THE THREE BODY PROBLEM AND THE SU_4 GROUP*

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ABSTRACT

The classification of states of three particles produced in a single event has been a subject of much discussion in recent publications[1,2,3]. *If the center of mass is taken out, the symmetry group of the Hamiltonian of three particles is the rotation group in six dimensions* R_6. *This group is isomorphic to the unitary unimodular group in four dimensions* SU_4. *Use is made of this fact for the explicit construction of the three particle states in configuration and momentum space.*

I. INTRODUCTION

In recent publications Kramer[1,2] and Dragt[3] have discussed the problem of the classification of states for three particles produced in a single event. They

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make extensive use of the fact that the symmetry group for three free particles, after elimination of the center of mass motion, is the rotation group in six dimensions R_6. In this paper we make explicit use of the isomorphism between R_6 and the unitary unimodular group in four dimensions SU_4, to construct the states of the three particle system.

Before proceeding to discuss our problem we shall review the well known equivalent problem for two particles in a way that will suggest the generalization of the procedure to the three body problem.

II. THE TWO BODY PROBLEM

Let us consider a system of two particles whose coordinates are ξ_i^α, $\alpha = 1, 2$ being the particle indice and $i = 1, 2, 3$ the vector index in three dimensional space. The corresponding moment will be designated as

$$\Pi_i^\alpha = \frac{1}{i} \frac{\partial}{\partial \xi_i^\alpha} \qquad (\hbar = 1) \tag{1}$$

The hamiltonian of this system of two particles, assuming for simplicity that their masses m are equal and choosing units such that $m = 1$, becomes

$$H = \frac{1}{2} \Pi_i^\alpha \Pi_i^\alpha \tag{2}$$

where repeated indices will be summed over their range of values. We introduce the center of mass and relative coordinates and momenta

$$X_i = \frac{1}{\sqrt{2}} (\xi_i^1 + \xi_i^2), \quad P_i = \frac{1}{\sqrt{2}} (\Pi_i^1 + \Pi_i^2)$$

$$x_i = \frac{1}{\sqrt{2}} (\xi_i^1 - \xi_i^2), \quad p_i = \frac{1}{\sqrt{2}} (\Pi_i^1 - \Pi_i^2), \tag{3}$$

and so

$$H = \frac{1}{2} (P_i P_i + p_i p_i) \tag{4}$$

We are interested in a solution in which the center of mass had a definite momentum K_i and the state of relative motion corresponds to definite energy E, angular momentum l and projection m. The solution is of course

$$\psi = \exp(iK_i X_i) r^{-\frac{1}{2}} J_{l+\frac{1}{2}} (\sqrt{2E}\, r)\ Y_{lm} (\theta, \varphi)$$

$$= \exp(iK_i X_i) r^{-l-\frac{1}{2}} J_{l+\frac{1}{2}} (\sqrt{2E}\, r)\ \mathcal{Y}_{lm} (x_i) \quad , \tag{5}$$

where the $\mathcal{Y}_{lm}(x_i)$ is a solid spherical harmonic in the x_i .

We want to derive the well known solution (5) in a way in which the groups theoretical ideas underlying the derivation become explicit so as to be able to extend the analysis to the three body system.

We first remark that invariance under translations implies that the total momentum K_i is a good quantum number and so we get rid of the plane wave and remain with the problem

$$H'\psi = \frac{1}{2} p_i p_i \psi(x_i) = -\frac{1}{2} \sum_i \frac{\partial^2}{\partial x_i^2} \psi = E\psi(x_i) \tag{6}$$

The hamiltonian (6) is clearly invariant under the group of rotations R_3 whose generators are given by the antisymmetric tensor

$$\Lambda_{ij} = (i/2)(x_i p_j - x_j p_i) \tag{7}$$

connected in a direct fashion with the components of angular momentum. Clearly then we could further specify the functions $\psi(x_i)$ by requiring that they would form a basis for an irreducible representation of R_3, which we achieve if we demand that ψ also be an eigenfunction of the Casimir operator of R_3 i.e.

$$\Phi\psi \equiv \Lambda_{ij}\Lambda_{ji}\psi = \kappa\psi \ . \tag{8}$$

The equation (8) is compatible with (6) as H^1 and Φ clearly commute.

To find the solution of (6) and (8) as well as to obtain the eigenvalue κ, let us first look into auxiliary problem of obtaining the homogeneous polynomials of degree l that satisfy the Laplace equation, i.e.

$$x_i \frac{\partial P}{\partial x_i} = lP \ , \quad \nabla^2 P = \frac{\partial^2 P}{\partial x_i \partial x_i} = 0 \tag{9a, b}$$

We shall derive these polynomials below, but first we shall show that P satisfies (8) and that with its help we can immediately determine $\psi(x_i)$. From (8) and the commutation relations of x_i, p_j we obtain

$$\Phi = -\frac{1}{4}(x_i p_j - x_j p_i)(x_j p_i - x_i p_j)$$

$$= \frac{1}{2} \{(\underline{r} \cdot \nabla)(\underline{r} \cdot \nabla + 1) - r^2 \nabla^2\} \quad , \tag{10}$$

where $\underline{r} \cdot \nabla = x_i \frac{\partial}{\partial x_i}$, $r^2 = x_i x_i$, $\nabla^2 = - p_i p_i$.

If we now apply Φ to a P that satisfies (9), we get

$$\Phi P = \frac{1}{2} \; l(l+1) \; P . \tag{11}$$

Furthermore, we have

$$\nabla^2 = \frac{1}{r^2} \; (\underline{r} \cdot \nabla)(\underline{r} \cdot \nabla + 1) - \frac{2}{r^2} \; \Phi \quad . \tag{12}$$

We propose for $\psi(x_i)$ the expression

$$\phi(x_i) = r^{-l} f(r) \; P(x_i) \tag{13}$$

where $f(r)$ is, so far, arbitrary. Substituting in (6) and using (11) and the fact that from (9a) $P(x_i)/r^l$ is independent of r, we get for $f(r)$ the equation

$$\frac{1}{r^2} \left(r \frac{d}{dr} \right) \left(r \frac{d}{dr} + 1 \right) f - \frac{l(l+1)}{r^2} \; f + 2E \; f = 0 \; , \tag{14}$$

or $f = r^{-\frac{1}{2}} J_{l+\frac{1}{2}} (\sqrt{2E}\; r)$ where J is a Bessel function.

We get then that the solution of (6) and (8) can be written as

$$\psi(x_i) = r^{-l-\frac{1}{2}} J_{l+\frac{1}{2}}(\sqrt{2E}\, r)\; P(x_i) \;, \tag{15}$$

where P satisfies (9).

As the polynomials that satisfy (9) also are eigenpolynomials of Φ, they form a basis for an irreducible representation of R_3 characterized by l. As the basis for an irreducible representation l has dimension $(2l+1)$, there are $(2l+1)$ linearly independent polynomials satisfying (9). We could further characterize these polynomials if we require that a subgroup of the group R_3 be explicitly reduced, for example the subgroup of rotations R_2 around the z axis. The polynomials is then characterized also by the equation

$$\Lambda_{12}\, P = (i/2)\; m\; P \;, \tag{9c}$$

which can be imposed on P as Λ_{12} clearly commutes with $\underline{r} \cdot \nabla$ and ∇^2. The parameter m is so far arbitrary.

To determine P we make the change of coordinates

$$x_+ = -(1/\sqrt{2})(x_1 + i x_2) \,,\;\; x_0 = x_3 \,,\;\; x_- = (1/\sqrt{2})(x_1 - i x_2) \tag{16}$$

and the equations (9) become

$$\left(x_+ \frac{\partial}{\partial x_+} + x_0 \frac{\partial}{\partial x_0} + x_- \frac{\partial}{\partial x_-} \right) P = l P \;,$$

$$\left(-2 \frac{\partial^2}{\partial x_+ \partial x_-} + \frac{\partial^2}{\partial x_0^2} \right) P = 0 \;,$$

$$\left(x_+ \frac{\partial}{\partial x_+} - x_- \frac{\partial}{\partial x_-} \right) P = m P \quad . \tag{17a,b,c}$$

The most general polynomial solution of (17) is

$$P = \sum_{n_+ n_0 n_-} A_{n_+ n_0 n_-} \; x_+^{n_+} x_0^{n_0} \; x_-^{n_-} \quad . \tag{18}$$

Equations (17a,c) require that

$$n_+ + n_0 + n_- = l \; , \quad n_+ - n_- = m \tag{19}$$

and so m must be an integer and P becomes

$$P = x_+^m \; x_0^{l-m} \quad \sum_{n_-} A_{n_-} \; (x_+ x_- / x_0^2)^{n_-} \quad . \tag{20}$$

As r^2 can be written as

$$r^2 = -2 \, x_+ x_- + x_0^2 \tag{20a}$$

we could replace $(x_+ x_- / x_0^2)$ by $\frac{1}{2} \; (1 - \frac{r^2}{x_0^2})$ and as A_{n_-} is arbitrary we could express P as

$$P = x_+^m \; x_0^{l-m} \; \sum_n \; A_n \; (r/x_0)^{2n} \tag{21}$$

If we now apply (17b) to (21), we obtain straightforwardly that A_n must obey the recurssion relation

$$\frac{A_{n+1}}{A_n} = -\frac{(l-m-2n)(l-m-2n-1)}{2(n+1)(2l-2n-1)} \tag{22}$$

We have then determined the polynomial satisfying (17) and from (22) it turns out to be, as we expect, the solid spherical harmonic

$$P(x_i) = \mathcal{Y}_{lm}(x_i) \quad . \tag{23}$$

The wave function $\psi(x_i)$ is then given in terms of a radial function $[J_{l+\frac{1}{2}}(\sqrt{2E}\, r)/r^{l+\frac{1}{2}}]$ and the polynomials that are basis for an irreducible representation l of R_3 in which the subgroup R_2 is explicitly reduced.

How does the solution (5) look in momentum space? We can write it in the form

$$\delta(\underline{P} - \underline{K})(2E)^{-\frac{1}{2}l}\ \delta(\tfrac{1}{2}p^2 - E)\ \mathcal{Y}_{lm}(p_i) \tag{24}$$

as the magnitude of p_i in the solid spherical harmonic cancels with $(2E)^{l/2}$ because of the δ function, and the Fourier transform of $\delta(\frac{1}{2}p^2 - E)\ Y_{lm}(\alpha, \beta)$, where α, β are the angles in momentum space, gives $r^{-\frac{1}{2}}\ J_{l+\frac{1}{2}}(\sqrt{2E}\, r)\ Y_{lm}(\theta, \varphi)$.

We shall now proceed to extend these results to the three body problem.

III. THE THREE BODY PROBLEM IN CONFIGURATION SPACE

We consider three particles of equal mass ($m = 1$) whose coordinates are ξ_i^α, $\alpha = 1,2,3$, $i = 1,2,3$, and whose momenta and hamiltonian are

$$\Pi_i^\alpha = \frac{1}{i} \frac{\partial}{\partial \xi_i^\alpha} \quad , \quad H = \frac{1}{2} \Pi_i^\alpha \Pi_i^\alpha \tag{25}$$

We make the transformation to center of mass and relative coordinates

$$X_i = \frac{1}{\sqrt{3}} (\xi_i^1 + \xi_i^2 + \xi_i^3), \; x_i^1 = \frac{1}{\sqrt{6}} (\xi_i^1 + \xi_i^2) - \sqrt{2/3}\, \xi_i^3, \; x_i^2 = \frac{1}{\sqrt{2}} (\xi_i^1 - \xi_i^2) \; . \tag{26}$$

The corresponding momenta are given by the usual definition

$$P_i = \frac{1}{i} \frac{\partial}{\partial X_i} \quad , \quad p_i^\alpha = \frac{1}{i} \frac{\partial}{\partial x_i^\alpha} \quad , \quad \alpha = 1,2$$

The hamiltonian is then

$$H = \frac{1}{2} \{ P_i P_i + p_i^\alpha p_i^\alpha \} \tag{27}$$

where repeated indices are summed and from now on α takes only the values 1,2.

The eigenfunction of H can be written as

$$\exp (iK_i X_i) \; \psi (x_i^\alpha) \tag{28}$$

where $\psi(x_i^\alpha)$ satisfies

$$H'\psi = \frac{1}{2} p_i^\alpha p_i^\alpha \psi(x_i^\alpha) = E\psi(x_i^\alpha) \quad . \tag{29}$$

We shall proceed to derive $\psi(x_i^\alpha)$ in analogy with the method followed for the two particle system. The hamiltonian H' is clearly invariant under orthogonal transformations in the six dimensional space of the x_i^α, the generators of this R_6 group being

$$\Lambda_{ij}^{\alpha\beta} = \frac{i}{2} (x_i^\alpha p_j^\beta - x_j^\beta p_i^\alpha) \; , \; \alpha, \beta = 1, 2, \; i, j = 1, 2, 3 \; . \tag{30}$$

Clearly then we could further specify the function $\psi(x_i^\alpha)$ by requiring that they form a basis for an irreducible representation of R_6, which we achieve if we demand that ψ should also be an eigenfunction of the Casimir operator of R_6 i.e.

$$\Phi\psi \equiv \Lambda_{ij}^{\alpha\beta} \Lambda_{ji}^{\beta\alpha} \psi = \kappa\psi \tag{31}$$

The equation (31) is compatible with (29) as Φ and H' clearly commute.

In analogy with what was done for the two body problem we can first determine the homogeneous polynomials of degree λ that satisfy the six dimensional Laplace equation

$$x_i^\alpha \frac{\partial P}{\partial x_i^\alpha} = \lambda P \; , \quad \nabla^2 P = \frac{\partial^2 P}{\partial x_i^\alpha \partial x_i^\alpha} = 0 \; . \tag{32a, b}$$

The polynomial P satisfies (31) as

$$\Phi = -\frac{1}{4}(x_i^\alpha p_j^\beta - x_j^\beta p_i^\alpha)(x_j^\beta p_i^\alpha - x_i^\alpha p_j^\beta)$$

$$= \frac{1}{2} \{(\underline{r}\cdot\nabla)(\underline{r}\cdot\nabla + 4) - r^2\nabla^2\} , \tag{33}$$

where here

$$\underline{r}\cdot\nabla \equiv x_i^\alpha \frac{\partial}{\partial x_i^\alpha} , \quad r^2 = x_i^\alpha x_i^\alpha , \quad \nabla^2 = - p_i^\alpha p_i^\alpha , \tag{34}$$

and so

$$\Phi P = \frac{1}{2}\lambda(\lambda + 4) P . \tag{35}$$

Furthermore we have that

$$\nabla^2 = \frac{1}{r^2}(\underline{r}\cdot\nabla)(\underline{r}\cdot\nabla + 4) - \frac{2}{r^2}\Phi \tag{36}$$

Proposing $\psi(x_i^\alpha)$ in the form

$$\psi(x_i^\alpha) = r^{-l} f(r) \; P(x_i^\alpha) , \tag{37}$$

we see that it is an eigenfunction of Φ as $\Lambda_{ij}^{\alpha\beta}$ commutes with r, and as from

(32 a) $P(x_i^\alpha)/r^\lambda$ is independent of r we get the equation

$$\frac{1}{r^2}\left(r\frac{d}{dr}\right)\left(r\frac{d}{dr}+4\right)f-\frac{\lambda(\lambda+4)}{r^2}f+2Ef=0 \ , \tag{38}$$

whose solution is

$$f = r^{-2}\, J_{\lambda+2}\,(\sqrt{2E}\ r) \tag{39}$$

where J is a Bessel function.

The solution of (29), (31) can then be written as

$$\psi(x_i^\alpha) = r^{-(2+\lambda)}\, J_{\lambda+2}\,(\sqrt{2E}\ r)\ P(x_i^\alpha)\ , \tag{40}$$

and so we have now the problem of determining the $P(x_i^\alpha)$ that satisfy (32 a, b), i.e. the set of polynomials that form a basis for the irreducible representation of R_6 characterized by λ.

As in the previous case, we must find a chain of subgroups that would help us to characterize the polynomial $P(x_i^\alpha)$ in analogy with the way the subgroup R_2 was used to characterize the polynomial $P(x_i)$ of the two body problem. We would like this chain of subgroups of R_6 to contain the subgroup

$$\begin{pmatrix} R_3 & 0 \\ 0 & R_3 \end{pmatrix} \tag{41}$$

of ordinary rotations of the reference frame in physical space as then we would have among our integrals of motion the relative angular momentum of the three particles.

To find this chain of subgroups that contains (41) we recall the fact that the R_6 and SU_4 groups have isomorphic Lie Algebras[4]. This implies that we could construct linear combinations of the generators of $\Lambda_{ij}^{\alpha\beta}$ that would have the commutation relations of the generators of SU_4. To obtain these linear combinations let us introduce the coordinates

$$x_i^+ \equiv (1/\sqrt{2})\,(x_i^1 + i x_i^2)\,, \quad x_i^- \equiv (1/\sqrt{2})\,(x_i^1 - i x_i^2)\,, \tag{42a}$$

and their corresponding momenta

$$p_i^+ = (1/\sqrt{2})\,(p_i^1 - i p_i^2)\,, \quad p_i^- = (1/\sqrt{2})\,(p_i^1 + i p_i^2)\,. \tag{42b}$$

We now define the operators

$$C_{ij} \equiv i\,\{(x_i^+ p_j^+ - x_j^- p_i^-) - \frac{1}{2}\,(x_k^+ p_k^+ - x_k^- p_k^-)\,\delta_{ij}\}\,, \tag{43a}$$

$$C_{k4} \equiv -\,\epsilon_{kij}\, x_i^- p_j^+\,, \tag{43b}$$

$$C_{4k} \equiv -\,\epsilon_{kij}\, x_i^+ p_j^-\,, \tag{43c}$$

$$C_{44} \equiv (i/2)\,(x_k^+ p_k^+ - x_k^- p_k^-)\,, \tag{43d}$$

where repeated indices i, j, k are summed over 1, 2, 3, and ϵ is the antisymmetric tensor.

The 16 operators $C_{\mu\nu}$, $\mu\nu = 1, 2, 3, 4$, are linearly independent except for $C_{44} = - C_{ii}$, and from (42) they can be expressed as linear combinations of $\Lambda_{ij}^{\alpha\beta}$. Furthermore from the commutation rules of $p_i^{\pm}$, $x_j^{\pm}$ one obtains immediately that

$$[C_{\mu\nu}, C_{\mu'\nu'}] = C_{\mu\nu'} \delta_{\mu'\nu} - C_{\mu'\nu} \delta_{\mu\nu'} , \tag{44}$$

which shows that $C_{\mu\nu}$ are the generators of a U_4 group, or rather an SU_4 group as the trace of $C_{\mu\nu}$ is zero.

The set of linearly independent polynomials P satisfying (32) would then be a basis for an irreducible representation of the group SU_4. We could characterize them further by the subgroup U_3 of SU_4 whose generators are the C_{ij} of (43a) and afterward by the subgroup R_3 of U_3 whose generators

$$\Lambda_{ij} \equiv \frac{1}{2} (C_{ij} - C_{ji})$$

$$= (i/2) \{ (x_i^+ p_j^+ - x_j^+ p_i^+) + (x_i^- p_j^- - x_j^- p_i^-) \} = \sum_{\alpha=1}^{2} \Lambda_{ij}^{\alpha\alpha} , \tag{45}$$

are clearly related with the components of the total angular momentum associated with two particles of coordinates x_i^{α} and momenta p_i^{α}.

The procedure of constructing the full set of polynomials that are basis for an irreducible representation of U_3 in the $U_3 \supset R_3$ chain from the highest weight polynomial in U_3 has been extensively discussed [5, 6, 7]. We need therefore only obtain the polynomials solution of (32) that are of highest weight[7] in U_3 i.e. P should also satisfy (the repeated index i is not summed in (46a).

$$C'_{ii} P = h'_i P , \quad C'_{ij} P = 0, \; i < j , \tag{46a, b}$$

where for simplicity, instead of the operators (43a), we take for the generators of our U_3 group the operators

$$\mathrm{C}'_{ij} = \mathrm{C}_{ij} + \mathrm{C}_{44}\ \delta_{ij} = i\,(x_i^+ p_j^+ - x_j^- p_i^-) \quad . \tag{47}$$

The numbers h_i' are integers that satisfy $h_1' \geqslant h_2' \geqslant h_3'$ but are not necessarily positive.

The most general polynomial $P(x_i^\pm)$ can be written as

$$P(x_i^\pm) = \Sigma A_{n_1 n_2 n_3 m_1 m_2 m_3} (x_1^+)^{n_1} (x_2^+)^{n_2} (x_3^+)^{n_3} (x_1^-)^{m_1} (x_2^-)^{m_2} (x_3^-)^{m_3} \quad , \tag{48}$$

If we apply the three equations (46a) we get

$$n_1 - m_1 = h_1' \,, \quad n_2 - m_2 = h_2' \,, \quad n_3 - m_3 = h_3' \quad , \tag{49}$$

so that the polynomial satisfying them takes the form

$$P(x_i^\pm) = (x_1^+)^{h_1'} (x_2^+)^{h_2'} (x_3^+)^{h_3'} \sum_{m_1 m_2 m_3} A_{m_1 m_2 m_3} (x_1^+ x_1^-)^{m_1} (x_2^+ x_2^-)^{m_2} (x_3^+ x_3^-)^{m_3} \quad . \tag{50}$$

We have now that

$$r^2 = x_i^\alpha x_i^\alpha = 2\,(x_1^+ x_1^- + x_2^+ x_2^- + x_3^+ x_3^-) \quad , \tag{51}$$

and so we could express $x_1^+ x_1^-$ in terms of r^2 and of $x_2^2\ x_2^-$, $x_3^+ x_3^-$.
As the A's are so far indetermined, we could write

$$P(x_i^{\pm}) = (x_1^+)^{b_1'} (x_2^+)^{b_2'} (x_3^+)^{b_3'} \sum_{m_1 m_2 m_3} A_{m_1 m_2 m_3} (r^2)^{m_1} (x_2^+ x_2^-)^{m_2} (x_3^+ x_3^-)^{m_3} \quad . \tag{52}$$

We now apply C_{13}' to (52) and, noticing that

$$C_{ij}' r^2 = 0 \, , \ C_{13}' x_1^+ = C_{13}' x_2^+ = C_{13}' x_2^- = C_{13}' x_3^- = 0 \tag{53}$$

we see that

$$C_{13}' P = \left(\frac{\partial P}{\partial x_3^+} \right) C_{13}' x_3^+ = x_1^+ \frac{\partial P}{\partial x_3^+} = 0 \tag{54}$$

Therefore P is independent of x_3^+ which means that

$$m_3 = - b_3' \tag{55}$$

The polynomial can now be written as

$$P(x_i^{\pm}) = (x_1^+)^{b_1'} (x_2^+)^{b_2'} (x_3^-)^{-b_3'} \sum_{m_1 m_2} A_{m_1 m_2} (r^2)^{m_1} (x_2^+ x_2^-)^{m_2} \quad . \tag{56}$$

We now apply C'_{12} to (56) and as

$$C'_{12}\, r^2 = C'_{12}\, x_1^+ = C'_{12}\, x_2^- = C'_{12}\, x_3^- = 0 \tag{57}$$

we get

$$C'_{12}\, P = \left(\frac{\partial P}{\partial x_2^+}\right) C'_{12}\, x_2^+ = x_1^+ \frac{\partial P}{\partial x_2^+} = 0 \tag{58}$$

and as P is then independent of x_2^+, we have

$$m_2 = -b'_2 \tag{59}$$

so that the polynomial becomes

$$P = (x_1^+)^{b'_1} (x_2^-)^{-b'_2} (x_3^-)^{-b'_3} \sum_{m_1} A_{m_1} (r^2)^{m_1} \; . \tag{60}$$

Finally we apply C'_{23}, and as

$$C'_{23}\, r^2 = C'_{23}\, x_1^+ = C'_{23}\, x_3^- = 0$$

we get

$$C'_{23}\, P = \left(\frac{\partial P}{\partial x_2^-}\right) C'_{23}\, x_2^- = -x_3^- \frac{\partial P}{\partial x_2^-} = 0 \; , \tag{61}$$

which means that

$$- b_2' = 0 \tag{62}$$

and so the polynomial is

$$P(x_i^{\pm}) = (x_1^+)^{b_1'} (x_3^-)^{-b_3'} \sum_{m_1} A_{m_1} (r^2)^{m_1} \quad , \tag{63}$$

where b_1' must be a non negative and b_3' a non positive integer.

So far we have not applied equations (32 a, b), which can be written as

$$x_i^+ \frac{\partial P}{\partial x_i^+} + x_i^- \frac{\partial P}{\partial x_i^-} = \lambda P \ , \quad \frac{\partial^2 P}{\partial x_i^+ \partial x_i^-} = 0 \ . \tag{64 a, b}$$

From (64 a) we immediately get that

$$b_1' - b_3' + 2m_1 = \lambda \tag{65}$$

and the polynomial becomes

$$(x_1^+)^{b_1'} (x_3^-)^{-b_3'} \ r^{\lambda - (b_1' - b_3')} \tag{65 a}$$

If we apply (64 b) to (65a), we get

$$\frac{1}{4} \ [(b_1' - b_3') + \lambda + 4] \ [\lambda - (b_1' - b_3')] \ (x_1^+)^{b_1'} (x_3^-)^{-b_3'} \ r^{\lambda - (b_1' - b_3') - 2} = 0, \tag{66}$$

and as both λ and $h_1' - h_3'$ are non-negative, we conclude that

$$\lambda = h_1' - h_3' \quad , \tag{67}$$

so that the polynomial satisfying (32,46) becomes

$$P = (x_1^+)^{k_1 - k_2} (x_3^-)^{k_2} \tag{68}$$

where

$$k_1 = h_1' - h_3' = \lambda \quad , \qquad k_2 = h_2' - h_3' = - h_3' \quad ,$$

are the numbers characterizing the irreducible representation[2] of SU_3. As the exponents in (68) must be non negative integers, we see that k_2 can take values

$$k_2 = \lambda, \lambda - 1, \ldots, 0 \quad . \tag{70}$$

We see then that the irreducible representation λ of R_6 is reducible under the subgroup SU_3 of R_6 and contains the irreducible representations of SU_3

$$(\lambda \lambda), (\lambda \lambda - 1), \ldots, (\lambda 0) \quad . \tag{71}$$

What is the representation of SU_4 corresponding to the representation λ of the isomorphic group R_6? In (68) we have the polynomial of the basis for an irreducible representation λ of R_6 which is of highest weight in U_3. If we require further that

$$C_{k4} P = 0 \ , \quad k = 1,2,3 \tag{72}$$

we get the polynomial of highest weight in the group SU_4. But from (43 b) we see that

$$C_{k4} P = - \epsilon_{ki1} x_i^- (\lambda - k_2)(x_1^+)^{\lambda - k_2 - 1} (x_3^-)^{k_2} = 0 \ , \tag{73}$$

which can only be satisfied if $\lambda - k_2 = 0$, and so the polynomial of highest weight in U_4 becomes

$$(x_3^-)^{\lambda} \ .$$

If we apply C_{ii}, C_{44} to (73), we get the eigenvalues

$$h_1 = \frac{\lambda}{2} \ , \ h_2 = \frac{\lambda}{2} \ , \ h_3 = -\frac{\lambda}{2} \ , \ h_4 = -\frac{\lambda}{2} \ . \tag{74}$$

The representation of SU_4 is given by

$$h_1 - h_4 = \lambda, \ \ h_2 - h_4 = \lambda, \ \ h_3 - h_4 = 0 \ . \tag{75}$$

Therefore the representation of SU_4 corresponding to the representation λ of R_6 is

$$[\lambda \lambda] \ , \tag{76}$$

and from (71) the representations of SU_3 contained in it, satisfy the inequalities[7]

$$\lambda \geqslant k_1 \geqslant \lambda \geqslant k_2 \geqslant 0$$

The solution to our three body problem corresponding to the highest weight representation of the subgroup U_3 of R_6 is then

$$\psi(x_i^{\alpha}) = r^{-2-\lambda}\, J_{\lambda+2}\,(\sqrt{2E}\,r)\,(x_1^{+})^{\lambda-k_2}\,(x_3^{-})^{k_2} \quad , \tag{78}$$

and the full set of solutions in the $U_3 \supset R_3$ chain can be obtained by applying to (78) the lowering operators[8] in the $U_3 \supset U_2 \supset U_1$ chain and the transformation brackets[6] that take us from this chain to the $U_3 \supset R_3$ one.

What are the operators, and their corresponding eigenvalues, that characterize ψ ? First we have the relative kinetic energy

$$H' = \frac{1}{2}\, p_i^{\alpha}\, p_i^{\alpha}\,, \quad H'\psi = E\psi\ . \tag{79}$$

Then the Casimir operator Φ of (31) with eigenvalue

$$\Phi\psi = \frac{1}{2}\,\lambda(\lambda+2)\,\psi \tag{80}$$

Afterwards we could consider the Casimir operator Γ of the SU_3 subgroup whose generators are (43a), i.e.

$$\Gamma = (C_{ij} - \frac{1}{3}\,\mathrm{tr}\,C\,\delta_{ij})(C_{ji} - \frac{1}{3}\,\mathrm{tr}\,C\,\delta_{ji})$$

$$= C_{ij} C_{ji} - \frac{1}{3} \left(\text{tr } C \right)^2, \text{ where tr } C = C_{ii} \quad . \tag{81}$$

The operator Γ clearly commutes with Φ as the SU_3 group is a subgroup of R_6. The eigenvalues of Γ are[5]

$$\Gamma\psi = \gamma\psi, \quad \gamma = \frac{2}{3} (k_1 + k_2)^2 - 2k_1(k_2 - 1) . \tag{82}$$

As $k_1 = \lambda$, we see that Γ determines the eigenvalue k_2 of our wave function.

We now look into the R_3 subgroup of SU_3 whose generators are given by the components of the total angular momentum

$$L_k = -2i \, \epsilon_{kij} \Lambda_{ij} , \text{ with } \Lambda_{ij} \text{ given by (45)} . \tag{83}$$

The integrals of motion are then

$$L^2 = 2 \Lambda_{ij} \Lambda_{ji} , \quad L_3 = -2i \Lambda_{12} \quad , \tag{84}$$

and the corresponding eigenvalues are

$$L(L+1) , \quad M \tag{85}$$

As is well known[5, 6] the R_3 subgroup does not completely determine the rows of an irreducible representation of U_3 in the $U_3 \supset R_3$ chain. The complete determination is achieved with the help of the operator Ω defined by

$$\Omega = Q_{ij} L_i L_j \text{ , where } Q_{ij} = \frac{1}{2}\left(C_{ij} + C_{ji}\right) - \frac{1}{3} \operatorname{tr} C \delta_{ij} \quad . \tag{86}$$

The eigenvalues of this operator will be denoted by ω and their determination is discussed in reference 5. The operators $H, \Phi, \Gamma, \Omega, L^2, L_3$ clearly commute among themselves.

The eigenfunction (28) of our three body problem can then be denoted by the ket

$$|K_1, K_2, K_3, E, \lambda, k_2, \omega, L, M>$$

$$= r^{-\lambda-2} J_{\lambda+2} (\sqrt{2E}\, r)\, P^{\lambda k_2}_{\omega L M} (x_i^\alpha) \tag{87}$$

where $P^{\lambda k_2}$ are the set of polynomials corresponding to a basis for an irreducible representation[5] of SU_3 characterized by (λk_2) whose rows are denoted by ωLM. The solution (87) brings out the correlations associated with the creation of three particles in a single event.

IV. THE THREE BODY PROBLEM IN MOMENTUM SPACE

If we want to find now the eigenfunction $\chi(p_i^\alpha)$ in momentum space associated with the hamiltonian H' of (29), the χ must clearly contain the δ function

$$\delta\left(\frac{1}{2} p_i^\alpha p_i^\alpha - E\right) = \delta\left(p_i^+ p_i^- - E\right) \quad , \tag{88}$$

and in fact, the most general solution of (29) will be

$$\chi(p_i^{\alpha}) = \delta(p_i^+ p_i^- - E) \; F(p_i^{\alpha}) \tag{89}$$

where $F(p_i^{\alpha})$ is an arbitrary function of p_i^{α}.

The hamiltonian H' of (29) is invariant under R_6 and so we could require that $F(p_i^{\alpha})$ would be a basis for an irreducible representation of R_6, i.e. it would satisfy

$$\Phi F = \kappa F , \tag{90}$$

where Φ is defined by (33) but now we have

$$x_i^{\alpha} = -\frac{1}{i} \frac{\partial}{\partial p_i^{\alpha}} \tag{91}$$

To find the F that satisfies (90) we could think first of an auxiliarly function G that satisfies

$$p_i^{\alpha} \frac{\partial G}{\partial p_i^{\alpha}} = \lambda G \; , \quad \frac{\partial^2 G}{\partial p_i^{\alpha} \partial p_i^{\alpha}} = 0 \; , \tag{92}$$

which implies that G is an homogeneous polynomial of degree λ in p_i^{α} that furthermore satisfies a Laplace equation in momentum space.

From (33) we have that

$$\Phi = \frac{1}{2} \{ (i\underline{p} \cdot \underline{r}) \; [(i\underline{p} \cdot \underline{r}) - 4] + p^2 r^2 \} \; , \tag{93}$$

where

$$i\underline{p}\cdot\underline{r} \equiv - p^i_\alpha \frac{\partial}{\partial p^i_\alpha} \ , \ r^2 \equiv - \frac{\partial^2}{\partial p_i^\alpha \, \partial p_i^\alpha} \ , \ p^2 \equiv p_i^\alpha p_i^\alpha \ , \tag{94}$$

and so

$$\Phi G = \frac{1}{2} \lambda(\lambda+4) \, G \tag{94a}$$

We now could further characterize the polynomial G by the restriction that it should be of highest weight in the subgroup U_3 of R_6 i.e., that it should satisfy (46a, b) where in $\mathcal{C}'_{ij}$ of (47)

$$x_i^{\pm} = - \frac{1}{i} \frac{\partial}{\partial p_i^{\pm}} \ , \tag{95}$$

i.e.

$$\mathcal{C}'_{ij} = -\left(p_j^+ \frac{\partial}{\partial p_i^+} - p_i^- \frac{\partial}{\partial p_i^-} \right) \ . \tag{97}$$

Clearly then we can repeat the analysis of section 3 just replacing

$$x_i^+ \to p_i^- \ , \ x_i^- \to p_i^+ \tag{97}$$

and so the $F(p_i^\alpha)$ corresponding to the highest weight in the subgroup U_3 becomes

$$F(p_i^\alpha) = (p_1^-)^{k_1 - k_2} (p_3^+)^{k_2} \tag{98}$$

where

$$p_i^- = (1/\sqrt{2})\,(p_i^1 + i\,p_i^2), \quad p_i^+ = (1/\sqrt{2})(p_i^1 - i\,p_i^2) \quad . \tag{99}$$

All states will be generated from (98) applying to them $C^{\dagger}_{ij}$ with $i > j$, and so the momentum space solution could be written as

$$(2E)^{-\lambda/2} \;\delta\left(\frac{1}{2}\, p_i^{\alpha}\, p_i^{\alpha} - E\right) \; P^{\lambda k_2}_{\omega L M}\;(p_i^{\alpha}) \tag{100}$$

where P is a polynomial in the components of the momentum that has the same interpretation that the polynomial in terms of the coordinates in (87). The Fourier transform of (100) gives, except for a multiplicative constant, the configuration space solution (87).

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