# Hermitian hamiltonians and imaginary eigenvalues beyond the Hilbert space

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In a quantum system initially in the $n$-th eigenstate, an adiabatic evolution of the Hamiltonian ensures that the system remains in the corresponding instantaneous eigenstate while acquiring a phase factor. This phase has two components: one resulting from standard time evolution and another associated with the dependence of the eigenstate on the varying Hamiltonian, known as the Berry phase. In this work, we explore the concept of geometric amplitudes in the context of a Hermitian Hamiltonian. We introduce the notion of geometric amplitude and provide a novel derivation of this concept. Our study reveals that a system undergoing cyclic evolution under adiabatic conditions acquires an additional amplitude factor of purely geometric origin. To illustrate this idea, we apply it to a concrete case: a generalized inverted harmonic oscillator. Although a pseudo-inner product can be introduced to make resonance states formally normalizable, this procedure relies on a non-unitary metric operator and defines a modified Hilbert space; it does not restore normalizability nor self-adjointness in the standard $L^2(R)$ framework.

In memory of my father, Maamache Leulmi-Amar, my mother, and my uncle, Djabou Zoulikha and Djabou Amar.

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## 1. Introduction

Quantum mechanics (QM), the fundamental theory governing microscopic phenomena, is built upon a rigorous mathematical framework that ensures its consistency and wide applicability across physics. One of its central objectives is the prediction of time evolution in Hamiltonian systems, a task essential for understanding a vast range of physical processes.

A cornerstone of the standard formulation of QM is the requirement that observables be represented by self-adjoint operators on a Hilbert space. This condition guarantees the reality of the spectrum and the unitarity of time evolution in closed systems [1]. However, it is important to emphasize that these properties are rigorously ensured only when the operator is truly self-adjoint on an appropriate domain [2, 3]. Formal Hermiticity alone is not sufficient.

The inverted harmonic oscillator provides a paradigmatic illustration of this subtlety [4–8]. Although its Hamiltonian is formally Hermitian, the associated potential is unbounded from below, which drastically modifies its spectral properties compared to the standard harmonic oscillator. While the conventional oscillator possesses a discrete, non-degenerate, equally spaced spectrum with square-integrable eigenfunctions, the inverted oscillator exhibits a purely continuous spectrum covering the entire real axis. Its generalized eigenstates are not square-integrable and therefore lie outside the Hilbert space $L^2(R)$.

In such contexts, the spectral problem must be formulated beyond the usual Hilbert space framework, typically within a rigged Hilbert space or resonance theory. Upon analytic continuation of the frequency parameter $\omega \to i\omega$, one formally obtains a discrete set of purely imaginary eigenvalues, $E_n = i\hbar\omega(n + 1/2)$.

These quantities do not correspond to stationary bound-state energies. Rather, they are associated with resonant (Gamow) states and appear as poles of the scattering matrix in the complex energy plane, encoding the intrinsic instability of the system.

This situation leads to a conceptual tension. A fundamental statement in quantum mechanics asserts that Hermitian (self-adjoint) operators have real eigenvalues [1]. Yet, in certain physically relevant extensions of the spectral problem, one encounters purely imaginary eigenvalues associated with formally Hermitian Hamiltonians. The resolution of this apparent paradox lies in the fact that these eigenvalues correspond to generalized, non-normalizable states that do not belong to the Hilbert space on which the operator is self-adjoint. Consequently, the usual expectation-value arguments ensuring spectral reality no longer apply.

If a Hermitian operator were to admit an eigenvalue

$$H\,|n\rangle = iE_n\,|n\rangle\,, \quad E_n \in \mathbb{R}, \tag{1}$$

within the Hilbert space, the expectation value $\langle\psi, H\psi\rangle$ would become purely imaginary, contradicting its guaranteed reality. The appearance of imaginary eigenvalues therefore signals that one is working outside the standard $L^2$-framework.

Despite the Hermiticity of $H$, such eigenvalues induce non-unitary time evolution. The evolution operator, $U(t) = \exp(-iHt)$ generates exponential decay or growth rather than a pure phase rotation. Accordingly, the quantum state evolves as:

$$|\Psi_n(t)\rangle = e^{E_n t}\,|n\rangle\,, \tag{2}$$

where we have set $\hbar = 1$ throughout. In this case, the norm of the state is not preserved: $\langle\Psi_n(t)\,|\Psi_n(t)\rangle \neq 1$. Such evolution is non-unitary and reflects dynamical instability rather than observable stationary energies. This clearly violates unitarity, which is expected for Hermitian dynamics. To restore a consistent probabilistic interpretation, one may introduce a modified scalar product using a metric operator $\eta$ such that the inner product $\langle\Psi_n(t)|\,\eta\,|\Psi_n(t)\rangle$ remains conserved in time and generalizes conventional quantum mechanics.

We emphasize that the imaginary eigenvalues discussed here do not originate from non-Hermiticity, $PT$ symmetry, or pseudo-Hermiticity. Unlike the non-Hermitian Hamiltonians introduced by Carl M. Bender and collaborators [9, 10] in the context of $PT$-symmetric quantum mechanics, whose spectral reality arises from unbroken $PT$ symmetry, the phenomena considered here emerge from the extension of the spectral problem to generalized states. Similarly, while pseudo-Hermitian formulations—developed notably by Ali Mostafazadeh [11]—allow one to define modified inner products ensuring spectral reality in certain non-Hermitian systems, our analysis starts from formally Hermitian Hamiltonians and attributes complex eigenvalues solely to the non-normalizable nature of the associated states.

The inverted harmonic oscillator thus serves as a canonical example of a Hermitian system exhibiting continuous spectra, resonance phenomena, and dynamically unstable modes. It plays an important role in diverse areas such as quantum instability, tunneling, scattering theory, and quantum chaos.

The situation becomes even more intriguing when the Hamiltonian is time-dependent. The adiabatic theorem—originally formulated by Paul Ehrenfest [12]—states that a system governed by a slowly varying Hamiltonian remains in its instantaneous eigenstate up to a phase factor, provided suitable spectral conditions are satisfied. The Hermiticity of the Hamiltonian is crucial in the standard proof of this theorem, as it ensures real instantaneous eigenvalues and orthonormal eigenstates.

In 1984, Michael Berry [13] demonstrated that adiabatic cyclic evolution leads to an additional geometric phase factor, now known as the Berry phase. This phase has profound geometrical and topological significance and has been observed in numerous physical systems. However, all standard formulations of the adiabatic theorem and geometric phase assume that the instantaneous eigenvalues are real. This naturally leads to the central question addressed in the present work. What becomes of adiabatic evolution and geometric phases when the Hamiltonian admits purely imaginary eigenvalues associated with generalized states?

The purpose of this paper is to clarify the mathematical status and physical interpretation of imaginary eigenvalues arising in formally Hermitian systems, to distinguish them from genuinely non-Hermitian spectral phenomena, and to analyze their implications for dynamical stability and adiabatic quantum evolution.

As mentioned, a fundamental fact in quantum mechanics is that Hermitian (or self-adjoint) operators have only real eigenvalues. However, one may encounter Hermitian Hamiltonians with imaginary eigenvalues.[i] Such cases involving imaginary eigenvalues have not been widely explored in the literature and will be one of the key topics discussed in this paper.

Before proceeding with the discussion, let us first recall a representative example that illustrates this theme: the inverted harmonic oscillator, characterized by the Hamiltonian

$$H = \frac{1}{2}\left[\frac{p^2}{m} - m\omega^2 q^2\right], \tag{3}$$

$\omega > 0$, $m > 0$ are real constants. This is the same form as the standard harmonic oscillator, but with a negative sign in front of the potential term. It represents an unstable (repulsive) potential: $-(1/2)m\omega^2 q^2$.

Mathematically, the generalized harmonic oscillator and its inverted counterpart exhibit closely related structures, allowing the solutions of one system to be derived almost directly from those of the other through appropriate transformations. This formal similarity, however, masks profound physical differences. The harmonic oscillator displays a discrete, non-degenerate, and equally spaced energy spectrum with normalizable eigenfunctions in the standard Hilbert space. By contrast, the inverted harmonic oscillator has a doubly degenerate continuous spectrum ranging from $-\infty \to +\infty$, and its eigenfunctions are not normalizable in the conventional sense. Nevertheless, the introduction of a nontrivial metric allows one to endow the state space with a well-defined pseudo-inner product, thereby providing a generalized notion of normalizability for the wave functions. We finally emphasize that the eigenvalues associated with the inverted harmonic oscillator are purely imaginary, reflecting the intrinsically unstable nature of this system.

Let's first look at the classical dynamics to understand the appearance of imaginary frequencies. The equation of motion associated with the Hamiltonian (3) is:

$$\ddot{q} = \omega^2 q \;\Rightarrow q(t) = Ae^{\omega t} + Be^{-\omega t}. \tag{4}$$

The classical solution is exponential, not oscillatory. This is equivalent to saying that the system has a purely imaginary frequency: $\omega \to \pm i\omega$. This is a hallmark of dynamical instability: the system does not return to equilibrium but instead undergoes unbounded growth or decay.

On the quantum level, solving the time-independent Schrödinger equation leads to solutions governed by the parabolic cylinder equation. The resulting wavefunctions are expressed in terms of Weber functions (or modified Airy functions at asymptotic limits). However, due to the inverted potential, these solutions grow exponentially at spatial infinity and are not square-integrable, the spectrum is continuous, not discrete.

This example challenges the standard assumption that Hermiticity guarantees unitary evolution. It also highlights the need for generalized mathematical frameworks, such as rigged Hilbert spaces or effective non-Hermitian formulations, even when starting from a formally Hermitian Hamiltonian.

When the Hamiltonian becomes explicitly time-dependent, the situation becomes even more intricate. In such cases, approximation techniques become essential. One particularly powerful method is the adiabatic theorem [12]. It applies to quantum systems whose Hamiltonian $H(t)$ varies slowly with time and asserts that the system remains in its instantaneous eigenstate up to a phase factor, provided the spectral gap condition is satisfied.

The Hermiticity of the Hamiltonian plays a crucial role in the validity of the adiabatic theorem, as shown in various foundational studies [14–17]. In the following, we build on this concept by introducing Berry's adiabatic phase [13] and the associated notion of geometric phase, which provides further insight into the topological and geometric aspects of quantum evolution in the presence of slow parameter variations. This phenomenon can be described by "global change without local change". Berry points out the geometrical character of this phase which is not negligible because of its non-integrable character. Since this much work has been done to this issue and the so-called Berry phase is now well established, theoretically as well as experimentally.

The adiabatic theorem, a fundamental result in quantum theory, states that for a Hamiltonian $H(t)$ that varies slowly over time, the solution of the time-dependent Schrödinger equation can be approximated using the eigenstates $|n(t)\rangle$ of the instantaneous Hamiltonian

$$H(t)\,|n(t)\rangle = E_n(t)\,|n(t)\rangle\,, \tag{5}$$

with energies $E_n(t)$ ($n = 0, 1, \ldots$). This eigenvalue equation implies no relation between the phases of the eigenstates $|n(t)\rangle$ at different $t$.

The adiabatic theorem concerns states $|\Psi_n(t)\rangle$ satisfying the time-dependent Schrödinger equation

$$i\frac{\partial}{\partial t}\,|\Psi_n(t)\rangle = H\,|\Psi_n(t)\rangle\,, \tag{6}$$

and asserts that if a quantum system with a time-dependent non-degenerate Hamiltonian $H\,(t)$ is initially in the $n$th eigenstate $|n(0)\rangle$ of $H\,(0)$, and if $H\,(t)$ evolves slowly enough, then the state at time $t$ remains in the $n$th instantaneous eigenstate of $H\,(t)$ up to a multiplicative phase factor.

This phase factor

$$\varphi_n\,(t) = -\int_0^t dt' \langle n(t')|H(t') - i\hbar\frac{\partial}{\partial t'}|n(t')\rangle \tag{7}$$

consists of two contributions: a dynamical phase and a geometric phase. Consequently, the quantum state $|\Psi_n(t)\rangle$ can be expressed as:

$$|\Psi_n(t)\rangle = e^{i\int_0^t dt' \left\langle n(t')\middle| i\frac{\partial}{\partial t'} - H(t')\middle| n(t')\right\rangle}\,|n(t)\rangle\,. \tag{8}$$

The geometric phase manifests itself in a wide variety of physical phenomena, such as the evolution of light polarization in optical fibers and the precession of neutrons subjected to magnetic fields. Furthermore, the concept of geometric phase has been extended to systems with continuous spectra [20, 21] as well as to non-Hermitian Hamiltonians [22–29].

Up to this point, all discussed cases assume that the energy eigenvalues are real, either discrete or continuous. A natural and intriguing question then arises: What happens when the eigenvalues of the Hamiltonian are imaginary?

The aim of this work is fourfold. First, we seek to clarify the mathematical status of imaginary eigenvalues in operators that are formally Hermitian, with particular attention to issues of self-adjointness and domain. Second, we aim to distinguish such situations from genuinely non-Hermitian frameworks, including those exhibiting PT symmetry. Third, we analyze the implications of imaginary eigenvalues for the time evolution of quantum states, especially regarding the breakdown of unitarity. Finally, we investigate how the notion of geometric phase, in particular the Berry phase, is modified in this nonstandard context.

## 2. Solution of the Schrödinger equation

We consider a quantum system described by a Hamiltonian $H(\vec{R}(t))$ that depends on a real time dependent parameter $\vec{R}(t)$ which parametrizes the environment of the system and such that $\vec{R}(T) = \vec{R}(0)$ . The time evolution is described by the time-dependent Schrödinger equation (6). At any instant, the natural basis consists of the eigenstates $|n(\vec{R}(t))\rangle$ associated with imaginary energies $iE_n(\vec{R}(t))$, defined by the eigenvalue equation

$$H(\vec{R}(t))|n(\vec{R}(t))\rangle = iE_n(\vec{R}(t))|n(\vec{R}(t))\rangle. \tag{9}$$

Assume that the system is initially prepared in the $n$-th eigenstate of the Hamiltonian, *i.e.*, $|\Psi_n(\vec{R}(0))\rangle = |n(\vec{R}(0))\rangle$. The adiabatic theorem states that if the Hamiltonian $H(\vec{R}(t))$ evolves slowly (adiabatically), the system remains in the corresponding instantaneous eigenstate $|n(\vec{R}(t))\rangle$ throughout the evolution. In other words, no transitions occur to other eigenstates, although the eigenenergy itself may change over time.

Under this adiabatic assumption, we consider the ansatz:

$$|\Psi_n(t)\rangle = C_n(t)|n(\vec{R}(t))\rangle, \tag{10}$$

where $C_n(t)$ is a time-dependent prefactor.

The eigenfunctions of the Hamiltonian $H(\vec{R}(t))$ do not generally belong to $\mathrm{L}^2$ and, as a result, do not form a complete set of orthonormal functions $\langle m(\vec{R}(t)|n(\vec{R}(t)\rangle \neq \delta_{mn}$. A key question arising from the current interest in Hermitian quantum mechanics with imaginary eigenvalues is: What are the necessary and sufficient conditions for obtaining normalizable solutions? To address this, we propose introducing a metric $\eta(t)$, inspired by the pseudo-Hermitian framework, to define a scalar product that guarantees a well-defined and positive norm,

$$\left\langle m(\vec{R}(t))\middle|\eta(t)\middle|n(\vec{R}(t))\right\rangle = \delta_{mn}. \tag{11}$$

Substituting the expression (10) into the time-dependent Schrödinger equation and projecting the result onto the eigenstate weighted by the time-dependent metric $\eta(t)$: *i.e.*, $\langle n(\vec{R}(t)|\eta(t)$, yields a differential equation that governs the evolution of $C_n(t)$.

As a result, the time-evolved wave function takes the form:

$$|\Psi_n(t)\rangle = e^{\int_0^t E_n(t)dti} e^{\gamma_n(t)} \big| n(\vec{R}(t)) \big\rangle. \qquad (12)$$

This expression includes both a dynamical term (from the imaginary eigenvalue $iE_n(t)$ and a geometric contribution arising from the nontrivial time dependence of the eigenbasis and the metric $\eta(t)$

$$\gamma_n(t) = \int_0^t \left\langle n(\vec{R}(t')) \right| \eta(t') \frac{\partial}{\partial t'} \left| n\left(t'\right) \right\rangle dt'. \qquad (13)$$

The total amplitude factor can thus be written as:

$$\phi = \underbrace{e^{\int_0^t E_n(t)dti}}_{\text{dynamical amplitude}} + \underbrace{e^{\gamma_n(t)}}_{\text{Geometrical amplitude}}. \qquad (14)$$

The geometric term $\gamma_n(t)$, which depends only on the path followed in Hilbert space, is analogous to the Berry phase encountered in Hermitian systems under cyclic adiabatic evolution. Meanwhile, the first exponential captures the familiar dynamical contribution.

The orthonormality relation in Eq. (11) does not directly extend to the time-evolved states $|\Psi_n(t)\rangle$, due to the time-dependent prefactor. To restore a consistent normalization framework, we define a rescaled metric: $\widetilde{\eta}(t) = \eta(t)e^{-2\int_0^t E_n(t)dt} e^{-2\gamma_n(t)}$. With this definition, the time-evolved states remain orthonormal in the modified inner product:

$$\langle \Psi_m(t) | \, \widetilde{\eta}(t) \, | \Psi_n(t) \rangle = \delta_{mn}. \qquad (15)$$

## 3. Example: inverted generalized harmonic oscillator

Consider the quantum system defined by the following Hamiltonian:

$$H(t) = \frac{1}{2}\left[Z\left(t\right)p^2 + Y\left(t\right)\left(pq + qp\right) + X\left(t\right)q^2\right]. \qquad (16)$$

usually called a generalized harmonic oscillator [30] which depends on the set of external parameters $(X, Y, Z) \in \mathbb{R}^3$. For fixed values of the parameters, $H$ corresponds to the Hamiltonian of the harmonic oscillator with purely imaginary frequency provided $XZ \langle \, Y^2$ and we assume henceforth that this remains the case as $X, Y, Z$ vary.

The Hamiltonian of the inverted generalized harmonic oscillator is defined by Eq. (16). In contrast to the standard harmonic oscillator, its spectral properties are markedly different due to the quadratic potential being unbounded from below. As a consequence, the system exhibits several distinctive features. First, its energy spectrum is continuous and extends over the entire real line $(E \in R)$. In particular, the Hamiltonian does not admit a ground state, since the potential diverges to $-\infty$ as $\mid q \mid \to \infty$.

The generalized eigenstates associated with this continuous spectrum are not square-integrable and therefore do not belong to the Hilbert space $L^2(R)$. Instead, they are understood as scattering states within an extended framework, such as the rigged Hilbert space formalism. Each energy value is, in general, doubly degenerate, a degeneracy that has a clear physical interpretation in terms of scattering solutions corresponding to left- and right-moving waves in the repulsive quadratic potential.

If one formally performs an analytic continuation of the frequency parameter ($\omega \to i\omega$), one obtains a discrete set of purely imaginary energy values of the form $E_n = i(n + [1/2])\omega$. However, these quantities do not represent physical energies of stationary bound states. Rather, they are associated with resonant (Gamow) states and correspond to poles of the scattering matrix in the complex energy plane, thereby encoding the intrinsic instability and decay properties of the system.

The inverted generalized harmonic oscillator therefore provides a paradigmatic example of a quantum system characterized by a continuous spectrum, non-normalizable eigenstates, and resonance phenomena. For this reason, it plays an important role in the study of quantum instability, tunneling processes, and various aspects of quantum chaos.

The eigenvalue equation takes the following form

$$\left[-\frac{Z}{2}\frac{\partial^2}{\partial q^2} - iYq\frac{\partial}{\partial q} + \left(\frac{Xq^2 - iY}{2}\right)\right]\psi_n = E_n\psi_n, \quad (17)$$

To determine the eigenfunctions $\psi_n$ and the eigenvalues, we introduce the following unitary transformation

$$\psi_n^{\mathrm{r}}(q) = U\psi_n(q) = \exp\left[\frac{iY}{2Z}q^2\right]\psi_n(q), \qquad (18)$$

(r indicates reversed) such as

$$U^\dagger q U = q, \qquad U^\dagger p U = p - \frac{Y}{Z}q\,, \qquad (19)$$

and let us note that the transformed Hamiltonian $H^{\mathrm{r}}$

$$H^{\mathrm{r}} = U^\dagger H U = \frac{1}{2}\left[Zp^2 + \frac{(XZ - Y^2)}{Z}q^2\right], \qquad (20)$$

corresponds to the Hamiltonian of the inverted harmonic oscillator with purely imaginary frequency $i\sqrt{(Y^2 - XZ)} = i\omega$.

Therefore, this replacement transforms the eigenfunctions $\psi_n$ into generalized eigenvectors $\psi_n^{\mathrm{r}}$ of the inverted harmonic oscillator (20)

$$\begin{aligned}\psi_n^{\mathrm{r}}(q) &= \left(\frac{i\omega m}{\pi}\right)^{\frac{1}{4}} \exp\left[-\frac{i}{2}\frac{Y}{Z}q^2\right]\chi_n\left[\left(\frac{i\omega}{2}\right)^{\frac{1}{2}}q\right] \\ &= \exp\left[-\frac{i}{2}\frac{Y}{Z}q^2\right]V^{-1}\left\{\left(\frac{\omega m}{\pi}\right)^{\frac{1}{4}}\chi_n\left[\left(\frac{\omega}{2}\right)^{\frac{1}{2}}q\right]\right\}, \end{aligned} \quad (21)$$

where the operator $V = \exp\left[-(\pi/8)(qp + pq)\right]$ changes the coordinate $q$ and momentum $p$ operators as

$$\exp\left[\frac{\pi}{8}(qp+pq)\right] q \exp\left[-\frac{\pi}{8}(qp+pq)\right] = qe^{-i\frac{\pi}{4}},$$
$$\exp\left[\frac{\pi}{8}(qp+pq)\right] p \exp\left[-\frac{\pi}{8}(qp+pq)\right] = pe^{i\frac{\pi}{4}}, \quad (22)$$

and its action on a wave function in the $q$ representation reads $Vf(q) = e^{i\frac{\pi}{8}} f(qe^{i\frac{\pi}{4}})$. The so-called $n$th Hermite function $\chi_n(q)$ satisfies the following equation:

$$\frac{\partial^2 \chi_n}{\partial q^2} + 2\left(n + 1 - q^2\right)\chi_n(x) = 0, \quad (23)$$

and the corresponding energies are exactly given by

$$E_n = \left(n + \frac{1}{2}\right)\omega = i\left(n + \frac{1}{2}\right)\sqrt{(Y^2 - XZ)}. \quad (24)$$

It is straightforward to verify that the norm associated with the solution (21) is not well-defined. Specifically, when calculating the squared norm of these functions, we observe that $\langle \psi_n^{\mathrm{r}} | \psi_n^{\mathrm{r}} \rangle \neq 1$. This suggest introducing the metric operator

$$\eta = \exp\left[-\frac{i}{2\hbar}\frac{Y}{Z}q^2\right]$$
$$\times \exp\left[\frac{\pi}{4}(xp + px)\right] \exp\left[\frac{i}{2\hbar}\frac{Y}{Z}q^2\right], \quad (25)$$

such that the squared norm $\langle \psi_n^{\mathrm{r}} | \eta | \psi_n^{\mathrm{r}} \rangle = 1$ is finite. Note that the metric operator is not unique, in the sense that infinitely many operators can possibly be implemented.

Inserting the wave function (18) and the metric (25) into the formula (13) defining the geometric quantity, one finds

$$\gamma_n(C) = -\left(n + \frac{1}{2}\right)\oint_C \frac{Z}{\omega} d\left(\frac{Y}{Z}\right), \quad (26)$$

where the following property of the Hermite functions $\int_{-\infty}^{+\infty} \zeta^2 \chi_n^2[\zeta]\, d\zeta = (n + [1/2])$,has been used. Thus $\gamma_n(C)$ given by a circuit integral in parameter space is imaginary. Thus the time-evolved wave can be written as:

$$|\Psi_n(q,t)\rangle = e^{\int_0^t E_n(t)dti} e^{\gamma_n(t)} \exp\left[-\frac{i}{2}\frac{Y}{Z}q^2\right]$$
$$\times V^{-1}\left\{\left(\frac{\omega m}{\pi}\right)^{\frac{1}{4}} \chi_n\left[\left(\frac{\omega}{2}\right)^{\frac{1}{2}} q\right]\right\}, \quad (27)$$

the first exponential is the dynamical amplitude and the second exponential (the geometrical amplitude) is the object of attention in this paper.

## 4. Conclusion

In quantum mechanics, the reality of the eigenspectrum plays a central role, as it guarantees the physical observability and stability of measurable quantities such as energy, momentum, and angular momentum. Within the conventional framework, this property follows from the Hermiticity of the Hamiltonian operator, which ensures a real spectrum and unitary time evolution. Real eigenvalues therefore represent physically meaningful states and form a fundamental basis for the predictive consistency of quantum theory.

Recent developments in non-Hermitian quantum mechanics have considerably broadened this perspective. It is now well established that certain classes of non-Hermitian operators, including those possessing $PT$ symmetry or belonging to the class of pseudo-Hermitian Hamiltonians, may still exhibit entirely real spectra under appropriate conditions. It is not generally true that all PT-symmetric Hamiltonians possess entirely real eigenvalue spectra; this holds only when the PT symmetry is unbroken. When PT symmetry is spontaneously broken, complex conjugate eigenvalue pairs can emerge. Conversely, the presence of a real spectrum does not imply that a Hamiltonian is PT-symmetric, as there exist non-PT-symmetric operators with entirely real eigenvalues. Therefore, PT symmetry is neither a necessary nor a sufficient condition for spectral reality. These extensions provide powerful theoretical tools for describing open quantum systems, dissipative dynamics, and gain–loss mechanisms, while maintaining a consistent probabilistic interpretation through generalized inner-product structures.

Within this context, the adiabatic theorem remains a fundamental principle governing the evolution of quantum states under slowly varying Hamiltonians. When a system initially prepared in an instantaneous eigenstate evolves adiabatically along a closed path in parameter space, the state acquires not only the usual dynamical phase but also an additional geometric contribution known as the Berry phase. This geometric phase depends exclusively on the geometry of the path traced in parameter space and has profound implications across many areas of physics.

In the present work, we revisited the geometric aspects of adiabatic evolution and explored their extension beyond the conventional phase structure. In particular, we introduced the notion of a geometric amplitude, which generalizes the concept of geometric phase in situations where the spectral or normalization properties of the eigenstates differ from those encountered in standard Hermitian systems. We have shown that, under cyclic and adiabatic evolution, the quantum state may acquire an additional amplitude factor of purely geometric origin. This contribution arises from the structure of the parameter space and from the normalization properties of the instantaneous eigenstates, and is therefore distinct from the dynamical evolution generated by the Hamiltonian.

To illustrate this mechanism, we applied the formalism to a generalized harmonic oscillator with time-dependent parameters and an effective negative frequency. This model provides a useful framework for investigating unstable quantum dynamics. In particular, it exhibits features such as continuous spectra, non-normalizable eigenstates, and resonance-like behavior, which are commonly encountered in problems involving quantum instability, tunneling processes, and aspects of quantum chaos.

Although the generalized harmonic oscillator and the inverted harmonic oscillator are mathematically related through suitable transformations, their physical interpretations remain fundamentally different. The ordinary harmonic oscillator possesses a discrete, non-degenerate, and equally spaced energy spectrum with square-integrable eigenfunctions forming a complete orthonormal basis in Hilbert space. By contrast, the inverted harmonic oscillator is characterized by a continuous spectrum extending from $-\infty$ to $+\infty$, and its eigenfunctions are not square-integrable in the conventional sense.

This apparent difficulty can be consistently addressed within extended frameworks such as rigged Hilbert spaces or generalized inner-product structures, which allow the proper treatment of non-normalizable states, including scattering and resonance states. Within this broader setting, the unstable nature of the inverted oscillator manifests through exponentially growing or decaying solutions, reflecting the intrinsic dynamical instability of the potential rather than a breakdown of the quantum formalism.

In summary, the present work highlights the rich geometric structure underlying adiabatic quantum evolution in systems that depart from the standard Hermitian paradigm. In particular, the introduction of the geometric amplitude extends the conventional framework of geometric effects in quantum mechanics and provides new insight into the role of geometry in the dynamics of unstable or non-standard quantum systems. Our main result shows that adiabatic cyclic evolution can generate not only a geometric phase but also a geometric amplitude, thereby revealing a deeper geometric structure in quantum dynamics beyond the traditional Berry-phase description.

I dedicate this work to my grandchildren Mohamed and Racim Chebouti, Taim-Abderahmane Maamache and Sadjed-Yazen Bendib

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$i$. A simple example may illustrate the issue: consider quantum mechanics on the real line and the quantum mechanical operator $O = q^3p + pq^3$ . This operator is hermitian (or symmetric) but it still has an eigenvalue $(-i\lambda)$ with a square-integrable eigenfunction $\chi_\lambda(q) = 1/|q|^{3/2}\mathrm{e}^{-\lambda\ /4q^2}$.


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