NORMALIZATION-COEFFICIENTS OF THE LOWERING AND RAISING OPERATORS AND MATRIX-ELEMENTS OF THE GENERATORS OF U_n.

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ABSTRACT

In this paper we derive in a self-contained way the normalization-coefficients of the lowering and raising operators of U_n *introduced previously by the authors. We then use the normalization-coefficients to derive in an abstract way the matrix-elements of the generators of* U_n *with respect to basis-vectors characterized by the canonical chain* $U_n \supset U_{n-1} \supset \ldots \supset U_2 \supset U_1$.

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1. INTRODUCTION

In a previous article[1], which shall be referred to as I and whose notation we shall be using, we introduced the concept of operators that lower or raise the irreducible vector-spaces of U_{n-1} contained in an irreducible vector-space of the unitary group U_n . The normalization-coefficients of these operators were determined with the help of the matrix-elements of the generators of U_n with respect to basis-vectors characterized be the chain of canonical subgroups

$$U_n \supset U_{n-1} \supset \cdots \supset U_2 \supset U_1 ,$$

obtained by Gelfand and Zetlin[2] and later rederived by Baird and Biedenharn[3] .

In this paper we shall turn the problem around and first give a self-contained derivation of the normalization-coefficients of the lowering and raising operators. We then use the normalization-coefficients to obtain the matrix-elements of a particular generator C^n_{n-1} of U_n with respect to basis-vectors characterized by the canonical chain

$$U_n \supset U_{n-1} \supset \cdots \supset U_2 \supset U_1 .$$

Using finally the commutation-relations of the generators we obtain the matrix-elements of C^n_m .

In this way we give a derivation of the Gelfand and Zetlin result, which is a purely abstract one as it only uses the commutation-relations of the generators. We have thus avoided the drawback of using explicit expressions[4] for the basis-vectors of the irreducible vector-spaces of U_n as was done in the derivation by Baird and Biedenharn.

2. THE UNITARY GROUP U_n

We shall start by reviewing the well-known properties of the unitary group U_n of n dimensions.

The generators of U_n, which we shall denote by $C_{\mu}^{\mu'}$ $1 \leq \mu, \mu' \leq n$, have the hermiticity properties

$$C_{\mu}^{\mu'\dagger} = C_{\mu'}^{\mu} \tag{2.1}$$

and fulfil the commutation-relations

$$\left[C_{\mu}^{\mu'}, C_{\mu''}^{\mu'''}\right] = \delta_{\mu''}^{\mu'} C_{\mu}^{\mu'''} - \delta_{\mu}^{\mu'''} C_{\mu''}^{\mu'} \tag{2.2}$$

from which one sees that $C_{\mu'}^{\mu}$ $\mu < \mu'$, C_{μ}^{μ} and $C_{\mu}^{\mu'}$ $\mu < \mu'$ are the lowering, weight and raising generators respectively.

An irreducible vector-space of U_n can be characterized by $[b_{\mu n}]$ $1 \leq \mu \leq n$ which is the highest of the weights of the basis-vectors of the vector-space. The $b_{\mu n}$ which are integers fulfil

$$b_{1n} \geq b_{2n} \geq \dots \geq b_{n-1n} \geq b_{nn} \tag{2.3}$$

For U_n a canonical subgroup is $U_{n-1} \dotplus 1$ whose generators are $C_{\mu}^{\mu'}$ $1 \leq \mu, \mu' < n$ and where $[b_{\mu n-1}]$ $1 \leq \mu < n$ characterizes an irreducible vector-space of U_{n-1}. The $b_{\mu n-1}$, besides satisfying

$$b_{1n-1} \geq b_{2n-1} \geq \dots \geq b_{n-2n-1} \geq b_{n-1n-1} \ ,$$

also fulfil

$$b_{\mu n} \geq b_{\mu n-1} \geq b_{\mu+1 n} \quad . \qquad (2.4)$$

A chain of canonical subgroups of U_n is then

$$U_n \supset U_{n-1} \supset \cdots \supset U_2 \supset U_1 \quad ,$$

so we can now completely characterize the normalized basis-vectors belonging to an irreducible vector-space of U_n by

$$\left| b_{\mu\nu} \right> \equiv \left| \begin{matrix} b_{1n} \cdots\cdots\cdots\cdots b_{nn} \\ b_{1n-1} \cdots\cdots b_{n-1n-1} \\ \cdots\cdots\cdots \\ b_{12} \quad b_{22} \\ b_{11} \end{matrix} \right\rangle \qquad 1 \leq \mu \leq \nu \leq n \; .$$

Defining

$$\left| \begin{matrix} b_\mu \\ q_\mu \end{matrix} \right\rangle \equiv \left| \begin{matrix} b_1 \cdots b_n \\ q_1 \cdots q_{n-1} \end{matrix} \right\rangle \equiv \left| \begin{matrix} b_1 \cdots\cdots\cdots\cdots b_n \\ q_1 \cdots\cdots\cdots q_{n-1} \\ q_1 \cdots\cdots q_{n-2} \\ \cdots\cdots\cdots \\ q_1 \quad q_2 \\ q_1 \end{matrix} \right\rangle$$

one has

$$C^{\mu}_{\mu} \begin{vmatrix} h_\lambda \\ q_\lambda \end{matrix} \Big\rangle = q_\mu \begin{vmatrix} h_\lambda \\ q_\lambda \end{matrix} \Big\rangle \quad , \quad 1 \le \mu < n \; , \tag{2.5}$$

$$C^{n}_{n} \Big| \begin{matrix} h_\mu \\ q_\mu \end{matrix} \Big\rangle = \sum_{\mu=1}^{n} h_\mu - \sum_{\mu=1}^{n-1} q_\mu \quad , \tag{2.6}$$

and

$$C^{\mu'}_{\mu} \Big| \begin{matrix} h_\lambda \\ q_\lambda \end{matrix} \Big\rangle = 0, \quad 1 \le \mu < \mu' < n \, .$$

3. THE LOWERING AND RAISING OPERATORS OF U_n .

In article I we defined the lowering operators L^m_n $1 \le m < n$ of U_n as operators which fulfil

$$[C^{\mu}_{\mu} , L^m_n] = - \delta^m_\mu L^m_n , \quad 1 \le \mu , m < n ,$$

and

$$L^m_n \Big| \begin{matrix} h_\mu \\ q_\mu \end{matrix} \Big\rangle \quad \alpha \quad \Big| \begin{matrix} h_\mu \\ q_\mu - \delta_{\mu m} \end{matrix} \Big\rangle \qquad 1 \le m < n \, .$$

They were there found to be given by

$$L^m_n = \left(\sum_{p=0}^{n-m-1} \sum_{\mu_p > \mu_{p-1} > \dots > \mu_2 > \mu_1 = m+1}^{n-1} C^m_{\mu_1} C^{\mu_1}_{\mu_2} \dots C^{\mu_{p-1}}_{\mu_p} C^{\mu_p}_n \prod_{i=1}^{p} \mathcal{E}^{-1}_{m\mu_1} \right) \prod_{\mu=m+1}^{n-1} \mathcal{E}_{m\mu} \; , \tag{3.1a'}$$

$$L_n^m = \prod_{\mu=m+1}^{n-1} \mathcal{E}_{m\mu} \sum_{p=0}^{n-m-1} \sum_{\mu_p > \mu_{p-1} > \ldots > \mu_2 > \mu_1 = m+1}^{n-1} \left(\prod_{i=1}^{p} \mathcal{E}^{-1}_{m\mu_i} \right) C_n^{\mu_p} \, C_{\mu_p}^{\mu_{p-1}} \ldots C_{\mu_2}^{\mu_1} \, C_{\mu_1}^{m} \tag{3.1b'}$$

The operators $\mathcal{E}_{\mu\mu'}$ are defined as

$$\mathcal{E}_{\mu\mu'} \equiv C_\mu^\mu - C_{\mu'}^{\mu'} + \mu' - \mu \, , \tag{3.2}$$

and have the properties

$$\mathcal{E}_{\mu'\mu} = - \mathcal{E}_{\mu\mu'} \tag{3.3}$$

and

$$\mathcal{E}_{\mu\mu'} \left| \begin{matrix} b_\lambda \\ q_\lambda \end{matrix} \right\rangle = q_{\mu\mu'} \left| \begin{matrix} b_\lambda \\ q_\lambda \end{matrix} \right\rangle \, . \tag{3.4}$$

The $q_{\mu\mu'}$ are defined as

$$q_{\mu\mu'} \equiv q_\mu - q_{\mu'} + \mu' - \mu \, ,$$

and fulfil

$$q_{\mu'\mu} = - q_{\mu\mu'} \tag{3.5}$$

and besides, as seen from (2.3),

$$q_{\mu\mu'} \gtreqless 0 \text{ for } \mu' \gtreqless \mu . \tag{3.6}$$

The raising operators R^n_m $1 \leq m < n$, which fulfil similarly

$$[C^\mu_\mu, R^n_m] = +\delta^n_m , \quad 1 \leq \mu, m < n$$

and

$$R^n_m \left| \begin{matrix} h_\mu \\ \\ q_\mu \end{matrix} \right\rangle \;\alpha\; \left| \begin{matrix} h_\mu \\ \\ q_\mu + \delta_{\mu m} \end{matrix} \right\rangle , \quad 1 \leq m < n ,$$

were found to be given by

$$R^n_m = \left(\sum_{p=0}^{m-1} \sum_{\mu_p > \mu_{p-1} > \ldots > \mu_2 > \mu_1 = 1}^{m-1} C^{\mu_p}_m C^{\mu_{p-1}}_{\mu_p} \ldots C^{\mu_1}_{\mu_2} C^n_{\mu_1} \prod_{i=1}^{p} \mathcal{E}^{-1}_{m\mu_i} \right) \prod_{\mu=1}^{m-1} \mathcal{E}_{m\mu} ,$$

(3.1a")

or

$$R^n_m = \prod_{\mu=1}^{m-1} \mathcal{E}_{m\mu} \sum_{p=0}^{m-1} \sum_{\mu_p > \mu_{p-1} > \ldots > \mu_2 \mu_1 = 1}^{m-1} \left(\prod_{i=1}^{p} \mathcal{E}^{-1}_{m\mu_i} \right) C^n_{\mu_1} C^{\mu_1}_{\mu_2} \ldots C^{\mu_{p-1}}_{\mu_p} C^{\mu_p}_m ,$$

(3.1b")

The lowering and raising operators given explicitly by Eqs. (3.1) satisfy furthermore

$$[L_n^m, L_n^{m'}] \left| \begin{matrix} b_\mu \\ q_\mu \end{matrix} \right\rangle = 0, \quad 1 \leq m < m' < n , \qquad (3.7')$$

$$[R_m^n, R_{m'}^n] \left| \begin{matrix} b_\mu \\ q_\mu \end{matrix} \right\rangle = 0, \quad 1 \leq m < m' < n , \qquad (3.7'')$$

$$[R_m^n, L_n^{m'}] \left| \begin{matrix} b_\mu \\ q_\mu \end{matrix} \right\rangle = 0, \quad 1 \leq m < m' < n , \qquad (3.8a)$$

$$[R_{m'}^n, L_n^m] \left| \begin{matrix} b_\mu \\ q_\mu \end{matrix} \right\rangle = 0, \quad 1 \leq m < m' < n , \qquad (3.8b)$$

but in general one has

$$[R_m^n, L_n^m] \left| \begin{matrix} b_\mu \\ q_\mu \end{matrix} \right\rangle \neq 0, \quad 1 \leq m < n . \qquad (3.9)$$

4. EXPANSION-FORMULAS FOR THE LOWERING AND RAISING GENERATORS

We shall here express the generators C_n^m and C_m^n in terms of lowering and raising operators of U_n and its unitary subgroups and the following operators $R_m^{n'}$ $1 \leq n' < m$

$$R_m^{n'} = \left(\sum_{p=0}^{m-1} \sum_{\mu_p > \mu_{p-1} > \ldots > \mu_2 > \mu_1 = n'+1}^{m-1} C_m^{\mu_p} C_{\mu_p}^{\mu_{p-1}} \ldots C_{\mu_2}^{\mu_1} C_{\mu_1}^{n'} \prod_{i=1}^{p} \mathcal{E}_{m\mu_i}^{-1} \right) \prod_{\mu=n'+1}^{m-1} \mathcal{E}_{m\mu} \,, \tag{4.1a}$$

or

$$R_m^{n'} = \prod_{\mu=n'+1}^{m-1} \mathcal{E}_{m\mu} \sum_{p=0}^{m-1} \sum_{\mu_p > \mu_{p-1} > \ldots > \mu_2 > \mu_1 = n'+1}^{m-1} \left(\prod_{i=1}^{p} \mathcal{E}_{m\mu_i}^{-1} \right) C_m^{\mu_p} C_{\mu_p}^{\mu_{p-1}} \ldots C_{\mu_2}^{\mu_1} C_{\mu_1}^{n'} \,, \tag{4.1b}$$

which however are not raising operators of U_n, as $1 \leq n' < m$. The formulas we shall prove are the following

$$C_n^m \prod_{\lambda > \kappa = m}^{n-1} \mathcal{E}_{\kappa\lambda} = \left(\sum_{\mu=m}^{n-1} R_\mu^m L_n^\mu \prod_{\substack{\nu=m \\ \nu \neq \mu}}^{n-1} \mathcal{E}_{\mu\nu}^{-1} \right) \prod_{\lambda > \kappa = m}^{n-1} \mathcal{E}_{\kappa\lambda} \,, \quad 1 \leq m < n \,, \tag{4.2a'}$$

$$\left(\prod_{\lambda > \kappa = m}^{n-1} \mathcal{E}_{\kappa\lambda} \right) C_n^m = \prod_{\lambda > \kappa = m}^{n-1} \mathcal{E}_{\kappa\lambda} \sum_{\mu=m}^{n-1} \left(\prod_{\substack{\nu=m \\ \nu \neq \mu}}^{n-1} \mathcal{E}_{\mu\nu}^{-1} \right) L_n^\mu R_\mu^m \,, \quad 1 \leq m < n \,, \tag{4.2b'}$$

$$C_m^n \prod_{\lambda > \kappa = 1}^{m} \mathcal{E}_{\kappa\lambda} = \left(\sum_{\mu=1}^{m} L_m^\mu R_\mu^n \prod_{\substack{\nu=1 \\ \nu \neq \mu}}^{m} \mathcal{E}_{\mu\nu}^{-1} \right) \prod_{\lambda > \kappa = 1}^{m} \mathcal{E}_{\kappa\lambda} \,, \quad 1 \leq m < n \,, \tag{4.2a''}$$

and

$$\left(\prod_{\lambda > \kappa = 1}^{m} \mathcal{E}_{\kappa\lambda} \right) C_m^n = \prod_{\lambda > \kappa = 1}^{m} \mathcal{E}_{\kappa\lambda} \sum_{\mu=1}^{m} \left(\prod_{\substack{\nu=1 \\ \nu \neq \mu}}^{m} \mathcal{E}_{\mu\nu}^{-1} \right) R_\mu^n L_m^\mu \,, \quad 1 \leq m < n \,, \tag{4.2b''}$$

where

$$L_m^m \equiv 1 \,, \tag{4.3'}$$

and

$$R_m^m \equiv 1 \,. \tag{4.3''}$$

We shall now prove explicitly formula (4.2a') . The other formulas can be proved in similar ways. From Eqs. (4.3"), (3.1b") and (3.1a') one obtains for the right-hand side of Eq. (4.2a')

$$\left\{ \sum_{\mu=m}^{n-1} R_\mu^m L_n^\mu \prod_{\substack{\nu=m \\ \nu\neq\mu}}^{n-1} \mathcal{E}_{\mu\nu}^{-1} \right\} \prod_{\lambda>\kappa=m}^{n-1} \mathcal{E}_{\kappa\lambda}$$

$$= \left\{ L_n^m \prod_{\nu=m+1}^{n-1} \mathcal{E}_{m\nu}^{-1} + \sum_{\mu=m+1}^{n-1} R_\mu^m L_n^\mu \prod_{\substack{\nu=m \\ \nu\neq\mu}}^{n-1} \mathcal{E}_{\mu\nu}^{-1} \right\} \prod_{\lambda>\kappa=m}^{n-1} \mathcal{E}_{\kappa\lambda}$$

$$= \left\{ \sum_{q=0}^{n-m-1} \sum_{\nu_q>\nu_{q-1}>\ldots>\nu_2>\nu_1=m+1}^{n-1} C_{\nu_1}^m C_{\nu_2}^{\nu_1} \ldots C_{\nu_q}^{\nu_{q-1}} C_n^{\nu_q} \prod_{i=1}^{q} \mathcal{E}_{m\nu_i}^{-1} \right.$$

$$+ \sum_{\mu=m+1}^{n-1} \left[\sum_{p=0}^{\mu-m-1} \sum_{\mu_p>\mu_{p-1}>\ldots>\mu_2>\mu_1=m+1}^{\mu-1} C_{\mu_1}^m C_{\mu_2}^{\mu_1} \ldots C_{\mu_p}^{\mu_{p-1}} C_\mu^{\mu_p} \right.$$

$$\left. \left. \times \left(\sum_{q=0}^{n-\mu-1} \sum_{\nu_q>\nu_{q-1}>\ldots>\nu_2>\nu_1=\mu+1}^{n-1} C_{\nu_1}^\mu C_{\nu_2}^{\nu_1} \ldots C_{\nu_q}^{\nu_{q-1}} C_n^{\nu_q} \prod_{i=1}^{q} \mathcal{E}_{\mu\nu_i}^{-1} \right) \prod_{i=1}^{p} \mathcal{E}_{\mu\mu_i}^{-1} \right] \mathcal{E}_{\mu m}^{-1} \right\} \prod_{\lambda>\kappa=m}^{n-1} \mathcal{E}_{\kappa\lambda}$$

$$= \left\{ C_n^m + \sum_{p=1}^{n-m-1} \sum_{\mu_p > \mu_{p-1} > \ldots > \mu_2 > \mu_1 = m+1}^{n-1} C_{\mu_1}^m C_{\mu_2}^{\mu_1} \ldots C_{\mu_p}^{\mu_{p-1}} C_n^{\mu_p} \left(\sum_{i=0}^{p} \prod_{\substack{j=0 \\ j \neq i}}^{p} \mathcal{E}_{\mu_i \mu_j}^{-1} \right) \right\} \prod_{\lambda > \kappa = m}^{n-1} \mathcal{E}_{\kappa\lambda} \,, \tag{4.4}$$

where

$$\mu_0 \equiv m \quad .$$

We shall now prove an identity-relation for the $\mathcal{E}_{\mu\mu'}$. Defining

$$\chi_i \equiv C_{\mu_i}^{\mu_i} - \mu_i$$

one obtains from the definition (3.2), using a well-known relation (see e.g. Ref.5),

$$\prod_{j>i=1}^{p} \mathcal{E}_{\mu_i \mu_j} = \prod_{j>i=1}^{p} (\chi_i - \chi_j) = \begin{vmatrix} 1 & 1 \ldots\ldots\ldots 1 & 1 \\ \chi_1 & \chi_2 \ldots\ldots\ldots \chi_{p-1} & \chi_p \\ \ldots & \ldots\ldots\ldots\ldots & \ldots \\ \chi_1^{p-2} & \chi_2^{p-2} \ldots\ldots \chi_{p-1}^{p-2} & \chi_p^{p-2} \\ \chi_1^{p-1} & \chi_2^{p-1} \ldots\ldots \chi_{p-1}^{p-1} & \chi_p^{p-1} \end{vmatrix} \equiv D(\chi_1 \cdots \chi_p) \,. \tag{4.5}$$

From Eqs. (3.3) and (4.5) it then follows that

$$\left(\sum_{i=1}^{p} \prod_{\substack{j=1 \\ j \neq i}}^{p} \mathcal{E}_{\mu_i \mu_j}^{-1} \right) \prod_{l>k=1}^{p} \mathcal{E}_{\mu_k \mu_l} = \sum_{i=1}^{p} (-)^{i-1} \prod_{\substack{l>k=1 \\ k,l \neq i}}^{p} \mathcal{E}_{\mu_k \mu_l}$$

$$= \sum_{i=1}^{p} (-)^{i-1} D(\chi_1 \cdots \chi_{i-1}\ \chi_{i+1} \cdots \chi_p)$$

$$= (-)^{p-1} \begin{vmatrix} 1 & 1 & \cdots\cdots & 1 & 1 \\ \chi_1 & \chi_2 & \cdots\cdots & \chi_{p-1} & \chi_p \\ \cdots & \cdots & \cdots\cdots & \cdots & \cdots \\ \chi_1^{p-2} & \chi_2^{p-2} & \cdots\cdots & \chi_{p-1}^{p-2} & \chi_p^{p-2} \\ 1 & 1 & \cdots\cdots & 1 & 1 \end{vmatrix} = 0 ,$$

which then shows that the sum over i in Eq. (4.4) vanishes and so Eq. (4.2a') is proved.

5. THE NORMALIZED LOWERING AND RAISING OPERATORS

The normalized lowering operators

$$\mathcal{L}\begin{pmatrix} q_\mu \\ h_\mu \\ q_\mu - \delta_{\mu m} \end{pmatrix} \equiv \mathcal{L}^{q_\mu}_{q_\mu - \delta_{\mu m}} \quad 1 \leq m < n ,$$

were in I defined to fulfil

$$\mathcal{L}^{q_\mu}_{q_\mu - \delta_{\mu m}} \left| \begin{matrix} h_\mu \\ q_\mu \end{matrix} \right\rangle \equiv \left| \begin{matrix} h_\mu \\ q_\mu - \delta_{\mu m} \end{matrix} \right\rangle \qquad 1 \leq m < n ,$$

and could hence in terms of a normalization-coefficient

$$N\begin{pmatrix} q_\mu \\ h_\mu \\ q_\mu - \delta_{\mu m} \end{pmatrix} \equiv N^{q_\mu}_{q_\mu - \delta_{\mu m}} \quad 1 \leq m < n \ ,$$

be written as

$$\mathfrak{L}^{q_\mu}_{q_\mu - \delta_{\mu m}} = \left(N^{q_\mu}_{q_\mu - \delta_{\mu m}} \right)^{-1} L^m_n \ . \qquad (5.1')$$

The normalized raising operators

$$\mathfrak{R}\begin{pmatrix} q_\mu \\ h_\mu \\ q_\mu + \delta_{\mu m} \end{pmatrix} \equiv \mathfrak{R}^{q_\mu}_{q_\mu + \delta_{\mu m}} \quad 1 \leq m < n \ ,$$

fulfil similarly

$$\mathfrak{R}^{q_\mu}_{q_\mu + \delta_{\mu m}} \left| \begin{matrix} h_\mu \\ q_\mu \end{matrix} \right\rangle \equiv \left| \begin{matrix} h_\mu \\ q_\mu + \delta_{\mu m} \end{matrix} \right\rangle^{*}$$

and can in terms of the normalization-coefficient

$$N\begin{pmatrix} q_\mu \\ h_\mu \\ q_\mu + \delta_{\mu m} \end{pmatrix} \equiv N^{q_\mu}_{q_\mu + \delta_{\mu m}} \quad 1 \leq m < n \ ,$$

be written as

$$\mathfrak{R}^{q_\mu}_{q_\mu + \delta_{\mu m}} = \left(N^{q_\mu}_{q_\mu + \delta_{\mu m}} \right)^{-1} R^n_m \ . \qquad (5.1'')$$

The two normalization-coefficients were in I found to fulfil the symmetry-relation

$$N^{q_\mu}_{q_\mu + \delta_{\mu m}} = \left(\prod_{\mu = 1}^{m-1} q_{m\mu} \Big/ \prod_{\mu = m+1}^{n-1} (q_{m\mu} + 1) \right) N^{q_\mu + \delta_{\mu m}}_{q_\mu} \quad . \tag{5.2}$$

We are now able to obtain with the help of the operators L^m_n and R^n_m any normalized basis-vector $\left| \begin{matrix} h_\mu \\ q'_\mu \end{matrix} \right\rangle$ from any given normalized basis-vector $\left| \begin{matrix} h_\mu \\ q_\mu \end{matrix} \right\rangle$.

Denoting the general normalization-coefficients by

$$N \begin{pmatrix} q_\mu \\ h_\mu \\ q'_\mu \end{pmatrix} \equiv N^{q_\mu}_{q'_\mu} \quad ,$$

we have

$$\left| \begin{matrix} h_\mu \\ q'_\mu \end{matrix} \right\rangle = \left(N^{q_\mu}_{q'_\mu} \right)^{-1} (L^1_n)^{q_1 - q'_1} \dots (R^n_{n-1})^{q'_{n-1} - q_{n-1}} \left| \begin{matrix} h_\mu \\ q_\mu \end{matrix} \right\rangle \tag{5.3}$$

where we have considered an example with $q'_1 < q_1 , \dots , q'_{n-1} > q_{n-1}$. In general, when $q'_\mu \leq q_\mu$ one has $(L^\mu_n)^{q_\mu - q'_\mu}$ and when $q'_\mu \geq q_\mu$ one has $(R^n_\mu)^{q'_\mu - q_\mu}$ in the product of the operators acting on $\left| \begin{matrix} h_\mu \\ q_\mu \end{matrix} \right\rangle$ in Eq. (5.3).

Due to Eqs. (3.7) and (3.8) the general normalization-coefficients in Eq. (5.3) is independent of the order of the lowering and raising operators and so, if we consider some values q''_μ where due to Eq. (3.9) either $q'_\mu \leq q''_\mu \leq q_\mu$ or $q'_\mu \geq q''_\mu \geq q_\mu$ $1 \leq \mu < n$, we obtain for the normalization-coefficients

$$N^{q_\mu}_{q'_\mu} = N^{q''_\mu}_{q'_\mu} N^{q_\mu}_{q''_\mu} \quad . \tag{5.4}$$

6. THE NORMALIZATION-COEFFICIENTS OF THE LOWERING AND RAISING OPERATORS

We shall now, by means of two recursion-relations, derive the explicit formulas for the normalization-coefficients. From Eq. (5.4), corresponding to Eq. (3.7'), we obtain the following relation among the normalization-coefficients

$$N^{q_\mu}_{q_\mu - \delta_{\mu m} - \delta_{\mu m'}}$$

$$= N^{q_\mu - \delta_{\mu m}}_{q_\mu - \delta_{\mu m} - \delta_{\mu m'}} \quad N^{q_\mu}_{q_\mu - \delta_{\mu m}} = N^{q_\mu - \delta_{\mu m'}}_{q_\mu - \delta_{\mu m} - \delta_{\mu m'}} \quad N^{q_\mu}_{q_\mu - \delta_{\mu m'}} \; , \quad 1 \leq m < m' < n \; . \tag{6.1}$$

Equation (3.7") does not give anything new due to the symmetry-relation (5.2). Corresponding to Eq. (3.8a) one obtains, using Eq. (5.2),

$$N^{q_\mu}_{q_\mu + \delta_{\mu m} - \delta_{\mu m'}}$$

$$= N^{q_\mu + \delta_{\mu m}}_{q_\mu + \delta_{\mu m} - \delta_{\mu m'}} \quad N^{q_\mu}_{q_\mu + \delta_{\mu m}} = \left(\prod_{\mu=1}^{m-1} q_{m\mu} \Big/ \prod_{\mu=m+1}^{n-1} (q_{m\mu}+1) \right) \; N^{q_\mu + \delta_{\mu m}}_{q_\mu + \delta_{\mu m} - \delta_{\mu m'}} \quad N^{q_\mu + \delta_{\mu m}}_{q_\mu}$$

$$= N^{q_\mu - \delta_{\mu m'}}_{q_\mu + \delta_{\mu m} - \delta_{\mu m'}} \quad N^{q_\mu}_{q_\mu - \delta_{\mu m'}}$$

$$= \Big((q_{mm'}+1) \big/ (q_{mm'}+2) \Big) \left(\prod_{\mu=1}^{m-1} q_{m\mu} \Big/ \prod_{\mu=m+1}^{n-1} (q_{m\mu}+1) \right) N^{q_\mu + \delta_{\mu m} - \delta_{\mu m'}}_{q_\mu - \delta_{\mu m'}} \; N^{q_\mu}_{q_\mu - \delta_{\mu m'}} \; , \; 1 \leq m < m' < n \; ,$$

from which follows

$$N^{q_\mu+\delta_{\mu m}-\delta_{\mu m'}}_{q_\mu-\delta_{\mu m'}} N^{q_\mu}_{q_\mu-\delta_{\mu m'}} = \left((q_{mm'}+2) \Big/ (q_{mm'}+1) \right) N^{q_\mu+\delta_{\mu m}}_{q_\mu+\delta_{\mu m}-\delta_{\mu m'}} \quad N^{q_\mu+\delta_{\mu m}}_{q_\mu} \; . \tag{6.2}$$

Inserting Eq. (6.2) into Eq. (6.1), one obtains the first part of the 1st recursion-relation for the normalization-coefficients

$$\left(N^{q_\mu-\delta_{\mu m}}_{q_\mu-\delta_{\mu m}-\delta_{\mu m'}} \right)^2 = \left((q_{mm'}+1) \Big/ q_{mm'} \right) \left(N^{q_\mu}_{q_\mu-\delta_{\mu m'}} \right)^2 , \quad 1 \leq m < m' < n \; . \tag{6.3a}$$

Corresponding to Eq. (3.8b) one obtains, in a similar way, the second part

$$\left(N^{q_\mu-\delta_{\mu m'}}_{q_\mu-\delta_{\mu m}-\delta_{\mu m'}} \right)^2 = \left((q_{mm'}+1) \Big/ q_{mm'} \right) \left(N^{q_\mu}_{q_\mu-\delta_{\mu m}} \right)^2 , \quad 1 \leq m < m' < n \; . \tag{6.3b}$$

Using Eqs. (6.3) successively one obtains in general

$$\left(N^{q_1' \cdots q_{m-1}' q_m q_{m+1}' \cdots\cdots q_{n-1}'}_{q_1' \cdots q_{m-1}' q_m-1\, q_{m+1}' \cdots q_{n-1}'} \right)^2$$

$$= \prod_{\mu=1}^{m-1} \frac{q_\mu - q_m + m - \mu + 1}{q_\mu' - q_m + m - \mu + 1} \prod_{\mu=m+1}^{n-1} \frac{q_m - q_\mu' + \mu - m}{q_m - q_\mu + \mu - m} \left(N^{q_1 \cdots q_{m-1} q_m q_{m+1} \cdots\cdots q_{n-1}}_{q_1 \cdots q_{m-1} q_m - 1\, q_{m+1} \cdots q_{n-1}} \right)^2 . \tag{6.4}$$

We shall now derive the 2nd recursion-relation which is needed for the normalization-coefficients. Using Eqs. (4.2a') and (3.4) and that, according to the inequalities (2.4), $b_{\mu+1}^{-}$ is the lowest value of q_μ^{-} and furthermore using Eq. (5.1') one obtains

$$C_n^m \left| \begin{matrix} b_1 \cdots\cdots\cdots\cdots b_n \\ q_1 \cdots q_m b_{m+2} \cdots b_n \end{matrix} \right\rangle = \left(\prod_{\mu=m+2}^{n} (q_m - b_\mu + \mu - m - 1) \right)^{-1} L_n^m \left| \begin{matrix} b_1 \cdots\cdots\cdots\cdots b_n \\ q_1 \cdots q_m b_{m+2} \cdots b_n \end{matrix} \right\rangle$$

$$= \left(\prod_{\mu=m+2}^{n} (q_m - b_\mu + \mu - m - 1) \right)^{-1} N_{q_1 \cdots q_m - 1\, b_{m+2} \cdots b_n}^{q_1 \cdots q_m b_{m+2} \cdots\cdots b_n} \left| \begin{matrix} b_1 \cdots\cdots\cdots\cdots\cdots b_n \\ q_1 \cdots q_m - 1\, b_{m+2} \cdots b_n \end{matrix} \right\rangle .$$

(6.5')

In a similar way one obtains from Eq. (4.2a"), using that b_μ is the highest value of q_μ and besides using the symmetry-relation (5.2)

$$C_m^n \left| \begin{matrix} b_1 \cdots\cdots\cdots\cdots b_n \\ b_1 \cdots b_{m-1} q_m \cdots q_{n-1} \end{matrix} \right\rangle$$

$$= \left(\prod_{\mu=m+1}^{n-1} (q_{m\mu} + 1) \right)^{-1} N_{b_1 \cdots b_{m-1} q_m \cdots\cdots q_{n-1}}^{b_1 \cdots b_{m-1} q_m + 1 \cdots q_{n-1}} \left| \begin{matrix} b_1 \cdots\cdots\cdots\cdots\cdots b_n \\ b_1 \cdots b_{m-1} q_m + 1 \cdots q_{n-1} \end{matrix} \right\rangle .$$

(6.5")

Now, taking the normalization-coefficients real, one obtains from Eqs. (6.5') and (6.5"), using Eqs. (2.1), (2.2), (2.5) and (2.6), the 2nd recursion-relation

$$\left(\prod_{\mu=m+2}^{n} (q_m - b_\mu + \mu - m - 1) \right)^{-2} \left(N_{b_1 \cdots b_{m-1} q_m - 1\, b_{m+2} \cdots b_n}^{b_1 \cdots b_{m-1} q_m b_{m+2} \cdots\cdots b_n} \right)^2$$

$$- \left(\prod_{\mu = m+2}^{n} (q_m - b_\mu + \mu - m) \right)^{-2} \left(N^{b_1 \cdots b_{m-1}\, q_m + 1\, b_{m+2} \cdots b_n}_{b_1 \cdots b_{m-1}\, q_m\, b_{m+2} \cdots\cdots b_n} \right)^2$$

$$= \left\langle \begin{matrix} b_1 \cdots\cdots\cdots\cdots\cdots\cdots b_n \\ b_1 \cdots b_{m-1}\, q_m\, b_{m+2} \cdots b_n \end{matrix} \right| [\, C^n_m \, , C^m_n \,] \left| \begin{matrix} b_1 \cdots\cdots\cdots\cdots\cdots\cdots b_n \\ b_1 \cdots b_{m-1}\, q_m\, b_{m+2} \cdots b_n \end{matrix} \right\rangle$$

$$= q_m - b_m + q_m - b_{m+1} \quad , \tag{6.6}$$

where

$$b_m \geq q_m \geq b_{m+1} \quad , \tag{6.7}$$

so that as boundary-conditions to Eq. (6.6) we have

$$N^{b_1 \cdots b_{m-1}\, b_{m+1}\, b_{m+2} \cdots\cdots b_n}_{b_1 \cdots b_{m-1}\, b_{m+1} - 1\, b_{m+2} \cdots b_n} = 0 \, , \tag{6.8'}$$

and

$$N^{b_1 \cdots b_{m-1}\, b_m + 1\, b_{m+2} \cdots b_n}_{b_1 \cdots b_{m-1}\, b_m\, b_{m+2} \cdots\cdots b_n} = 0 \quad . \tag{6.8''}$$

The unique solution to the linear recursion-relation (6.6) with the boundary-condition (6.8') or (6.8") is easily seen to be

$$\left(N^{b_1 \cdots b_{m-1}\, q_m\, b_{m+2} \cdots\cdots b_n}_{b_1 \cdots b_{m-1}\, q_m - 1\, b_{m+2} \cdots b_n} \right)^2 = - \prod_{\mu = m+2}^{n} (q_m - b_\mu + \mu - m - 1) \prod_{\mu = m}^{n} (q_m - b_\mu + \mu - m - 1) \; . \tag{6.9}$$

From Eqs. (6.4) and (6.9) one then finally obtains for the normalization-coefficients of the lowering operators

$$\left[N\begin{pmatrix} q_\mu \\ b_\mu \\ q_\mu - \delta_{\mu m} \end{pmatrix} \right]^2$$

$$= \left(\prod_{\mu=m+1}^{n-1} (q_{m\mu} \Big/ \prod_{\mu=1}^{m-1} (q_{\mu m}+1) \right) \prod_{\mu=1}^{m} (b_\mu - q_m + m - \mu + 1) \prod_{\mu=m+1}^{n} (q_m - b_\mu + \mu - m - 1) \ . \tag{6.10}$$

We shall now choose our phase-convention for the normalization-coefficients as

$$N\begin{pmatrix} q_\mu \\ b_\mu \\ q_\mu - \delta_{\mu m} \end{pmatrix} > 0 \ , \tag{6.11}$$

for all q_μ , b_μ and m . From Eqs. (6.10) and (6.11) one then gets for the normalization-coefficients of the lowering operators

$$N\begin{pmatrix} q_\mu \\ b_\mu \\ q_\mu - \delta_{\mu m} \end{pmatrix}$$

$$= \left[\left(\prod_{\mu=m+1}^{n-1} (q_{m\mu} \Big/ \prod_{\mu=1}^{m-1} (q_{\mu m}+1) \right) \prod_{\mu=1}^{m} (b_\mu - q_m + m - \mu + 1) \prod_{\mu=m+1}^{n} (q_m - b_\mu + \mu - m - 1) \right]^{\frac{1}{2}}$$

$$= \left[- \left(\prod_{\mu=m+1}^{n-1} q_{m\mu} \Big/ \prod_{\mu=1}^{m-1} (q_{m\mu} - 1) \right) \prod_{\mu=1}^{n} (q_m - b_\mu + \mu - m - 1) \right]^{\frac{1}{2}} \ . \tag{6.12'}$$

Using Eq. (5.4) successively and Eq. (6.12') one now obtains the general normalization-coefficients of the lowering operators as

$$N\begin{pmatrix} q_\mu \\ b_\mu \\ q'_\mu \end{pmatrix}$$

$$= \left[\prod_{\mu>\lambda=1}^{n-1} \frac{(q_\lambda - q_\mu + \mu - \lambda)!}{(q'_\lambda - q'_\mu + \mu - \lambda)!} \prod_{\mu>\lambda=1}^{n-1} \frac{(b_\lambda - q'_\mu + \mu - \lambda)!}{(b_\lambda - q_\mu + \mu - \lambda)!} \prod_{\mu>\lambda=1}^{n} \frac{(q_\lambda - b_\mu + \mu - \lambda - 1)!}{(q'_\lambda - b_\mu + \mu - \lambda - 1)!} \right]^{\frac{1}{2}} \quad q'_\mu \leq q_\mu \ .$$

For the normalization-coefficients of the raising operators one obtains from the symmetry-relation (5.2) and Eq. (6.12'), also using Eqs. (3.5) and (3.6),

$$N\begin{pmatrix} q_\mu \\ b_\mu \\ q_\mu + \delta_{\mu m} \end{pmatrix}$$

$$= \left[\left(\prod_{\mu=m+1}^{n-1} (q_{m\mu} \Big/ \prod_{\mu=1}^{m-1} (q_{\mu m} + 1) \right) \prod_{\mu=1}^{m} (b_\mu - q_m + m - \mu + 1) \prod_{\mu=m+1}^{n} (q_m - b_\mu + \mu - m - 1) \right]^{\frac{1}{2}}$$

$$= (-)^{m-1} \left[- \left(\prod_{\mu=1}^{m-1} q_{m\mu} \Big/ \prod_{\mu=m+1}^{n-1} (q_{m\mu} + 1) \right) \prod_{\mu=1}^{n} (q_m - b_\mu + \mu - m) \right]^{\frac{1}{2}} , \qquad (6.12'')$$

from which one sees, that the phase-convention (6.11) does not imply that the normalization-coefficients of the raising operators are also all positive.

From (6.12") one obtains in the same way as for the lowering operators

$$N\begin{pmatrix} q_\mu \\ b_\mu \\ q'_\mu \end{pmatrix} = (-)^{\sum_{\mu=1}^{n-1}(\mu-1)(q'_\mu - q_\mu)}$$

$$\times\left[\prod_{\mu>\lambda=1}^{n-1}\frac{(q_\lambda - q_\mu + \mu - \lambda)!}{(q'_\lambda - q'_\mu + \mu - \lambda)!}\prod_{\mu>\lambda=1}^{n-1}\frac{(b_\lambda - q_\mu + \mu - \lambda)!}{(b_\lambda - q'_\mu + \mu - \lambda)!}\prod_{\mu>\lambda=1}^{n}\frac{(q'_\lambda - b_\mu + \mu - \lambda - 1)!}{(q_\lambda - b_\mu + \mu - \lambda - 1)!}\right]^{\frac{1}{2}}, \quad q'_\mu \geq q_\mu \, . \tag{6.13''}$$

7. THE MATRIX-ELEMENTS OF C^n_{n-1}.

In I one saw that the normalization-coefficients of the lowering operators can be expressed in terms of matrix-elements of the lowering generators C^m_n. The converse is also true, i.e. that the matrix-elements of C^{n-1}_n can be expressed in terms of normalization-coefficients and hence from Eqs. (6.13) can be calculated explicitly. From the matrix-elements of C^{n-1}_n, $C^{n-2}_{n-1}, \ldots C^2_3$, C^1_2 one can then, using the commutation-relations (2.2), derive the matrix-elements of all the generators C^m_n.

We shall now proceed to derive the matrix-elements of C^n_{n-1}, rather than of C^{n-1}_n. Defining

$$\left|\begin{matrix} b_\mu \\ q_\mu \\ r_\mu \end{matrix}\right\rangle \equiv \left|\begin{matrix} b_1 \ldots\ldots\ldots\ldots\ldots b_n \\ q_1 \ldots\ldots\ldots q_{n-1} \\ \ldots\ldots\ldots\ldots \\ r_1 \ldots\ldots r_{n-2} \\ \ldots\ldots \\ r_1 \end{matrix}\right\rangle$$

using Eq. (5.3) and the fact that C^n_{n-1} commutes with $L^{\varkappa}_{n-1}$, and besides using Eq. (4.2a"), we obtain

$$\left\langle \begin{matrix} b_\mu \\ q_\mu + \delta_{\mu l} \\ r_\mu \end{matrix} \right| C^n_{n-1} \left| \begin{matrix} b_\mu \\ q_\mu \\ r_\mu \end{matrix} \right\rangle$$

$$= \left[N \begin{pmatrix} q_\mu \\ q_\mu \\ r_\mu \end{pmatrix} \right]^{-1} \left\langle \begin{matrix} b_\mu \\ q_\mu + \delta_{\mu l} \\ r_\mu \end{matrix} \right| C^n_{n-1} \prod_{\varkappa=1}^{n-2} \left(L^{\varkappa}_{n-1} \right)^{q_\varkappa - r_\varkappa} \left| \begin{matrix} q_\mu \\ q_\mu \\ q_\mu \end{matrix} \right\rangle$$

$$= \left[N \begin{pmatrix} q_\mu \\ q_\mu \\ r_\mu \end{pmatrix} \right]^{-1} \sum_{\nu=1}^{n-1} \left(\prod_{\substack{\lambda=1 \\ \lambda \neq \nu}}^{n-1} q_{\nu\lambda} \right)^{-1} \left\langle \begin{matrix} b_\mu \\ q_\mu + \delta_{\mu l} \\ r_\mu \end{matrix} \right| \prod_{\varkappa=1}^{n-2} \left(L^{\varkappa}_{n-1} \right)^{q_\varkappa + \delta_{\varkappa\nu} - r_\varkappa} R^n_\nu \left| \begin{matrix} b_\mu \\ q_\mu \\ q_\mu \end{matrix} \right\rangle \quad , \; l < n \; . \tag{7.1}$$

Now noticing that C^n_{n-1} is a scalar with respect to U_{n-2}, which follows from the special case of the commutation-relations (2.2)

$$[\, C^{\mu'}_{\mu} \, , \; C^n_{n-1} \,] = 0 \, , \quad 1 \leq \mu, \mu' < n-1 \, ,$$

one sees that the matrix of C^n_{n-1} is diagonal with respect to the basis-vectors of an irreducible vector-space of U_{n-2} and furthermore does not depend on these (see e.g. Ref. 6). Using this fact one then finally obtains from Eq. (7.1) for the general matrix-element of C^n_{n-1}, changing nótation and also using Eqs. (5.1"), (5.3) and (6.12"), (6.13'),

$$\left\langle b_{\mu\nu} + \delta_{\mu l}\, \delta_{\nu n-1} \left| C^{n}_{n-1} \right| b_{\mu\nu} \right\rangle$$

$$= \left(\prod_{\substack{\lambda=1 \\ \lambda \neq l}}^{n-1} b_{ln-1,\, \lambda n-1} \right)^{-1} N \begin{pmatrix} b_{\mu n-1} \\ b_{\mu n} \\ b_{\mu n-1} + \delta_{\mu l} \end{pmatrix} N \begin{pmatrix} b_{\mu n-1} + \delta_{\mu l} \\ b_{\mu n-1} + \delta_{\mu l} \\ b_{\mu n-2} \end{pmatrix} \left[N \begin{pmatrix} b_{\mu n-1} \\ b_{\mu n-1} \\ b_{\mu n-2} \end{pmatrix} \right]^{-1}$$

$$= \left[- \frac{\prod_{\lambda=1}^{n-2} (b_{ln-1,\, \lambda n-2} + 1)}{\prod_{\substack{\lambda=1 \\ \lambda \neq l}}^{n-1} (b_{ln-1,\, \lambda n-1} + 1)} \cdot \frac{\prod_{\lambda=1}^{n} b_{ln-1,\, \lambda n}}{\prod_{\substack{\lambda=1 \\ \lambda \neq l}}^{n-1} b_{ln-1,\, \lambda n-1}} \right]^{\frac{1}{2}} > 0, \quad 1 \leq l < n, \tag{7.2}$$

where

$$b_{\mu\nu,\, \mu'\nu'} \equiv b_{\mu\nu} - b_{\mu'\nu'} + \mu' - \mu ,$$

which fulfil, as seen from the inequalities (2.4),

$$b_{\mu_n n,\, \mu_{n-1} n-1} \gtreqless 0 \text{ for } \mu_{n-1} \gtreqless \mu_n . \tag{7.3}$$

Formula (7.2) coincides with the result obtained in Refs. 2) and 3), including the phase-factor according to the phase-convention by Gelfand and Zetlin[2] for the matrix-elements of the generators which is

$$\left\langle b'_{\mu\nu} \left| C^{n}_{n-1} \right| b_{\mu\nu} \right\rangle \geq 0 , \tag{7.4}$$

so we see that our phase-convention (6.11) for the normalization-coefficients agrees with the convention (7.4) .

8. THE MATRIX-ELEMENTS OF C_m^n

We shall now show by induction in m that the following formula for the general matrix-element of C_m^n is valid

$$\left\langle b_{\mu\nu} + \delta_{\mu l_\nu} \sum_{\nu'=m}^{n-1} \delta_{\nu\nu'} \left| C_m^n \right| b_{\mu\nu} \right\rangle$$

$$= \prod_{\lambda=m+1}^{n-1} S(l_{\lambda-1} - l_\lambda) \left[b_{l_\lambda\lambda, l_{\lambda-1}\lambda-1} \left(b_{l_\lambda\lambda, l_{\lambda-1}\lambda-1} + 1 \right) \right]^{-\frac{1}{2}}$$

$$\times \prod_{\lambda=m+1}^{n} \left\langle b_{\mu\nu} + \delta_{\mu l\nu}\, \delta_{\nu\lambda-1} \left| C_{\lambda-1}^{\lambda} \right| b_{\mu\nu} \right\rangle$$

$$= \prod_{\lambda=m+1}^{n-1} S(l_{\lambda-1} - l_\lambda) \left[b_{l_\lambda\lambda, l_{\lambda-1}\lambda-1} \left(b_{l_\lambda\lambda, l_{\lambda-1}\lambda-1} + 1 \right) \right]^{-\frac{1}{2}}$$

$$\times \prod_{\lambda=m}^{n-1} \left[- \frac{\prod_{\varkappa=1}^{\lambda-1} \left(b_{l_\lambda\lambda, \varkappa\lambda-1} + 1 \right)}{\prod_{\substack{\varkappa=1 \\ \varkappa \neq l_\lambda}}^{\lambda} \left(b_{l_\lambda\lambda, \varkappa\lambda} + 1 \right)} \; \frac{\prod_{\varkappa=1}^{\lambda+1} b_{l_\lambda\lambda, \varkappa\lambda+1}}{\prod_{\substack{\varkappa=1 \\ \varkappa \neq l_\lambda}}^{\lambda} b_{l_\lambda\lambda, \varkappa\lambda}} \right]^{\frac{1}{2}}, \quad 1 \leq l_\lambda \leq \lambda, \qquad (8.1'')$$

where use has been made of (7.2) and

$$S(x) \equiv \begin{cases} 1 & \text{for } x \geq 0 , \\ 1 & \text{for } x < 0 \end{cases}$$

and where for $m = n-1$ the product over λ from n to $n-1$ is defined as unity.

We first note that formula (8.1") indeed is valid for $m = n-1$, as it becomes an identity. We shall now prove that if it is valid for m then it is also valid for $m-1$.

Using the commutation-relations (2.2) we obtain

$$\left\langle h_{\mu\nu} + \delta_{\mu l_\nu} \sum_{\nu'=m-1}^{n-1} \delta_{\nu\nu'} \,\middle|\, C^n_{m-1} \,\middle|\, h_{\mu\nu} \right\rangle$$

$$= \left\langle h_{\mu\nu} + \delta_{\mu l_\nu} (\delta_{\nu m-1} + \delta_{\nu m}) \,\middle|\, C^m_{m-1} \,\middle|\, h_{\mu\nu} + \delta_{\mu l_m} \delta_{\nu m} \right\rangle$$

$$\times \left\langle h_{\mu\nu} + \delta_{\mu l_\nu} \sum_{\nu'=m}^{n-1} \delta_{\nu\nu'} \,\middle|\, C^n_m \,\middle|\, h_{\mu\nu} \right\rangle$$

$$- \left\langle h_{\mu\nu} + \delta_{\mu l_\nu} \sum_{\nu'=m-1}^{n-1} \delta_{\nu\nu'} \,\middle|\, C^n_m \,\middle|\, h_{\mu\nu} + \delta_{\mu l_{m-1}} \delta_{\nu m-1} \right\rangle$$

$$\times \left\langle h_{\mu\nu} + \delta_{\mu l_{m-1}} \delta_{\nu m-1} \,\middle|\, C^m_{m-1} \,\middle|\, h_{\mu\nu} \right\rangle$$

where we have used that in the summation over all intermediate basis-vectors only one term survives.

Now, using Eq. (8.1") to express those matrix-elements whose kets are different from $b_{\mu\nu}$ in terms of matrix-elements whose kets are equal to $b_{\mu\nu}$ one obtains also using (7.2), (7.3) and again (8.1"),

$$\left\langle b_{\mu\nu} + \delta_{\mu l_\nu} \sum_{\nu'=m-1}^{n-1} \delta_{\nu\nu'} \left| C^n_{m-1} \right| b_{\mu\nu} \right\rangle$$

$$= \left\{ \frac{\left\langle b_{\mu\nu} + \delta_{\mu l_\nu} (\delta_{\nu m-1} + \delta_{\nu m}) \left| C^m_{m-1} \right| b_{\mu\nu} + \delta_{\mu l_m} \delta_{\nu m} \right\rangle}{\left\langle b_{\mu\nu} + \delta_{\mu l_{m-1}} \delta_{\nu m-1} \left| C^m_{m-1} \right| b_{\mu\nu} \right\rangle} \right.$$

$$\left. - \frac{\left\langle b_{\mu\nu} + \delta_{\mu l_\nu} (\delta_{\nu m-1} + \delta_{\nu m}) \left| C^{m+1}_m \right| b_{\mu\nu} + \delta_{\mu l_{m-1}} \delta_{\nu m-1} \right\rangle}{\left\langle b_{\mu\nu} + \delta_{\mu l_m} \delta_{\nu m} \left| C^{m+1}_m \right| b_{\mu\nu} \right\rangle} \right\}$$

$$\times \left\langle b_{\mu\nu} + \delta_{\mu l_{m-1}} \delta_{\nu m-1} \left| C^m_{m-1} \right| b_{\mu\nu} \right\rangle \left\langle b_{\mu\nu} + \delta_{\mu l_\nu} \sum_{\nu'=m}^{n-1} \delta_{\nu\nu'} \left| C^n_m \right| b_{\mu\nu} \right\rangle$$

$$= \left\{ \left[\frac{b_{l_{m-1}\, m-1,\, l_m m} - 1}{b_{l_{m-1}\, m-1,\, l_m m}} \right]^{\frac{1}{2}} - \left[\frac{b_{l_m m,\, l_{m-1}\, m-1}}{b_{l_m m,\, l_{m-1}\, m-1} + 1} \right]^{\frac{1}{2}} \right\}$$

$$\times \left\langle b_{\mu\nu} + \delta_{\mu l_{m-1}} \delta_{\nu m-1} \left| C^m_{m-1} \right| b_{\mu\nu} \right\rangle \left\langle b_{\mu\nu} + \delta_{\mu l_\nu} \sum_{\nu'=m}^{n-1} \delta_{\nu\nu'} \left| C^n_m \right| b_{\mu\nu} \right\rangle$$

$$= S(l_{m-1} - l_m)\left[b_{l_m m,\, l_{m-1}\, m-1}(b_{l_m m,\, l_{m-1}\, m-1} + 1)\right]^{-\frac{1}{2}}$$

$$\times \langle b_{\mu\nu} + \delta_{\mu l_{m-1}}\delta_{\nu m-1} \,|\, C^{m}_{m-1} \,|\, b_{\mu\nu}\rangle \quad \langle b_{\mu\nu} + \delta_{\mu l_\nu}\sum_{\nu'=m}^{n-1}\delta_{\nu\nu'} \,|\, C^{n}_{m} \,|\, b_{\mu\nu}\rangle$$

$$= \prod_{\lambda=m}^{n-1} S(l_{\lambda-1} - l_\lambda)\left[b_{l_\lambda \lambda,\, l_{\lambda-1}\, \lambda-1}(b_{l_\lambda \lambda,\, l_{\lambda-1}\, \lambda-1} + 1)\right]^{-\frac{1}{2}}$$

$$\times \prod_{\lambda=m}^{n} \langle b_{\mu\nu} + \delta_{\mu\nu}\delta_{\nu\lambda-1} \,|\, C^{\lambda}_{\lambda-1} \,|\, b_{\mu\nu}\rangle \qquad ,$$

which is again of the form (8.1") and which then proves formula (8.1").

As a special case of formula (8.1") we have

$$\left\langle \begin{matrix} b_\mu \\ \\ q_\mu + \delta_{\mu l} \end{matrix} \left| C^{n}_{m} \right| \begin{matrix} b_\mu \\ \\ q_\mu \end{matrix} \right\rangle = \left[- \frac{\prod\limits_{\lambda=1}^{n}(q_l - b_\lambda + \lambda - l)}{\prod\limits_{\substack{\lambda=m \\ \lambda\neq l}}^{n-1}(q_{l\lambda} + 1)\prod\limits_{\substack{\lambda=1 \\ \lambda\neq l}}^{m} q_{l\lambda}} \right]^{\frac{1}{2}} > 0, \; 1 \leq l \leq m \, .$$

(8.2)

From formula (8.1") we obtain, for the general matrix-elements of C^{m}_{n}, using Eq. (2.1),

$$\langle b_{\mu\nu} - \delta_{\mu l_\nu}\sum_{\nu'=m}^{n-1}\delta_{\nu\nu'} \,|\, C^{m}_{n} \,|\, b_{\mu\nu}\rangle$$

$$= \prod_{\lambda=m+1}^{n-1} S(l_{\lambda-1}-l_\lambda)\left[b_{l_\lambda\lambda,\, l_{\lambda-1}\,\lambda-1}(b_{l_\lambda\lambda,\, l_{\lambda-1}\,\lambda-1}+1)\right]^{-\frac{1}{2}}$$

$$\times \prod_{\lambda=m}^{n-1}\left[-\frac{\prod_{\varkappa=1}^{\lambda-1} b_{l_\lambda\lambda,\,\varkappa\lambda-1}}{\prod_{\substack{\varkappa=1\\ \varkappa\neq l_\lambda}}^{\lambda} b_{l_\lambda\lambda,\,\varkappa\lambda}}\cdot\frac{\prod_{\varkappa=1}^{\lambda+1}(b_{l_\lambda\lambda,\,\varkappa\lambda+1}-1)}{\prod_{\substack{\varkappa=1\\ \varkappa\neq l_\lambda}}^{\lambda}(b_{l_\lambda\lambda,\,\varkappa\lambda}-1)}\right]^{\frac{1}{2}}, \quad 1\leq l_\lambda\leq\lambda\,. \tag{8.1'}$$

It should be noted that the matrix-elements of the generators (8.1) have in this work been obtained in a purely abstract way using only the commutation-relations of the generators (2.2), contrary to Baird and Biedenharn[3] who, besides the commutation-relations, also used explicit expressions[4] for the basis-vectors.

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