractional derivative in 2015 characterized by a non-singular and nonlocal exponential kernel. This formulation preserves the memory effect inherent to fractional calculus while ensuring improved regularity and computational tractability. Caputo and Fabrizio proposed changing the kernel $(t-s)^{-\alpha}$ in the Caputo fractional derivative by the exponential function $\exp[-(\alpha t/(1-\alpha))]$ and $1/\Gamma(1-\alpha)$ by $(2-\alpha)M(\alpha)/2(1-\alpha)$, to obtain the following definition for the fractional derivative with non singular kernel of the Caputo-Fabrizio type given by

$$ {}^{CF}D_t^{\alpha} f(t) = \frac{(2-\alpha)M(\alpha)}{2(1-\alpha)} \int_0^t \frac{df(s)}{ds} \times \exp\left[-\frac{\alpha(t-s)}{1-\alpha}\right] ds, \quad (3) $$

where $M(\alpha)$ is a normalization function such that $M(\alpha) = 2/(2-\alpha)$.

The Caputo–Fabrizio derivative satisfies several fundamental properties that make it suitable for theoretical and applied mathematical physics. The Caputo–Fabrizio operator satisfies the following mathematical properties

Linearity. For any functions $f, g \in C^1([0,T])$ and constants $a, b \in \mathbb{R}$,

$$ {}^{CF}D_t^{\alpha} \left(af(t) + bg(t)\right) = a\, {}^{CF}D_t^{\alpha} f(t) + b\, {}^{CF}D_t^{\alpha} g(t). \quad (4) $$

Derivative of a constant. If $f(t) = C$, where C is a constant, then

$$ {}^{CF}D_t^{\alpha} C = 0, \quad (5) $$

which is consistent with classical differentiation and advantageous for physical modeling.

Limit cases. The Caputo–Fabrizio derivative interpolates between classical operators:

$$ \lim_{\alpha \to 0} {}^{CF}D_t^{\alpha} f(t) = f(t) - f(0), $$
$$ \lim_{\alpha \to 1} {}^{CF}D_t^{\alpha} f(t) = \frac{df(t)}{dt}. \quad (6) $$

Laplace transform. Let $\mathcal{L}\{f(t)\}(s) = F(s)$. The Laplace transform of the Caputo–Fabrizio derivative is given by

$$ \mathcal{L}\left\{ {}^{CF}D_t^{\alpha} f(t) \right\}(s) = \frac{1}{1-\alpha} \frac{sF(s) - f(0)}{s + \frac{\alpha}{1-\alpha}}, \quad (7) $$

which facilitates the analytical solution of fractional differential equations.

Due to its non-singular kernel, well-defined initial conditions, and favorable analytical properties, the Caputo–Fabrizio derivative has been extensively employed in the formulation and analysis of fractional differential equations. Applications include stability analysis of dynamical systems, reaction–diffusion processes, and control problems, where it provides a mathematically consistent and physically interpretable alternative to classical fractional operators. The operator (3) is nonlocal, as the value of the derivative at time t depends on the entire history of the function over $[0,t]$. However, unlike power-law kernels, the exponential kernel induces a memory with finite persistence, which is often more realistic in applied models.

From Eq. (3) we see that the Caputo-Fabrizio fractional derivative is a dimensionless operator, therefore in order to write down the corresponding fractional differential operator we need to introduce a new dimensionless parameter given by

$$ \tau(t,\alpha) = \int_0^t \frac{dt}{\varphi(t,\alpha)}, \quad (8) $$

where $\varphi(t,\alpha)$ is a function that has to have the dimension of time in order for τ to be a dimensionless function and also has to be chosen in such a way so that $\tau(t,\alpha)$ is a strictly monotonic increasing function. Therefore, we can replace the ordinary time derivative operator by the following one

$$ \frac{d}{dt} \rightarrow \frac{d\tau}{dt}\frac{d}{d\tau} \rightarrow \frac{1}{\varphi(t(\tau),\alpha)} {}^{CF}D_{\tau}^{\alpha}. \quad (9) $$

In order to obtain the ordinary time derivative in Eq. (9) when $\alpha = 1$ we must impose the following condition:

$$ \frac{1}{\varphi(t,1)} \frac{d}{d\tau} = \frac{d}{dt}. \quad (10) $$

Therefore, if we would like to solve the fractional differential equation, using the Caputo-Fabrizio fractional derivative operator, equivalent to a first order differential equation with constant coefficients given by

$$\frac{dx}{dt} + \mathrm{P}x(t) = \mathrm{Q}, \tag{11}$$

where P and Q are constants, we proceed to solve the following fractional differential equation

$${}^{CF}D_\tau^\alpha x(\tau) + \mathrm{P}\varphi(t(\tau),\alpha)x(\tau) = \mathrm{Q}\varphi(t(\tau),\alpha). \tag{12}$$

Substituting Eq. (3) into Eq. (12) we have

$$\frac{e^{-\alpha\tau/(1-\alpha)}}{1-\alpha}\int_0^\tau e^{\alpha u/(1-\alpha)}x'(u)du + \mathrm{P}\varphi(t(\tau),\alpha)x(\tau)x(\tau) = \mathrm{Q}\varphi(t(\tau),\alpha). \tag{13}$$

Differentiating both sides of Eq. (13) with respect to τ and simplifying, we get

$$\frac{dx}{d\tau} = (1-\alpha)\frac{d}{d\tau}\left[\mathrm{Q}\varphi(t(\tau),\alpha) - \mathrm{P}\varphi(t(\tau),\alpha)x(\tau)\right] + \alpha\left[\mathrm{Q}\varphi(t(\tau),\alpha) - \mathrm{P}\varphi(t(\tau),\alpha)x(\tau)\right], \tag{14}$$

which can be rearranged as

$$\begin{gathered}\left[(1-\alpha)\mathrm{P}\varphi(t(\tau),\alpha)+1\right]x'(\tau) + \\ \left[(1-\alpha)\mathrm{P}\varphi(t(\tau),\alpha)\dot{\varphi} + \alpha\mathrm{P}\varphi(t(\tau),\alpha)\right]x(\tau) - \\ \left[(1-\alpha)\mathrm{Q}\varphi(t(\tau),\alpha)\dot{\varphi} + \alpha\mathrm{Q}\varphi(t(\tau),\alpha)\right] = 0,\end{gathered} \tag{15}$$

where $\dot{\varphi} = d\varphi/dt$. Equation (15) has the following integrating factor [11]

$$\mu(\tau) = \exp\int \frac{\alpha\mathrm{P}\varphi(t(\tau),\alpha)}{1+(1-\alpha)\mathrm{P}\varphi(t(\tau),\alpha)}d\tau. \tag{16}$$

Therefore, multiplying Eq. (15) by $\mu(\tau)$ and integrating we arrive to the solution of Eq. (12) which is given by

$$x(\tau) = \Xi(\tau)\bigg(C + \int \mu(\tau)[(1-\alpha)\mathrm{Q}\dot{\varphi} + \alpha\mathrm{Q}]\varphi(t(\tau),\alpha)d\tau\bigg), \tag{17}$$

where

$$\Xi(\tau) = \frac{1}{\left[(1-\alpha)\mathrm{P}\varphi(t(\tau),\alpha)+1\right]\mu(\tau)}, \tag{18}$$

3. Example: Analysis of an RC circuit

We are now interested in the modeling and study of a simple electrical circuit consisting of a resistor and a capacitor within the framework of the fractional derivative with a non singular kernel [12-17]. Using the Kirchhoff voltage law in the circuit shown in Fig. 1, we get the following fractional differential equation

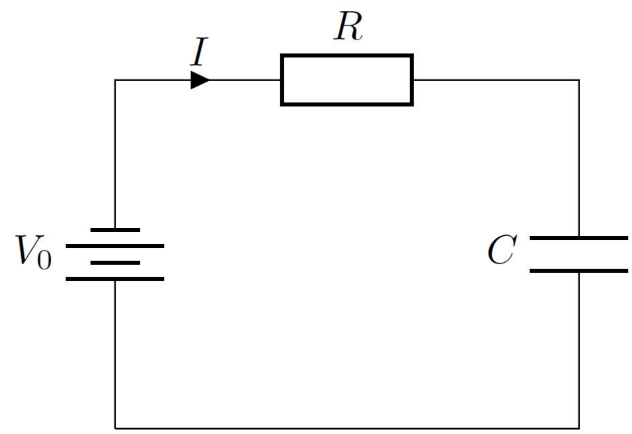

FIGURE 1. The figure shows the RC circuit.

$${}^{CF}D_t^\alpha q(t) + \Gamma q(t) = \Gamma q_0, \tag{19}$$

where $\Gamma = 1/RC$ and $q_0 = V_0 C$.

In order to have dimensional consistency in Eq. (19), several authors include an auxiliary parameter σ, which has the dimension of seconds and is associated with the temporal components in the system to maintain the dimensional consistency of the fractional differential equation, which is valid for the case of singular kernels [18-24].

For the case of differential equations with non singular kernels we have to introduce an auxiliary function φ in order to have dimensional consistency in Eq. (19), therefore in this example we are going to use our method by using two different dimensionless functions $\tau(t,\alpha)$. For the first case we are going to use $\varphi(t,\alpha) = e^{-(1-\alpha)\Gamma t}/\Gamma$ and for the second case we are going to use $\varphi(t,\alpha) = (1+\Gamma(1-\alpha)t)/\Gamma$, respectively.

3.1. Case 1

We are going to use the following function $\varphi(t,\alpha) = e^{-(1-\alpha)\Gamma t}/\Gamma$ which has the unit of second and using Eq. (8) we get the following dimensionless function

$$\tau(t,\alpha) = \frac{e^{(1-\alpha)\Gamma t}-1}{1-\alpha}. \tag{20}$$

Note that $\tau(t,\alpha)$ is a strictly monotonic increasing function and satisfies Eq. (10). Using Eq. (12) we have the following fractional differential equation for the RC circuit

$${}^{CF}D_\tau^\alpha q(\tau) + \frac{q}{1+(1-\alpha)\tau} = \frac{q_0}{1+(1-\alpha)\tau} \quad \text{where} \quad 0 < \alpha \leq 1. \tag{21}$$

Using Eqs. (17) and (18) we find the solution of the fractional differential equation given in Eq. (21) which is given by

$$q(\tau,\alpha) = q_0 + C_0(1+(1-\alpha)\tau) \times \left(2-\alpha+(1-\alpha)\tau\right)^{1/(\alpha-1)}. \tag{22}$$

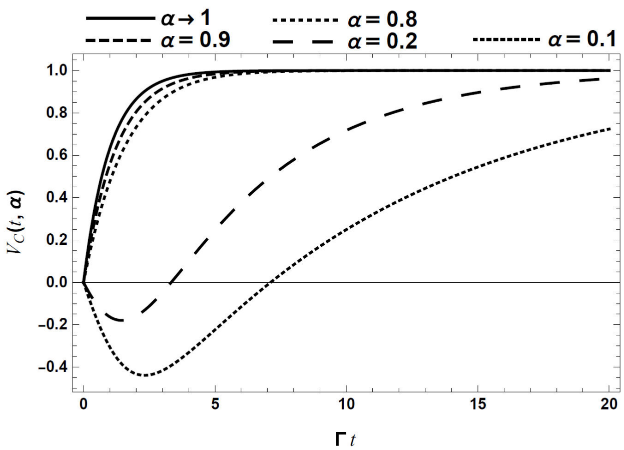

FIGURE 2. The figure shows the voltage across the capacitor for the case when the DC voltage is $V_0 = 1$ V.

Using the initial condition $q(0,\alpha) = 0 = q_0 + C_0(2-\alpha)^{1/(\alpha-1)}$ and using Eq. (20) we can write the solution in terms of the ordinary time and the initial condition which is given by

$$q(t,\alpha)=q_0 \times \left(1-e^{(1-\alpha)\Gamma t}\left[\frac{2-\alpha}{(1-\alpha)+e^{(1-\alpha)\Gamma t}}\right]^{1/(1-\alpha)}\right). \quad (23)$$

The voltage across the capacitor while it is charging is given by

$$V_C(t,\alpha) = V_0 \times \left(1 - e^{(1-\alpha)\Gamma t}\left[\frac{2-\alpha}{(1-\alpha)+e^{(1-\alpha)\Gamma t}}\right]^{1/(1-\alpha)}\right). \quad (24)$$

It is easy to verify the solution by taking $\alpha \to 1$ in Eq. (24) in order to get

$$V_C(t,1) = V_0\left(1-e^{-\Gamma t}\right), \quad (25)$$

which is the expected solution for an RC circuit connected to a DC battery.

In Fig. 2 we show the graph of the voltage across the capacitor given by Eq. (24) for different values of α. The results given in Fig. 2 shows that when $\alpha \to 1$ the system displays the classical behavior. However, for values of $0 < \alpha < 1$, the solution describes a dissipative system due to the fact that the fractional differentiation with respect to time represents a non-local effect of dissipation of energy (internal friction) represented by the fractional order α.

3.2. Case 2

We are going now to use the following function $\varphi(t,\alpha) = (1+\Gamma(1-\alpha)t))/\Gamma$ which has the unit of time and using Eq. (8) we get the following dimensionless function

$$\tau(t,\alpha) = \frac{1}{1-\alpha}\ln\left[1+\Gamma(1-\alpha)t\right]. \quad (26)$$

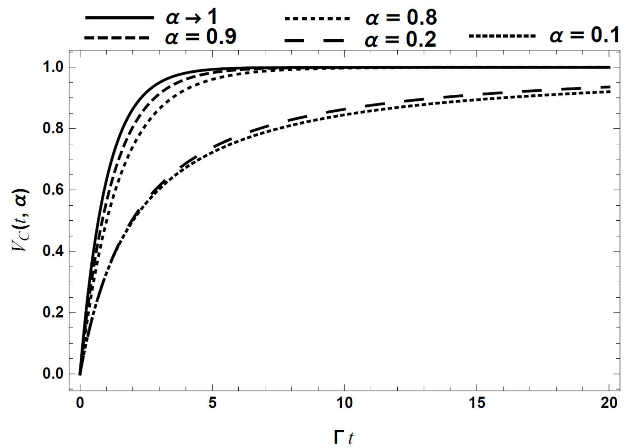

FIGURE 3. The figure shows the voltage across the capacitor for the case when the DC voltage is $V_0 = 1$ V.

Note that $\tau(t,\alpha)$ satisfies Eq. (10). Using Eq. (12) we have the following fractional differential equation for the RC circuit

$$^{CF}D_\tau^\alpha q(\tau) + qe^{(1-\alpha)\tau} = q_0 e^{(1-\alpha)\tau} \quad \text{where} \quad 0<\alpha\le 1. \quad (27)$$

Using Eqs. (17) and (18) we find the solution of the fractional differential equation given in Eq. (27) which is given by

$$q(\tau,\alpha) = q_0 + \frac{C_0}{\left[1+(1-\alpha)e^{(1-\alpha)\tau}\right]^{1+\alpha/(1-\alpha)^2}}. \quad (28)$$

Using the initial condition $q(0,\alpha) = 0 = q_0 + C_0(2-\alpha)^{-1-\alpha/(1-\alpha)^2)}$ and using Eq. (26) we can write the solution in terms of the ordinary time and the initial condition which is given by

$$q(t,\alpha) = q_0\left(1 - \frac{1}{\left[1+\frac{\Gamma(1-\alpha)^2}{(2-\alpha)}t\right]^{1+\alpha/(1-\alpha)^2}}\right). \quad (29)$$

The voltage across the capacitor while it is charging is given by

$$V_C(t,\alpha) = V_0\left(1 - \frac{1}{\left[1+\frac{\Gamma(1-\alpha)^2}{(2-\alpha)}t\right]^{1+\alpha/(1-\alpha)^2}}\right). \quad (30)$$

It is easy to verify the solution by taking $\alpha \to 1$ in Eq. (30) in order to get

$$V_C(t,1) = V_0\left(1-e^{-\Gamma t}\right), \quad (31)$$

which is the expected solution for an RC circuit connected to a DC battery.

4. Conclusions

A systematic way to construct fractional differential equations with non singular kernels has been proposed. The relevant aspect of this work is the method for setting up the fractional differential equations with nonsingular kernels while keeping the dimensionality of the physical parameters. A new function has been introduced, that depends on the order of the derivative being considered *i.e*. $0 < \alpha \leq 1$, to keep the dimensionality of the physical parameters. An example of this new approach has been given for the case of an RC circuit connected to a DC battery. The results presented in this paper can be easily extended to non singular fractional derivatives of higher order. We should point out that there might be other functions φ that could result in a different fractional differential equation and that will result in other classes of solutions with different behavior.

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