SOLUTIONS FOR A FEYNMAN-GELL-MANN PARTICLE IN A COULOMB FIELD

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ABSTRACT

Solutions of the Feynman-Gell-Mann equation in the case of a Coulomb field are given both for bound and continuum states. It turns out that the eigenfunctions are mixtures consisting mainly of the state they are assigned in Dirac's theory together with small percentages of states of opposite parity.

RESUMEN

Se dan soluciones para la ecuación de Feynman y Gell-Mann para estados tanto ligados como del continuo. Resultó que las eigenfunciones son mezclas que contienen principalmente el estado que se les asigna en la teoría de Dirac, junto con pequeñas fracciones de estados de paridad opuesta.

I. INTRODUCTION

The two-component second-order wave equation for fermions proposed some years ago by Feynman and Gell-Mann[1] introduces a violation of parity conservation in a very appealing way. By assuming that all the fermions interacting through a Fermi coupling (gradients excluded) obey the equation, instead of the four-component Dirac equation, one is lead to a unique weak four-fermion coupling, of $V-A$ type.

In this work, attention is given to the solutions of the Feynman-Gell-Mann equation in the Coulomb case. For an external electromagnetic field, this equation is not invariant under space reflections[2]. Therefore, the solutions contain mixtures of different parities. These solutions are briefly indicated both for the bound and the continuum-states.

It turns out that the eigenfunctions are mixtures consisting mainly of the states they are assigned in Dirac's theory together with smaller percentages of states of opposite parity. The degree of parity mixing is given. As in the Dirac equation, there appears a double accidental degeneracy in the energy levels.

One of the features of the equation is its simplicity, the radial equation being of the same type of the non-relativistic Schrödinger-Coulomb equation, though with irrational "orbital quantum numbers". The continuum-state solutions are of interest for the consideration of the Coulomb-field distortion of the lepton wave function in weak-decay processes like β-decay[2].

II. THE SOLUTIONS

The electromagnetic Feynman-Gell-Mann equation is (with $\hbar = c = 1$) :

$$[D_\mu^2 - \frac{e}{2}\sigma_{\mu\nu}F_{\mu\nu} - m_0^2]\,\chi = 0, \text{ where } D_\mu = \partial_\mu - ieA_\mu \tag{2.1}$$

with the condition $\rho_1\chi = -\chi$. (2.2)

(2.2) implies $\chi = \begin{pmatrix} \varphi \\ -\varphi \end{pmatrix}$, φ being a two-component spinor which obeys

$$\left[D_\mu^2 + e\vec{\sigma}\cdot(H + iE) - m_0^2\right]\varphi = 0 \tag{2.3}$$

where the σ's are the Pauli matrices.

If the external field is a point Coulomb field, (2.3) becomes

$$\left[\frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial}{\partial\rho}\right) - \frac{L^2 - \zeta^2 + i\zeta\vec{\sigma}\cdot\vec{n}}{\rho^2} + \frac{\lambda}{\rho} - \frac{1}{4}\right]\varphi = 0 \tag{2.4}$$

where:

$$\begin{aligned}
\zeta &= Ze^2 \\
\rho &= kr \\
k^2 &= 4(m_0^2 - E^2) \\
\lambda &= \frac{2E\zeta}{k}
\end{aligned} \tag{2.5}$$

(2.4) admits $J = L + \frac{1}{2}\vec{\sigma}$ and J_z as integrals of motion, and contains scalar and pseudo-scalar terms under space reflections.

To solve (2.4), we put forward the Ansatz

$$\varphi = a(\rho)\left(c\,\Omega_{JlM} + \Omega_{Jl'M}\right) , \tag{2.6}$$

where c is a constant to be determined later; here $l = J + \frac{1}{2}$ and $l' = J - \frac{1}{2}$; the Ω_{JLM} are spinor-harmonics defined by

$$\Omega_{JLM} = \sum_{m,m_\sigma} (Lm\,\tfrac{1}{2}\,m_\sigma | JM)\, Y_L^m\, \chi_{m_\sigma}^m \quad , (L = l \text{ or } l') \tag{2.7}$$

with obvious notation[3].

So one sees that (2.6) is a superposition of angular states of the same J and M, but different parities $[= (-)^L]$.

The spinor-harmonics satisfy

$$\int \Omega^*_{JLM} \, \Omega_{J'L'M'} \, d\omega = \delta_{JJ'} \, \delta_{LL'} \, \delta_{MM'} \tag{2.8}$$

and obey further the relations [3]

$$\vec{\sigma} \cdot \vec{n} \, \Omega_{JJ\pm\frac{1}{2}M} = - \Omega_{JJ\mp\frac{1}{2}M} \tag{2.9}$$

Substitution of (2.6) into (2.4) leads to an equation envolving $a(\rho)$, Ω_{JlM} and $\Omega_{Jl'M}$. Multiplying it separately by Ω^*_{JlM} and by $\Omega^*_{Jl'M}$ and using (2.8) one gets two equations in $a(\rho)$, that are made to coincide into

$$\frac{1}{\rho^2} \frac{d}{d\rho} \left(\rho^2 \frac{da(\rho)}{d\rho} \right) - \frac{J^2 - \frac{1}{4} - \zeta^2 - i\zeta c_{\pm}}{\rho^2} \, a(\rho) + \left(\frac{\lambda}{\rho} - \frac{1}{4} \right) a(\rho) = 0 \tag{2.10}$$

by suitable choice of c:

$$c_{\pm} = \frac{i}{\zeta} \left\{ (J + \frac{1}{2}) \mp \left[(J + \frac{1}{2})^2 - \zeta^2 \right]^{\frac{1}{2}} \right\} \tag{2.11}$$

Equation (2.10) has the form of the radial non-relativistic Schrödinger equation, where the orbital angular momentum is replaced by an irrational quantum number s which obeys

$$s_{\pm} (s_{\pm} + 1) = J^2 - \frac{1}{4} - \zeta^2 - i\zeta c_{\pm} \quad . \tag{2.12}$$

From (2.11), (2.12) and (2.13) one arrives at

$$s_\pm (s_\pm + 1) = \gamma(\gamma + 1) \text{ with } \gamma = \left[\left(J + \frac{1}{2} \right)^2 - \zeta^2 \right]^{\frac{1}{2}} ; \tag{2.13}$$

this gives four possibilities to s . But equation (2.10), in the form

$$\left[\frac{1}{\rho^2} \frac{d}{d\rho} \left(\rho^2 \frac{d}{d\rho} \right) - \frac{s(s+1)}{\rho^2} + \frac{\lambda}{\rho} - \frac{1}{4} \right] a(\rho) = 0$$

has the well-known solutions of the form

$$a(\rho) = N\rho^s e^{-\frac{1}{2}\rho} F(s + 1 - \lambda, 2s + 2; \rho) . \tag{2.14}$$

The usual boundary conditions at the origin impose $s \geqslant 0$, eliminating two of the four possibilities, remaining

$$s_+ = \gamma - 1 \text{ and } s_- = \gamma , \tag{2.15}$$

of which the former presents an "admissible" singularity.

III. BOUND-STATE SOLUTIONS

We now consider the case of real and positive k. The condition at infinity on (2.14) is now

$$s_\pm + 1 - \lambda = -n', \; n' = 0, 1, 2, 3, \ldots \tag{3.1}$$

This, combined with (2.5), gives the energy eigenvalues:

$$E_{n's_\pm JM} = \frac{m_0}{[1 + \zeta^2 (n' + s_\pm + 1)^{-2}]^{1/2}} \tag{3.2}$$

An expansion in ζ, up to ζ^4 gives us

$$E_{ns_+ JM} = m_0 \left[1 - \frac{\zeta^2}{2n^2} - \frac{\zeta^4}{2n^4} \left(\frac{n}{J + \frac{1}{2}} - \frac{3}{4} \right) \right] \quad \text{and} \tag{3.3}$$

$$E_{ns_- JM} = m_0 \left[1 - \frac{\zeta^2}{2(n+1)^2} - \frac{\zeta^4}{2(n+1)^2} \left(\frac{n+1}{J + \frac{1}{2}} - \frac{3}{4} \right) \right] \quad , \tag{3.4}$$

where $n = n' + J + \frac{1}{2} = 1, 2, 3, \ldots$ is the "principal quantum number" of the non-relativistic theory.

The two admissible values for s, given by (2.15), show the existence of a degeneracy similar to that found in the Dirac spectrum.

The normalized eigenfunctions are

$$\varphi_{ns_\pm JM} = \frac{\pi^{1/2} \rho^{s_\pm} e^{-1/2\rho} \; F(s_\pm + 1 - \lambda, 2s_\pm + 2; \rho)}{2^{2s_\pm + 1} \Gamma(s_\pm + \frac{3}{2})(1 + |c_\pm|^2)^{1/2}} \left[c_\pm \Omega_{JlM} + \Omega_{Jl'M} \right] \tag{3.5}$$

The confluent hypergeometrical function is a (non-Laguerre) polynomial of degree $\lambda - s_\pm - 1$[6]. One sees furthermore that these solutions show mixtures of different

	$s_+ = \gamma - 1$	$s_- = \gamma$
$n = 3$	$\sim 100\%\ d_{5/2} + \sim 0.15 \times 10^{-3}\%\ p_{5/2}$	
	$\sim 100\%\ p_{3/2} + \sim 0.3 \times 10^{-3}\%\ d_{3/2}$	$\sim 100\%\ d_{3/2} + \sim 0.3 \times 10^{-3}\%\ p_{3/2}$
	$\sim 100\%\ s_{1/2} + \sim 1.3 \times 10^{-3}\%\ p_{1/2}$	$\sim 100\%\ p_{1/2} + \sim 1.3 \times 10^{-3}\%\ s_{1/2}$
$n = 2$	$\sim 100\%\ p_{3/2} + \sim 0.3 \times 10^{-3}\%\ d_{3/2}$	
	$\sim 100\%\ s_{1/2} + \sim 1.3 \times 10^{-3}\%\ p_{1/2}$	$\sim 100\%\ p_{1/2} + \sim 1.3 \times 10^{-3}\%\ s_{1/2}$
$n = 1$	$\sim 100\%\ s_{1/2} + \sim 1.3 \times 10^{-3}\%\ p_{1/2}$	

Fig. 1. The energy levels of $z = 1$. The degenerate levels, up to $n = 3$, corresponding to s_+ and s_- are drawn in separate columns and the degree of parity mixing is shown.

parities. For fixed values of n, J and s, the probabilities of $l = J + \frac{1}{2}$ and $l' = J - \frac{1}{2}$ are in the ratio $|c_{\pm}|^2 : 1$. The $|c_{\pm}|^2$, therefore, may be defined as a parity-mixing coefficient. It is found that $|c_{\pm}|^2$ increases with z^2 and $|c_-|^2$ decreases in the same proportion, so that the mixture grows for greater z's. In the case $z = 1$, $J = \frac{1}{2}$, one has $|c_+|^2 \approx 1.3 \times 10^{-5} + \mathcal{O}(\zeta^4)$ and $|c_-|^2 \approx 7.5 \times 10^4 + \mathcal{O}(\zeta^2)$; therefore, the state n, J, s_+ is an almost pure state of parity, since $\Omega_{Jl'M}$ strongly predominates (proportion $1 : 1.3 \times 10^{-5}$). The same thing happens in the state s_-, with opposite parity. All the states are mixed, as shown in fig. 1.

The integral of Biedenharn[2,4], $\Gamma = \vec{\sigma} \cdot \vec{L} - i\zeta\vec{\sigma} \cdot n$, may be shown to have eigenvalues $\pm\gamma$.

IV. CONTINUUM STATES

In this case we may put $k = i\kappa$, with real κ, the radial solution normalized in the κ-scale being[5]:

$$a(\rho) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \kappa e^{\pi/2\kappa} \frac{|\Gamma(s_{\pm} + 1 - \lambda)|}{|\Gamma(2s_{\pm} + 2)|} |\rho|^{s_{\pm}} e^{-\frac{1}{2}\rho} F(s_{\pm} + 1 - \lambda, 2s_{\pm} + 2; \rho) \tag{4.1}$$

There is, of course, no more restriction like (3.1). The spectrum extends continuously from m_0 to infinity.

The free case is illustrative: if $\zeta = 0$, $\lambda = 0$ from (2.5), and equation (2.4) becomes a spherical Bessel equation, with solutions $j_{s_{\pm}}(\rho)$ and $n_{s_{\pm}}(\rho)$, where now $\rho = (E^2 - m^2)^{\frac{1}{2}} \pi$; the latter are singular at the origin. It is clear that in this case the term $\vec{\sigma} \cdot \vec{n}$ disappears from (2.4), and both $a(\rho)\,\Omega_{JlM}$ and $a(\rho)\,\Omega_{Jl'M}$ are independently solutions.

It is tempting to speculate about the smallness of the degree of mixing involved in the Feynman-Gell-Mann solutions in connection with the hydrogen atom. A full discussion of the problem, however, involves the consideration of the Lamb shift effect and would go beyond the limited scope of the present work.

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REFERENCES

1. R.P. Feynman and M. Gell-Mann, Phys. Rev., **109**, 193 (1958).
2. L.C. Biedenharn, Nuovo Cim., **22**, 1097 (1961).
3. M.E. Rose, "Relativistic Electron Theory", John Wiley, New York, 1961.
4. L.C. Biedenharn, Phys. Rev., **126**, 845 (1962).
5. L.D. Landau and E.M. Lifshitz, "Quantum Mechanics", Pergamon Press, London, 1959, p. 125-6.
6. Ibid., p. 499, formula (d.11).

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