ENDOMORPHIC FORMS OF FINITE AUTOMATA

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ABSTRACT

Using relational algebra, certain transformation properties of automata are examined, both under external stimuli and, in general, under endomorphic mappings. An isomorphism is shown to exist between the lattice of endomorphs and the set of relations generating the endomorphic images, by virtue of which it is possible to associate to each lattice operation a corresponding operation of relational algebra.

The lattice of an automaton can be interpreted as a program showing the manner by which the automata realizes a given operation, according to a process of abstraction or description. Along this process, according to the direction in which the lattice is described, the steps can be interpreted as association or classification processes leading to the formation of the "concepts" required for the operation.

Outputs can be considered as properties of the state and by adding the con-

dition that endomorphisms should retain the output, one arrives at a sublattice of "acceptable" endomorphisms having a universal lower bound which corresponds to the minimal form of Huffman. It is to be noticed that the whole set of endomorbisms has a meaning independently of output properties and that the concept of redundant states with respect to an operation has a meaning only when the outputs are specified.

RESUMEN

Usando el álgebra relacional se examinan las propiedades de transformación de los autómatas bajo los estímulos externos y, en general, las transformaciones endomórficas. Se muestra que existe un isomorfismo entre la latíz de endomorfos y las relaciones que los determinan, siendo posible asociar a cada operación del álgebra laticial una correspondiente operación relacional. La latíz de un autómata se interpreta como un programa que corresponde a una operación realizada por éste, de acuerdo con un proceso de abstracción o de descripción a lo largo del cual se forman "conceptos" por asociación o clasificación, según el sentido en que se describe la latíz.

Las salidas, consideradas como propiedades del estado, restringen los endomorfismos, cuando se impone la condición de que la salida se mantenga invariante. Queda una sublatíz cuya cota universal inferior corresponde a la forma mínima de Huffman. Se hace notar que el concepto de redundancia de estados solamente tiene sentido en relación con un sistema de salidas.

1. Introduction

In a previous paper[1] an automaton A was considered with an input alphabet $\mathfrak{X}$ consisting of m symbols $x_1, x_2 \ldots . x_m$; a phase space $\mathfrak{Y}$ formed by n inner states $y_1, y_2, \ldots . y_n$ and an output alphabet $\mathfrak{Z}$ of p symbols $z_1, z_2 \ldots . z_p$. If x, y, z are variables over the sets $\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z}$ respectively, A was assumed to fulfill a set of canonical equations

$$y_1 = f(x, y)$$
$$z = g(y) \qquad (1)$$

giving the next state y_1 as a single-valued function $f(x, y)$ of the previous stimulus x and state y, and the output symbol z as a single-valued function $g(y)$ of the state. Further, $f(x, y)$ is assumed to be defined for all $x \in \mathfrak{X}$ and for all $y \in \mathfrak{Y}$ with values $y_1 \in \mathfrak{Y}$. Again, $g(y)$ is defined for all $y \in \mathfrak{Y}$ with values $z \in \mathfrak{Z}$.

As it is well known, (1) defines for each x a many-to-one transformation

$$y \xrightarrow{x} y_1 \qquad (2)$$

which we translated into the formal language of logic, by a relation, $x(y_1, y)$ meaning, y_1 is the x-sequent of y. The relation itself, x, is the set of all ordered pairs of states, $y_1 ; y$, in which y_1 is a x-sequent of y. The inverse relation $\breve{x}(y, y_1)$ means that y is a x-precedent of y_1.

With the aid of these relations, one can transform classes of states. Thus, if K is an arbitrary set of states, $x'' K$, the *projection* of K by x, is the set of all states which are x-sequents of the states of K: the x of the K's. $\breve{x}'' K$ the *retrojection* of K by x is the set of all states being x-precedents of the states of K. In particular, if $K = \mathfrak{Y}$ is the whole phase space of A, $x'' \mathfrak{Y} = x''$ are the x-sequents, $\breve{x}'' \mathfrak{Y} = \breve{x}''$ the x-precedents. Again if $K = \iota y$ is the unitary class of y, the class having y as its sole member, $x'' \iota y$ are the x-sequents of y, $\breve{x}'' \iota y$ are the x-precedents of y. Because of the single-valued character of (1), $x'' \iota y$ contains one element, y_1, the x-sequent of y. This element is denoted by $x' y$: the x of y.

Finally we recall that from any relation such as x, two relations can be formed:

$\breve{x} x$: a co-reference relation holding between two states when both have the same x-sequent

$x\breve{x}$: a corelation relation holding between two states when both are x-sequents of the same states.

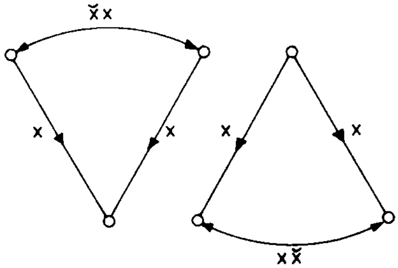


Fig.1

Now, single-valuedness of x, means that two states cannot be corelate, unless they are equal:

$$(y_1)(y_2)\,[\vdash x\breve{x}(y_1, y_2) \supset \cdot\, \mathcal{I}(y_1, y_2)]$$

where $\mathcal{I}$ is the identical relation, namely, $\mathcal{I}(y_1, y_2)$ means $y_1 = y_2$ and could be read, y_1 is a equal of y_2.

The above property we translate into the formal relational language by $x\breve{x} \subseteq \mathcal{I}$.

The above mentioned assumptions upon the nature of the canonical equations (1) can be translated into our language by saying that A is

I) defined:	$x'' \subseteq \mathcal{U}$	(a)	
II) determined:	$\breve{x}'' = \mathcal{U}$	(b)	(3)
III) causal:	$x\breve{x} \subseteq \mathcal{I}$	(c)	

By defined, $x'' \subseteq \mathcal{U}$ we mean that any x-sequent of any state of $\mathcal{U}$, is again a state of $\mathcal{Y}$.

By determined, $\breve{x}" = \mathcal{Y}$ we mean that every state y is, for any x, a x-precedent.

By causal, $x\breve{x} \subseteq \mathcal{I}$ we mean that for any x, each state has a single x-sequent.

Graphically, in the kinematic diagram of A, (3a) means that all arrows end on circles of the diagram. There are no arrows leaving the diagram to end in some unknown place. (3b) means that each circle of the diagram is the starting point of m arrows, one for each x. In other words, that we know what the automaton will do under any circumstances, i.e. when placed in any of its possible states is subjected to any possible stimulus. Finally (3c) means that from each state emerges only one x-arrow.

For convenience we shall call the coreference relation $\breve{x}x$, the x-partnership relation. Two states related by $\breve{x}x$, i.e. having a common x-sequent, will be called *x-partners*.

Clearly, each state is, for any x, its own x-partner. This we formally write as $\mathcal{I} \subseteq \breve{x}x$, which, when combined with (3c) yields the result,

$$x\breve{x} \subseteq \mathcal{I} \subseteq \breve{x}x \qquad (4)$$

x-partnership is obviously

reflexive:	$\mathcal{I} \subseteq \breve{x}x$	(a)	
symmetric:	$(\breve{x}x)^{\smile} = \breve{x}x$	(b)	(5)
transitive:	$(\breve{x}x)^2 \subseteq \breve{x}x$	(c)	

It should be observed that transitivity, (5c) can be shown to be a consequence of causality. In fact, the latter states that $x\breve{x} \subseteq \mathcal{I}$. Multiplication of this expression at left by $\breve{x}$ gives $\breve{x}x\breve{x} \subseteq \breve{x}$ and now at right by x yields $\breve{x}x\breve{x}x \subseteq \breve{x}x$ or

$(\breve{x}x)^2 \subseteq \breve{x}x$.

Moreover, reflexivity $\mathcal{I} \subseteq \breve{x}x$ implies, as it is seen multiplying by $\breve{x}x$ that $\breve{x}x \subseteq (\breve{x}x)^2$. Hence we get the stronger form

$$(\breve{x}x)^2 = \breve{x}x \tag{6}$$

$\breve{x}x$ is therefore an identity relation and as such effects a partition of the phase space $\mathcal{U}$ of A in x-equivalence classes: two states belong to the same x-class if and only if they have the same x-sequent. States belonging to different x-classes have different x-sequents.

We recall from[1] the sequence relation

$$s = x_1 \cup x_2 \cup \dots \cup x_m = \bigcup x_k \tag{7}$$

the l'th degree sequence relation

$$s^l = \bigcup x_{k_1} x_{k_2} \dots x_{k_l} \tag{8}$$

the succession relation

$$\sigma = \# s = \mathcal{I} \cup s \cup s^2 \cup s^3 \cup \dots = \bigcup_{l=0}^{\infty} s^l \tag{9}$$

and the connection relation

$$c = \sigma \cap \breve{\sigma} \tag{10}$$

reflexive, symmetric and transitive, determining a partition of the states of A in equivalence classes which we called *groups* of states. A group $\mathcal{G}$ was defined as a set of connected states

$$\text{Group:} \qquad c''\mathcal{G} = (\sigma \cap \breve{\sigma})''\mathcal{G} = \mathcal{G} \tag{11}$$

Groups were classified as

initial: $\breve{s}''\mathcal{G} \subseteq \mathcal{G}$ or $\breve{\sigma}''\mathcal{G} \subseteq \mathcal{G}$

final: $s''\mathcal{G} \subseteq \mathcal{G}$ or $\sigma''\mathcal{G} \subseteq \mathcal{G}$

transient: $\mathcal{G}' \cap \sigma''\mathcal{G} \neq \emptyset$

passing: $\mathcal{G}' \cap \sigma''\mathcal{G} \neq \emptyset$ and $\mathcal{G}' \cap \breve{\sigma}''\mathcal{G} \neq \emptyset$

isolated: $(\sigma \cup \breve{\sigma})''\mathcal{G} \subseteq \mathcal{G}$

In general groups fullfill

$$\sigma''\mathcal{G} \cap \breve{\sigma}''\mathcal{G} = (\sigma \cap \breve{\sigma})''\mathcal{G} = \mathcal{G} \tag{12}$$

All initial as well as all passing groups are transient. An isolated group is simultaneously initial and final.

Groups themselves can be related by sequence:

$$s(\mathcal{G}_1, \mathcal{G}) \equiv . \; s''\mathcal{G} \cap \mathcal{G}_1 \neq \emptyset \tag{13}$$

by l'th order sequence s^l and by succession:

$$\Sigma = \mathscr{I} \cup S \cup S^2 \cup S^3 \cup \ldots = \bigcup_{l=0}^{\infty} S^l \tag{14}$$

We recall that Σ fullfills

$$\Sigma \cap \breve{\Sigma} = 0 \tag{15}$$

and that groups are organized in families in a tree-like manner defined through the family relation

$$F = \bigcup_{k=0}^{\infty} (S \cup \breve{S})^k = \left(\Sigma \cup \breve{\Sigma}\right)^{\infty} \tag{16}$$

Each such family is a sub-automaton of A.

2. Endomorphisms and their construction.

As in (I) we denote by t a transformation of the phase space $\mathcal{Y}$ of A into itself. In general we want to think of t as a many-to-one transformation assigning to each $y \in \mathcal{Y}$ a unique transform $\eta \in \mathcal{Y}$ so that the entire $\mathcal{Y}$ is mapped onto a subset H of itself.

We are to consider t as a relation, $t(\eta, y)$ meaning that η is the transform of y. The assumptions concerning t are that

I) the transforms are elements of $\mathcal{Y}$.

$$t'' \subseteq \mathcal{Y} \tag{17a}$$

II) the transformation is defined for all states of $\mathcal{Y}$,

$$\breve{t}'' = \mathcal{Y} \tag{17b}$$

III) each state y has a unique transform

$$t\breve{t} \subseteq \mathcal{I} \tag{17c}$$

These properties are formally identical to those we found exist among x-relations. Hence, their formal consequences are the same: we can introduce a t-partnership relation $\breve{t}t$ holding between two states when and only when both have the same transform. This relation fullfills the analogues of (4) and (6), namely

$$t\breve{t} \subseteq \mathcal{I} \subseteq \breve{t}t \tag{18}$$

$$(\breve{t}t)^2 = \breve{t}t \tag{19}$$

The unique image η of a state y under t, is $\eta = t'y$, the t of y. The subset $H = \{\eta\}$ of images of $\mathcal{Y}$ will be denoted by $H = t''\mathcal{Y}$. Conversely, each η may have several t-inverses. The set of all t-inverses of the same η is the class $\breve{t}''\iota\eta$. This is one of the equivalence classes of t, whose elements are t-partners. For brevety we designate these classes as b_{η}. Namely

$$b_{\eta} = \breve{t}''\iota\eta = \hat{y}t(\eta, y) = \hat{y}\breve{t}(y, \eta) \tag{20}$$

$\hat{y}t(\eta, y)$ denoting the class of y's such that η is a t of y.

Each transformation t determines a partition Π_t of $\mathcal{Y}$ into equivalence classes b_{η}, the members of the partition. This we represent by

$$\Pi_t = [\breve{t}''\iota\eta] = [b_{\eta}] \tag{21}$$

We recall now that a transformation t of A was called an endomorphism of A, if t preserved all sequence relations x of A, that is if the commutator of x and t vanishes:

$$[x, t] = x t \Lambda t x = \theta \qquad (22)$$

It was shown in[1] that an endomorphism fullfills

$$\begin{array}{lll} x t = t x & \ldots\ldots\ldots & \text{(a)} \\ \breve{x}\breve{t} = \breve{t}\breve{x} & \ldots\ldots\ldots & \text{(b)} \\ t\breve{x} \subseteq \breve{x} t & \ldots\ldots\ldots & \text{(c)} \\ x\breve{t} \subseteq \breve{t} x & \ldots\ldots\ldots & \text{(d)} \end{array} \qquad (23)$$

Multiplying orderly (23c) by (23d) one gets

$$x \breve{t} t \breve{x} \subseteq \breve{t} x \breve{x} t$$

but, by virtue of (3c),

$$\breve{t} x \breve{x} t \subseteq \breve{t} t$$

hence

$$x \breve{t} t \breve{x} \subseteq \breve{t} t \qquad (24)$$

that is, x-transforms of t-partners are t-partners.

Because of the fact that the four expressions (23) are symmetric in x and t, one may interchange x and t in (24) getting

$$t x \breve{x} \breve{t} \subseteq \breve{x} x \tag{25}$$

which states that t-transforms of x-partners are x-partners.

Again, since $\mathcal{J} \subseteq \breve{t} t$, multiplying at left by $\breve{x}$, at right by x and using the commutativity properties (23a) and (23b) one obtains

$$\breve{x} x \subseteq \breve{x} \breve{t} t x = \breve{t} \breve{x} x t$$

thus

$$\breve{x} x \subseteq \breve{t} \breve{x} x t \tag{26}$$

Interchanging x and t:

$$\breve{t} t \subseteq \breve{x} \breve{t} t x \tag{27}$$

that is, x-partners are t-transforms of x-partners, and t-partners are x-sequents of t-partners.

Endomorphisms then are such that all members of a given t-class undergo under any x a transformation sending all of them into the same t-class. We state this property by saying that t-classes are *covariant.*

Because of covariance, the set $H = \{\eta\}$ of t-transforms of the y's, can be interpreted as the phase space of a new automaton, $t'A$, the t-endomorphic image of A. Moreover, the image states η can be identified with their corresponding covariant classes $\breve{t}" \lfloor \eta = b_\eta$ in the sense that any element of b_η can equally well represent the whole class. In this sense we can interpret the members b_η of the partition Π_t determined by t as states of $t'A$. The original relation x in $\mathcal{U}$ induces a corresponding relation x in H defined through

$$\bar{\bar{x}}(\eta_1, \eta) \equiv \cdot\ t(\eta_1, y_1) \cdot t(\eta, y) \cdot x(y_1, y) \tag{28}$$

which by mere reordering of factors is seen to be

$$x = t x \breve{t} \tag{29}$$

In fact, paraphrasing a terminology very common in algebra one might say $\mathring{t}t$ defines a congruence modulo t between states, so that

$$y_1 = = y_2 \quad \text{if and only if} \quad \breve{t}t(y_1, y_2) \tag{30}$$

Then, the states of the endomorph $t'A$ are merely the states of A modulo t:

$$H = \mathcal{Y} \ (\mathrm{mod}\ t) \tag{31}$$

and the automaton itself, $t'A$, is the original A, modulo t:

$$t'A = A \ (\mathrm{mod}\ t) \tag{32}$$

Now, the covariance of t-classes immediately suggests a method for finding endomorphic images of A practically identical to that proposed by Huffman[2] in connection with elimination od redundant states of automata. The relation between the present point of view and that of Huffman will be discussed in the next section.

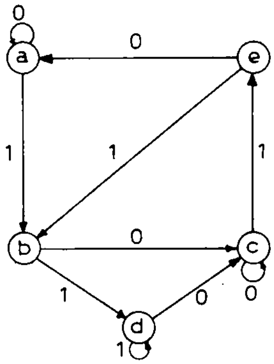
For the present purpose we illustrate the method by means of an example.

Example 1.-

Consider the automaton whose transition table and kinematic diagram are shown below :

	x: 0	1
a	a	b
b	c	d
c	c	e
d	c	d
e	a	b

Fig.2



First, we try to combine pairs of states to form covariant classes.

Suppose we try a and b. We write down the proposed partition, numbering its members and writing below the elements of the proposed combination the number of the class to which it goes under each stimulus, using always a fixed order for the stimuli.

1		2	3	4
a	b	c	d	e
1	2			
1	3			

States a and b, because of the different transformation properties cannot be combined unless class 2 and class 3 become 1. Hence we next try

1				2
a	b	c	d	e
1	1	1	1	
1	1	2	1	

Again, because of c, one requires to identify classes 1 and 2, thereby getting

1					
a	b	c	d	e	
1	1	1	1	1	partition (abcde)
1	1	1	1	1	

a and *b* cannot be combined unless all states are fused.
Try now *a* and *c*.

1		2	3	4
a	c	b	d	e
1	1			
2	4			

the combination can be made covariant if classes 2 and 4 become the same. Hence we try

1		2		3
a	c	b	e	d
1	1	1	1	
2	2	3	2	

Now, the combination *ac* becomes of course satisfactory, but we still need test the new combination *be*. It is seen that this can be made covariant if classes 2 and 3 are combined. We then try

1		2	
a	c	bde	
1	1	111	partition (ac, bde)
2	2	222	

The partition (ac, bde) corresponds to an endomorphic image.

Now one tries *a* and *d* :

1		2	3	4
a	d	b	c	e
1	3			
2	1			

requiring the fusion of 2 and 3 with 1. However, in so doing one gets *abcd*, *e*. The first member contains *ab* which is known to require the fusion of all states. Hence we get (abcde).

Try now *a* and *e* :

1		2	3	4	
a	e	b	c	d	
1	1				partition (ae, b, c, d)
2	2				

This partition leads to a possible endomorph.

Calculations are better performed in a table, as shown below. On each row there appear the different steps of combination. The last partition appearing on each row is the simplest endomorph containing the states combined.

1	2	3	4	1	2	3	1	2	1
ab 12 13	c	d	e				abcd 1111 1121	e	abcde 11111 11111
ac 11 24	b	d	e	ac 11 22	be 11 32	d	ac 11 22	bde 111 222	
ad 13 21	b	c	e				abcd	e	abcde
ae 11 22	b	c	d						
a	bc 22 34	d	e	a	bc 22 33	de 21 32			abcde
a	bd 33 22	c	e						
a	be 31 42	c	d				ac	bde	
a	b	cd 33 43	e	a	b	cde 331 332			abcde
a	b	ce 31 32	d				abce	d	abcde
a	b	c	de 31 42				ac	bde	

1	2	3	1	2	1
aeb	c	d			abcde
aec 111 221	b	d	abce	d	abcde
aed	b	c			abcde
ae	bc 22 31	d	aed	bc	abcde
ae	bd 33 22	c ✓			
ae	b	cd	acde	b	abcde
abd	c	e			abcde
ac	bd	e	ac	bde	
ae	bd	c			
a	bdc	e			abcde
a	bde	c	ac	bde	
a	bd	ce			abcde
			aebd	c	abcde
			aec	bd	abcde
			ae	bdc	abcde
			ac	bde	abcde

Each new endomorph is marked with a sign ✓. One takes next each of the 4-state endomorphs trying to combine pairs of states remembering, however, the limitations imposed by the different degrees of combineability, in order to reduce the amount of work.

In our example, we work on (ae, b, c, d) and (a, bd, c, e).

Next one tries the 3-state endomorphs, in our case being (ae, bd, c) and finally the two state endomorphs whose treatment is, however, trivial.

The set of all marked partitions is the set of all endomorphic images. In our case, including the original automaton, the set consists of:

$A = A_0 = (a, b, c, d, e)$

$A_1 = (ae, b, c, d)$

$A_2 = (a, bd, c, e)$

$A_3 = (ae, bd, c)$

$A_4 = (ac, bde)$

$A_5 = (abcde)$

Kinematic diagrams can be directly obtained from that of A, by carrying on the required combinations.

In our case, one obtains the results shown in Fig. 3

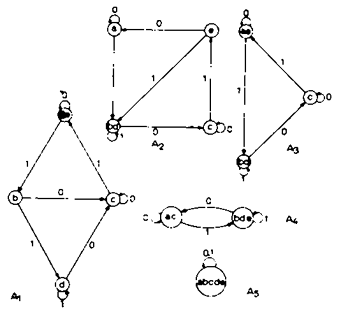


Fig 3

The general method should, by now, be quite clear: from an n-state A, by trying pairs of states one obtains a set of endomorphs. One works now in a similar fashion with the (n-1)-state images, then with the (n-2)-state images, and so on.

3. The Lattice of Endomorphs

We have seen that to any endomorphism $t'A$ of A, there corresponds a covariant partition Π_t of the phase-space $\mathcal{Y}$ of A, and conversely, to any covariant partition Π_t there corresponds an endomorphic image $t'A$. There exists a one-to-one correspondence

$$t'A \longleftrightarrow \Pi_t \longleftrightarrow \breve{t}t \tag{33}$$

between endomorphs and covariant partitions.

Consider now the set $\mathcal{L}(A)$ of all endomorphs of A. This corresponds to the set $\mathcal{L}(\mathcal{Y})$ of covariant partitions of $\mathcal{Y}$, which in turn is a subset $\mathcal{L}(\mathcal{Y}) \subseteq \mathcal{L}_0(\mathcal{Y})$ of the set $\mathcal{L}_0(\mathcal{Y})$ of all possible partitions of $\mathcal{Y}$.

Any automaton A has at least two trivial endomorphisms: First, the *identical endomorphism*, $\mathcal{J}$, sending each state onto itself, $\eta = \mathcal{J}'y = y$. Its associated partitions $\Pi_{\mathcal{J}} = \mathcal{Y}$ has as members the states of A. Further, the endomorph $\mathcal{J}'A = A$ is A itself.

Second, the *total endomorphism* $\mathbb{O}$, sending every state of A onto a single state $I = \mathbb{O}'y$. The associated partition $\Pi_{\breve{\mathbb{O}}}$ consists of a single member I, containing all states of A, and the image $\mathbb{O}'A$ consists of a single state.*

Hence, the set $\mathcal{L}(A)$ is never void. It will contain at least the *improper endomorphisms*, $\mathcal{J}$ and $\mathbb{O}$.

A, however, may possess other endomorphisms t which we might call *proper*. In order to study their mutual relations, it is convenient to remind the

*For detailed proof of these statements, see Appendix I.

reader some well known properties of partitions[3] which we illustrate with our covariant partitions Π_t.

A partition Π_t consists of a number of sets $b_1, b_2, \ldots\ldots b_q$, its *members* such that

I) each member is a subset of $\mathcal{U}$: $b_r \subseteq \mathcal{U}$ $(r = 1, \ldots\ldots q)$

II) each element of $\mathcal{U}$ belongs to one of the members;

$$b_1 \cup b_2 \cup \ldots\ldots \cup b_q = \mathcal{U} \qquad \text{(exhaustive)}$$

III) two different members have no elements in common;

$$b_i \cap b_j = \mathcal{O} \text{ if } i \neq j \qquad \text{(exclusive)}$$

The partition is represented by

$$\Pi_t = [b_1, b_2, \ldots\ldots b_q]$$

and is said to be of order q.

A partition $\Pi_{t_1} = [\mu_1, \mu_2, \ldots\ldots \mu_p]$ is said to be included in a partition $\Pi_{t_2} = [\nu_1, \nu_2, \ldots\ldots \nu_q]$, in symbols $\Pi_{t_1} \subseteq \Pi_{t_2}$, if, given any member μ_j of Π_{t_1}, there exists a member ν_k of Π_{t_2}, such that $\mu_j \subseteq \nu_k$. In short, $\Pi_{t_1} \subseteq \Pi_{t_2}$ means that every member of Π_{t_1} is a subset of some member of Π_{t_2}. Inclusion is obviously reflexive and transitive.

Two partitions are equal, $\Pi_{t_1} = \Pi_{t_2}$, if each is included in the other, i.e., if both $\Pi_{t_1} \subseteq \Pi_{t_2}$ and $\Pi_{t_2} \subseteq \Pi_{t_1}$ are true. This requires that both be of the same order and have the same members.

Given again two partitions

$$\Pi_{t_1} = [\mu_1, \mu_2, \ldots\ldots \mu_p] \text{ and } \Pi_{t_2} = [\nu_1, \nu_2, \ldots\ldots \nu_q],$$

the partition whose members are the intersections

$$\lambda_{ij} = \mu_i \cap \nu_j \quad (i = 1, \ldots p; j = 1, \ldots . q)$$

of the members of Π_{t_1} and Π_{t_2}, is the intersection of Π_{t_1} and Π_{t_2}.

$$\Pi_{t_1} \cap \Pi_{t_2} = [\lambda_{11}, \lambda_{12}, \ldots . \lambda_{pq}]$$

Now, $\Pi_{t_1} \cap \Pi_{t_2}$ is a lower bound (l.b.) of Π_{t_1} and Π_{t_2}, i.e., is contained in both :

$$\Pi_{t_1} \cap \Pi_{t_2} \subseteq \Pi_{t_1} \qquad\qquad \Pi_{t_1} \cap \Pi_{t_2} \subseteq \Pi_{t_2}$$

However, if Π_t is any other l.b., i.e., if

$$\Pi_t \subseteq \Pi_{t_1} \qquad\qquad \Pi_t \subseteq \Pi_{t_2}$$

then $\Pi_t \subseteq \Pi_{t_1} \cap \Pi_{t_2}$. This is why $\Pi_{t_1} \cap \Pi_{t_2}$ is called the ***greatest lower bound*** (G.L.B.) of Π_{t_1} and Π_{t_2}.

We recall the reader that two sets α and β are *conjoint*, if they have common elements, i.e., if $\alpha \cap \beta \neq \emptyset$.

Now, two sets α and β of a given class $\mathcal{C}$ of sets are said to be *chain-connected*, in symbols $\alpha \backsim \beta$, if there is a finite number n, say, of sets $\gamma_1, \gamma_2 \ldots . \gamma_n$ of $\mathcal{C}$, such that $\gamma_1 = \alpha$, $\gamma_n = \beta$ and for every $k(k = 1, 2, n-1)$, γ_k and γ_{k+1} are conjoint.

Chain-connection is illustrated in Fig. 4.

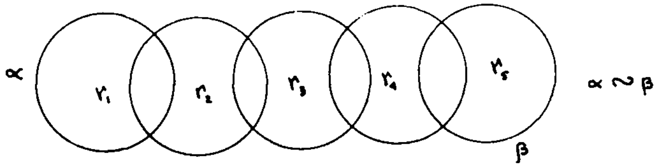


Fig.4

Chain-connection is obviously

reflexive: $\alpha \; \mathbf{co} \; \alpha$

symmetric: if $\alpha \; \mathbf{co} \; \beta$, then $\beta \; \mathbf{co} \; \alpha$

transitive: if $\alpha \; \mathbf{co} \; \beta$ and $\beta \; \mathbf{co} \; \gamma$ then $\alpha \; \mathbf{co} \; \gamma$.

Hence, if one has an arbitrary collection of sets, chain-connection effects a partition of the sets in classes. Two sets belong to the same class if and only if they are chain-connected. Two sets belonging to different classes are not chain-connected.

Now, given two partitions $\Pi_{t_1} = [\mu_1, \mu_2 \ldots \mu_p]$ and $\Pi_{t_2} = [\nu_1, \nu_2, \ldots \nu_q]$, form the class of sets $(\mu_1, \mu_2, \ldots \mu_p, \nu_1, \nu_2, \ldots \nu_q)$, consisting of the members of both partitions. Let now $\lambda_1, \lambda_2 \ldots \lambda_r$ be the sets formed by chain-connection of the sets of the class. The partition whose members are $\lambda_1, \lambda_2, \ldots \lambda_r$, i.e., $[\lambda_1, \lambda_2, \ldots \lambda_r]$ is the union

$$\Pi_{t_1} \cup \Pi_{t_2} = [\lambda_1, \lambda_2, \ldots \lambda_r]$$

of Π_{t_1} and Π_{t_2}.

$\Pi_{t_1} \cup \Pi_{t_2}$ is an *upper bound* (u.b.) of Π_{t_1} and Π_{t_2}, i.e., contains both,

Π_{t_1} and Π_{t_2} :

$$\Pi_{t_1} \subseteq \Pi_{t_1} \cup \Pi_{t_2} , \qquad \Pi_{t_2} \subseteq \Pi_{t_1} \cup \Pi_{t_2}$$

and further, it is the *least upper bound* (l.u.b.), i.e., if Π_t is any u.b.,

$$\Pi_{t_1} \subseteq \Pi_t \text{ and } \Pi_{t_2} \subseteq \Pi_t$$

then

$$\Pi_{t_1} \cup \Pi_{t_2} \subseteq \Pi_t \quad .$$

What we want to show now, is that these relations and operations of partitions are beautifully described by corresponding relations and operations of endomorphisms. In fact, that the whole algebra of partitions can be translated to the algebra of relations in a rather simple manner.

To begin with, assume that t_1 and t_2 are two endomorphisms and Π_{t_1} and Π_{t_2} their associated partitions. It is easily seen* that

$$\breve{t}_1 \, t_1 \subseteq \breve{t}_2 \, t_2 \tag{34}$$

if and only if

$$\Pi_{t_1} \subseteq \Pi_{t_2} \tag{34a}$$

Moreover, in this case, $t_1' A$ and $t_2' A$ are such that they are not only endomorphs of A, but $t_2' A$ is in turn an endomorph of $t_1' A$. We thus write

*For a detailed proof of these statements see Appendix I.

$$t_1' A \subseteq t_2' A \tag{34b}$$

meaning now that $t_2' A$ is an endomorph of $t_1' A$. That is, there exists now an endomorphism τ of $t_1' A$ such that

$$t_2' A = \tau' t_1' A \tag{35}$$

where

$$\tau \subseteq t_2 \breve{t}_1 \tag{36}$$

Conversely, if $t_1' A$ is an endomorph of A and τ an endomorphism of $t_1' A$, the image

$$\tau' t_1' A = t_2' A \tag{35a}$$

is an endomorph of A given by

$$t_2 = \tau t_1 \tag{36a}$$

and $t_1' A \subseteq t_2' A$.

Endomorphic relation $\subseteq$ is then transitive: an endomorph of an endomorph is an endomorph. It is trivially reflexive since every A is an endomorph of itself.

On the other hand every endomorphism t fullfills (*)

$$\breve{\mathcal{I}}\mathcal{I} \subseteq \breve{t}t \subseteq \breve{\mathbb{O}}\mathbb{O} \tag{37}$$

---

*(*) See Appendix I.

and every partition Π_t

$$\Pi_{\mathcal{I}} \subseteq \Pi_t \subseteq \Pi_{\mathbb{O}} \tag{37a}$$

with the corresponding property for endomorphs:

$$A \subseteq t'A \subseteq \mathbb{O}'A \tag{37b}$$

The one-state endomorph $\mathbb{O}'A$ is endomorph of every endomorph of A. Hence, A and $\mathbb{O}'A$ are universal bounds for the set of all endomorphs of A.
A series of successive endomorphs

$$A = A_0 \subsetneq A_1 \subsetneq A_2 \subsetneq \ldots\ldots \subsetneq A_s = \mathbb{O}'A \tag{38}$$

in which for each $j\,(j = 1, \ldots. n)$, A_j is an endomorph of A_{j-1}, will be called an ***endomorph chain***.

In such a chain,

$$A_j = t_j'A = \tau_j'A_{j-1} \tag{39}$$

is an endomorph of A and of all A_k with $k < j$. t_j maps A onto A_j and τ_j maps A_{j-1} onto A_j. Clearly in a chain

$$\left.\begin{aligned} t_j &= \tau_j \tau_{j-1} \ldots\ldots \tau_2 \cdot \tau_1 \\ \tau_j &\subsetneq t_j \overset{\cup}{t}_{j-1} \\ t_1 &= \tau_1 \end{aligned}\right\} \tag{40}$$

We may say that A_j is *maximal* in A_{j-1} if $A_{j-1} \subsetneq A_j$ and there is no endomorph of A containing A_{j-1} and included in A_j.

An endomorphic chain (38) starting with A, ending with the one-state endomorph $\mathbb{C}'A$ and such that for each j, A_j is maximal in A_{j-1}, will be called a *composition series* of A.

An automaton will be called *simple* if it has no composition series other than the trivial $A \subsetneq \mathbb{C}'A$.

Operation with partitions can be described by corresponding operations with relations and provide corresponding operations of automata.

Thus, if t_1 and t_2 are two endomorphisms of A, the relation

$$\breve{t}_1 t_1 \cap \breve{t}_2 t_2$$

describes a partition

$$\Pi_{t_1} \cap \Pi_{t_2}$$

to which there corresponds an endomorphism t such that *

$$\breve{t}\, t = \breve{t}_1 t_1 \cap \breve{t}_2 t_2 \tag{41a}$$

$$\Pi_t = \Pi_{t_1} \cap \Pi_{t_2} \tag{41b}$$

$$t'A = t_1'A \cap t_2'A \tag{41c}$$

$t'A$ as given by (41c) has both $t_1'A$ and $t_2'A$ as endomorphic images and is an endomorph of every other endomorph having the same property, in other words it is the G.L.B. of $t_1'A$ and $t_2'A$. Formally

$$t'A \subseteq t_1'A \qquad\qquad t'A \subseteq t_2'A$$

---

*See Appendix I.

$$t'A \subseteq t'_1 A \qquad\qquad t'A \subseteq t'_2 A$$

and if $\tau' A$ is any endomorph of A such that

$$\tau' A \subseteq t'_1 A \qquad\qquad \tau' A \subseteq t'_2 A$$

then $$\tau' A \subseteq t' A \ .$$

Again, let t_1 and t_2 be two endomorphisms of A and form the ancestral

$$\#(\breve{t}_1 t_1 \cup \breve{t}_2 t_2)$$

If Π_{t_1} and Π_{t_2} are the partitions determined by t_1 and t_2, it can be shown* that this ancestral determines a partition which is precisely

$$\Pi_{t_1} \cup \Pi_{t_2}$$

and that there exists an endomorphism t of A such that

$$\breve{t} t = \#(\breve{t}_1 t_2 \cup \breve{t}_2 t_2) \qquad (42a)$$

$$\Pi_t = \Pi_{t_1} \cup \Pi_{t_2} \qquad (42b)$$

$$t'A = t'_1 A \cup t'_2 A \qquad (42c)$$

this last expression being a definition of the union of two endomorphs.

*See Appendix I.

$t'A$, as defined by (42c) is the l.u.b. of $t_1'A$ and $t_2'A$, namely,

$$t_1'A \subseteq t'A \qquad\qquad t_2'A \subseteq t'A$$

and if $\tau'A$ is any other endomorph such that

$$t_1'A \subseteq \tau'A \qquad\qquad t_2'A \subseteq \tau'A$$

then

$$t'A \subseteq \tau'A \quad .$$

The set $\mathcal{L}(A)$ of endomorphs of A and the set $\mathcal{L}(\mathcal{U})$ of covariant partitions of $\mathcal{U}$ have in common the properties

I) of being partially ordered through a reflexive and transitive relation $\subseteq$,

II) being such that any two elements

$$\Pi_{t_1}, \Pi_{t_2} \in \mathcal{L}(\mathcal{U}) \qquad \text{or} \qquad t_1'A, t_2'A \in \mathcal{L}(A)$$

have a G.L.B.

$$\Pi_{t_1} \cap \Pi_{t_2} \qquad \text{or} \qquad t_1'A \cap t_2'A$$

and a l.u.b.

$$\Pi_{t_1} \cup \Pi_{t_2} \qquad \text{or} \qquad t_1'A \cup t_2'A$$

which properties define a special type of set called by mathematitians a *lattice*.

Thus, $\mathcal{L}(\mathfrak{U})$ and $\mathcal{L}(A)$ are lattices, and in fact isomorphic lattices. Moreover, since these lattices have the further property

IV) there exist two elements

$$\Pi_{\mathfrak{I}}\,,\ \Pi_{\mathfrak{O}} \in \mathcal{L}(\mathfrak{U}) \qquad\qquad A\,,\ \mathfrak{O}'A \in \mathcal{L}(A)$$

such that for every

$$\Pi_t \in \mathcal{L}(\mathfrak{U}) \qquad\qquad t'A \in \mathcal{L}(A)$$

one has

$$\Pi_{\mathfrak{I}} \subseteq \Pi_t \subseteq \Pi_{\mathfrak{O}} \qquad\qquad A \subseteq t'A \subseteq \mathfrak{O}'A \quad .$$

$\mathcal{L}(\mathfrak{U})$ and $\mathcal{L}(A)$ have ***universal bounds*** and consequently are ***bounded.***

The foregoing arguments suggest the simultaneous treatment of $\mathcal{L}(\mathfrak{U})$ and $\mathcal{L}(A)$ as an abstract algebra of endomorphisms themselves. Thus, corresponding to (34), (34a), and (34c) we define a new form of inclusion as

$$t_1 \dot{\subseteq} t_2 \quad \text{if and only if} \quad \breve{t}_1 t_1 \subseteq \breve{t}_2 t_2 \tag{43}$$

In accordance with (41a), (41b), (41c) we introduce a new form of intersection by

$$t = t_1 \dot{\cap} t_2 \quad \text{if and only if} \quad \breve{t}_1 t_1 \cap \breve{t}_2 t_2 \tag{44}$$

and a new form of union through

$$t = t_1 \dot{\cup} t_2 \quad \text{if and only if} \quad \breve{t} t = \#(\breve{t}_1 t_1 \cup \breve{t}_2 t_2) \tag{45}$$

Clearly, under relation (43) and operation (44) and (45) our endomorphisms form a lattice $\mathcal{L}$, isomorphic to $\mathcal{L}(A)$ and $\mathcal{L}(\mathcal{Y})$. $\mathcal{L}$ is bounded since (37) implies

$$\mathcal{I} \dot{\subseteq} \iota \dot{\subseteq} \mathbb{O} \quad (46)$$

From lattice algebra we know that $\dot{\cap}$ and $\dot{\cup}$ are idempotent, commutative, associative and satisfy the laws of absorption and coherence. Distributive law, however, has to be substituted by the weaker semidistributive law :

$$\left.\begin{array}{l} \iota_1 \dot{\cap} (\iota_2 \dot{\cup} \iota_3) \dot{\supseteq} (\iota_1 \dot{\cap} \iota_2) \dot{\cup} (\iota_1 \dot{\cap} \iota_3) \\ \\ \iota_1 \dot{\cup} (\iota_2 \dot{\cap} \iota_3) \dot{\supseteq} (\iota_1 \dot{\cup} \iota_2) \dot{\cap} (\iota_1 \dot{\cup} \iota_3) \end{array}\right\} \quad (47)$$

with similar expressions for partitions and endomorphs.

The possibility of extending (43), (44) and (45) to relations in general, whether such relations are endomorphisms or not, should be apparent by now. A detailed discussion of this matter as well as some interesting consequences, will be treated elsewhere.

For the moment, we shall avail ourselves with the graphical representation of lattices well known in the literature[3]. The diagram of $\mathcal{L}$ is formed as follows : One draws a small circle on the top of the diagram, representing the one-state automaton $\mathbb{O}'A$. Then, on a row below, one draws small circles representing endomorphs in which $\mathbb{O}'A$ is maximal. This row contains certainly all two-state endomorohs. and all endomorphs having no endomorphic images other than $\mathbb{O}'A$. One draws now a line from $\mathbb{O}'A$ down to each of the small circles of the second row. Then, one examines the endomorphs of the second row and draws small circles on a third row, representing endomorphs in which the endomorphs of the second row are maximal, drawing a line between two circles when one endomorph is included in the other. One next forms a fourth row in a similar fashion with endomorphs in which those of the third are maximal, and so on. At the bottom of the diagram appears A alone joined with lines to all small circles of the previous row.

Example 2.-

We illustrate the method by showing the diagram of Example 1.- This is shown in Fig. 5

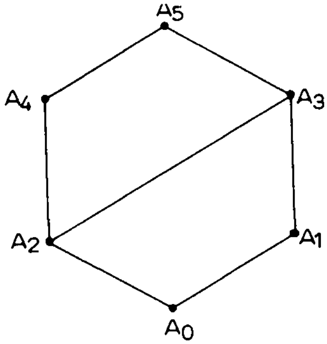


Fig.5

All lattice relations can be read from the diagram. Thus, an endomorph A_i is included in A_j (A_j is an endomorph of A_i !) if there is a path going always up, from A_i to A_j . Further, A_j will be maximal in A_i if the ascending path consists of a single segment.

The intersection $A_i \cap A_j$ is obtained by finding the highest possible point in which a descending path from A_i can meet a descending path from A_j . If $A_i \subseteq A_j$ clearly, $A_i \cap A_j = A_i$ itself. Further $A_i \cap A_i = A_i$.

The union, $A_i \cup A_j$ of A_i and A_j is obtained by finding the lowest point in which an ascending path from A_i can meet an ascending path from A_j . Again, if $A_i \subseteq A_j$

$$A_i \cup A_j = A_j \qquad \text{and} \qquad A_i \cup A_i = A_i \ .$$

Composition series are represented by ascending paths going from the bottom to the top circle. In Ex. 2 we find three, namely,

$$A_0 \subseteq A_1 \subseteq A_3 \subseteq A_5$$

$$A_0 \subseteq A_2 \subseteq A_3 \subseteq A_5$$

.

$$A_0 \subseteq A_2 \subseteq A_4 \subseteq A_5$$

Example 3.-

Consider the automaton whose transition table and kinematic diagram are shown in Fig. 6

	x = 0	x = 1
a	a	b
b	c	d
c	c	d
d	a	b

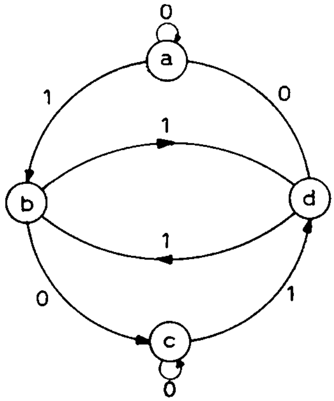


Fig. 6

Its endomorphs, found by the method of section 2, are :

A_0 = (a, b, c, d)

A_1 = (ad, b, c)

A_2 = (a, bc, d)

A_3 = (ad, bc)

A_4 = (ac, bd)

A_5 = (abcd)

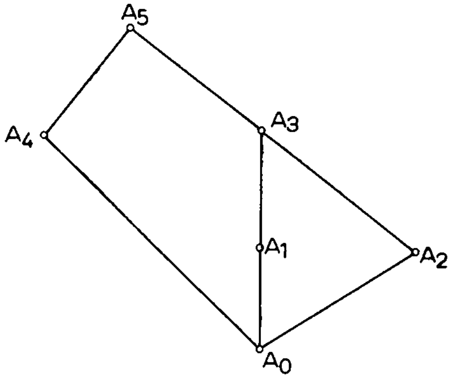


Fig. 6a

We obtain the following composition series :

$$A_0 \subseteq A_1 \subseteq A_3 \subseteq A_5$$

$$A_0 \subsetneq A_2 \subseteq A_3 \subseteq A_5$$

$$A_0 \subseteq A_4 \subseteq A_5$$

The considerations of this section can be easily extended to automata having no input lines. Such automata have a phase space $\mathcal{Y} = \{y\}$, an output alphabet $\mathcal{Z} = \{z\}$ but their input alphabet is void $\mathcal{X} = \mathcal{O}$. Nevertheless, the operation of the automaton is governed by means of a rule specifying for each state y, which state will be the next. The rule is usually exhibited in a transition table, or a kinematic diagram.

In our formalism, the above procedure amounts to define the automaton A by a sequence relation s, so that $s(y_1, y)$ means that y_1 is a sequent of y. If A is to be defined, determined and causal, s must fullfill

$$s" \subseteq \mathcal{Y}$$

$$\breve{s}" = \mathcal{Y} \qquad (48)$$

$$s\breve{s} \subseteq \mathcal{I}$$

and satisfies

$$s\breve{s} \subseteq \mathcal{I} \subseteq \breve{s}s \qquad (49)$$

$\breve{s}s$ being reflexive, symmetric and transitive. Indeed

$$(\breve{s}s)^2 = \breve{s}s \qquad (50)$$

is idempotent.

Now, an endomorphism t of A is a many-to-one mapping of A such that

$$[s, t] = 0 \qquad (51)$$

the endomorph is $t'A$.

Having s the same formal properties as external stimuli, the whole theory can be applied to the present situation. The set $\mathcal{L}(A)$ of endomorphs of A is a bound lattice; these endomorphs can be found by the procedure outlined in Sec. 2.

4. Interpretation of Endomorphs.

In order to introduce ourselves to an understanding of the meaning of endomorphisms in automata, we shall start by studying first, a few simple cases.

Consider the automaton illustrated in Fig. 7. It is assumed to have an input x capable of assuming the values 0 and 1. Its purpose is to recognize the sequence 01.

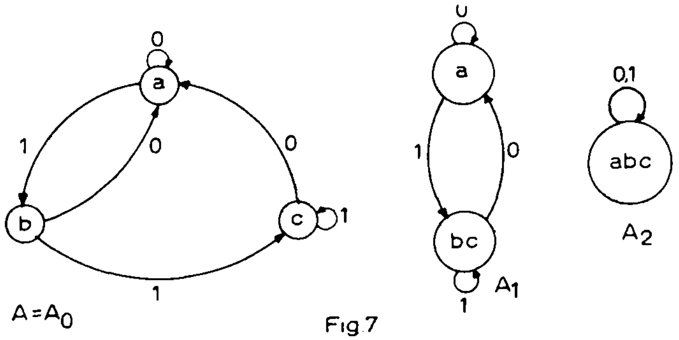


Fig.7

The original automaton appears at the left of Fig. 7, its two endomorphs being shown at right. The endomorph lattice as shown at the left of Fig. 7a, is seen to consist of a single chain giving the composition series

(a b c)

(a, b c)

(a, b, c)

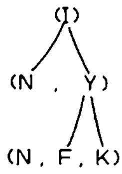


Fig.7a

The single state of A_2 always present irrespective of what the input might be, can be interpreted as a concept of the universal event I, as shown at the top right of Fig. 7a. In A_1, state *a* is always reached after a 0 appears on the input, whereas state *b* responds to a 1. These states then, can be considered as concepts of the events x = 0, which we shall call *N*, and x = 1, denoted by *Y*, as shown on the second row of Fig. 7a. Finally, in A, state *a* is reached whenever a 0 appears in the input, state *b* is the result of a sequence 01 and state *c* responds to the sequence 11. Again, the states correspond to the concepts of the events N: 0, F: 01 and K: 11, as shown on the third row of the figure.

Consider now the state of affairs represented by the bottom line: concepts of three objects appear, namely the events N = no, F = fires, K = keeps firing. Then, as we ascend to the middle row, we find that *N* appears again, but *F* and *K* have been associated to form the more general concept Y = yes, common to *F* and *K*, a property of *F* and *K*. Finally, in reaching the uppermost level, we find *N* and *Y* associated to form the most general concept I = is. Ascension up the chain, corresponds then to a process of abstraction or association which, starting from the bottom with certain objects, abstracts common properties each time more general.

Let us follow now, the opposite process starting this time from the top of the diagram. In so doing, let us keep in mind the idea that our purpose is to recognize the sequence 01. In order to realize such purpose, as a primary requirement, we need, first of all, the ability of perception, the capacity of perceiving an object. This state corresponds to (I). Then, we have to distinguish two kinds of objects, and recognize each one separately as such: 0 and 1. This is the second stage, (N, Y). In the next step we do not find necessary to refine our knowledge of 0, so we just leave it as it is. As to the 1 however, we can use our previous knowledge of 0 and 1 in order to distinguish between two types of 1's: those preceded by a 0, and those preceded by a 1. This is how we separate 01 from 11.

Descending down the chain, amounts then to a process of description by distinction, by separation. Moreover, out of all the possible events, we have effected a selection with a view to the attainment of an end. We have developed a scheme of operation, a program whose instructions read:

I) perceive objects.

II) Distinguish a 0 from a 1.

III) Distinguish a 1 preceeded by 0, from a 1 preceeded by 1.

In following such a program, we find that in spite of the fact that our sole purpose was that of detecting the sequence 01, we were forced (by the program), to detect also a 0 and a 11. Thus, we end up with a knowledge of three objects, 0, 01 and 11. Since nobody is asking us about anything other than 01, we keep the extra information we possess for ourselves, and produce a signal for the outside world, only when a 01 appears. Consequently we assign an output to each state, for ex., 0 to a, 1 to b, 0 to c. In this manner what we really do, is merely to specify the overt behaviour of the automaton.

As a second illustration, suppose a machine, having an input x and an output z, is to operate as follows: all things being considered from a certain initial moment onwards then, if x has never fired, z does not fire. If x has fired at least once, z fires whenever x is not firing.

A kinematic diagram for such a machine is shown on Fig. 8, together with its endomorphs.

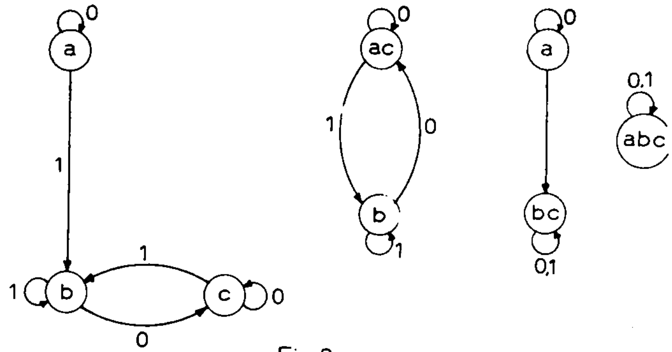


Fig.8

The lattice of endomorphs is shown in Fig. 8a.

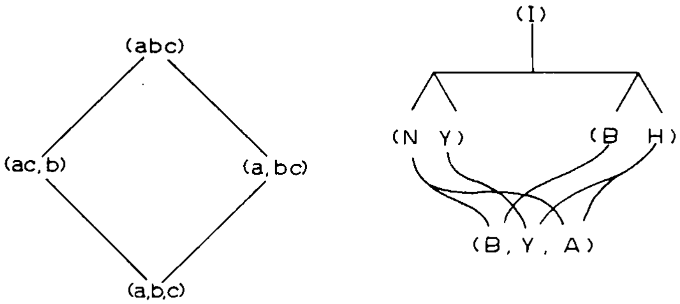


Fig.8a

A_3 corresponds to the universal event I, represented by the compound state abc. A_1 performs the familiar distinction between a 0, event *N* represented by ac, and 1, event *Y* represented by *b*. This time, however, a new feature appears, corresponding to A_2. From the kinematic diagram of A_2 one sees that *a* corresponds to a situation in which no change in the state of the line has ever occurred, whereas bc represents the situation after a first change took place. Consequently *a* represents the event B = before, in the sense of "nothing has happened", whereas bc represents the event H = something happened. On the bottom of the diagram appears A_0. State *a* stands again for the event *B*. This time, however, because of the fact that *a* proceeds also from state ac of A_1, the same *B* appears under a new light: the notion "before", included in *N*, appears refined in the sense that "nothing has happened" means the same as "*x* has never fired". Next state *b*, as derived from A_1, stands for the event *Y*, x = 1. However, because of its inclusion in bc, it is related to H. "Something happened" refers now to the appearance of a first 1 on the line. From there on, the 1's possess a new meaning, approximately conveyed by expressions such as "while *x* is firing", "during the time *x* is firing". Then comes the third state *c*, corresponding to the event we were looking for: it is included in the notions of "no" and "happened". Indeed it could be translated by A = after, meaning "after *x*".

The reader will have certainly noticed, that the requirements on the machine were such that in order to perform the desired operation, a mere distinction between 0's and 1's was not enough. Two more general notions corresponding to "not yet" and "already", or to "before" and "after", were required. This is reflected in the lattice structure which shows now two branches representing two different types of notions, two types of associations or distinctions: one refers to the kind of symbol, whereas the other refers mainly to periods of time. In this branch, the particular shape of symbol is irrelevant, symbols being merely taken as time-markers. Each branch of the lattice represents a different point of view and a different set of concepts.

Besides the processes of abstraction and description, or association and distinction represented in the lattice one can formulate a formal language whose primary symbols are the members of the partitions appearing in the lattice and whose theorems are syntactic truths concerning facts shown by the lattice.

To illustrate the point, let us treat the symbols on the right of Fig. 8a as classes of elements, the elements being the states *a*, *b*, *c*.

The composition of such classes is shown in the following table:

	I	N	Y	B	H	A
a	1	1	0	1	0	0
b	1	0	1	0	1	0
c	1	1	0	0	1	1

Examples of syntactic truths are;
From the bottom line partition,

$$B + Y + A = I \qquad BY + BA + YA = 0$$

From A_1 partition:

$$Y + N = I \qquad Y \cdot N = 0$$

From inclusion relations in the lattice we have abstractive associations:

$$N = B + A \qquad H = Y + A$$

descriptions:

$$B = N \cdot B, \quad Y = Y \cdot H, \quad A = N \cdot H$$

inclusions (implications):

$$B \subseteq N, \quad A \subseteq N$$

$$Y \subseteq H, \quad A \subseteq H$$

and so on.

In general now, if we consider the behaviour of an automaton A following its canonical equations of motion, we can associate each state y to some event E_y.

The set of internal states can be interpreted as a set of names for events. Phase space of A, $\mathcal{Y} = \{y\}$ can be interpreted as a universal class or universe $\mathcal{U}_A = \{E_y\}$ of events, the *universe of A*. $\mathcal{U}_A$ is the set of all events E_y that A can recognize or distinguish. This set can not contain more objects recognized as different, than internal states A has.

In the diagram of $\mathcal{L}(A)$ these objects appear on the bottom or first level, as states of $A_0 = A$. If one follows a path ascending up the lattice, as one reaches higher levels of $\mathcal{L}$, what one finds are endomorphs having less states but whose states are classes of the original objects, formed by association. Events represented by these states are associations of primary events and thus of a more general character. One may then consider ascending paths as abstraction processes, whereby objects that from some point of view or other possess common features, are put together in the same category.

On the contrary, if one follows a descending path, as one comes to lower levels in the lattice one finds that events that appeared as individual units dissociate in two or more events, necessarily of a more particular character. This corresponds to a description process by refinement or addition of distinctive features, that is, of distinction.

Abstractional ascending paths terminate when one reaches a stage such that the only common feature is that of being an event, the identical event I corresponding to the one-state endomorph $\mathbb{O}'A$. Descriptional descending paths terminate when the number of added distinctive features is such as to restrict classes to unit classes containing only single individuals.

Branching of paths in $\mathcal{L}(A)$ corresponds to different possibilities or different points of view. Ascending by abstraction one encounters different alternative associations, and descending by description, alternative distinctions may appear. Thus, in forming a composition series one has, in general, to make a choice among the different alternatives.

The partition Π_t associated to endomorphism t can be considered as a classification of the primary objects into a number of categories equal to the order of the partition. Given two endomorphs t_1 and t_2, then $\Pi_{t_1} \subseteq \Pi_{t_2}$ means that Π_{t_1} is a refinement of Π_{t_2}. Again given t_1 and t_2, $\Pi_{t_1} \cap \Pi_{t_2}$ is the simplest classification that can be made according to the characteristics used in Π_{t_1} *and* the characteristics used in Π_{t_2}. On the other hand, $\Pi_{t_1} \cup \Pi_{t_2}$ is the simplest classification in which the characteristics of Π_{t_1} and Π_{t_2} can be associated as a common feature.

In this manner $\mathcal{L}(A)$ shows the different degrees of associability of the primary events E_y. The higher the level of $\mathcal{L}(A)$ in which two events E_1 and E_2 appear for the first time associated, the lesser will be their associability. Now, according to the point of view of this section one might say that the degree of associability of E_1 and E_2 depends on their nature. Events that under some point of view (branch) have common features, are rapidly associated in low levels of $\mathcal{L}(A)$. Events of dissimilar nature are associated for the first time on higher levels, even on the top level I. On the other hand, from the result of Sec. 2 one sees that associability of states depends on similarities or differences of *variance*, that is, of the kind of transitions undergone by the states under different stimuli. Hence, variance of states, as shown by the kinematic diagram must be related to the nature of the events they represent. In other words, the topology of kinematic diagrams appears to depend on deep laying properties of events.

Another point of interest is that from each lattice one can easily derive a formal language of events with certain syntactic truths as theorems. In a future report it will be shown that all formal languages corresponding to different lattices have common features that allow their inclusion in a formalism of logical operators involving an algebra of events.

As a final remark to this section we want to call the reader's attention upon a fact that perhaps has been somewhat obscured in our analysis. When we have spoken of an automaton A and its endomorphs $t'A$, we have presented kinematic diagrams of the endomorphs and talked about them as if they were independent automata separated from A. The unfamiliar reader should be warned against such point of view, for it would be wrong. As a matter of fact, A contains all the endomorphs, as constituents, not as separate automata.

This can easily be seen as follows: Imagine the kinematic diagram of A and consider the partition Π_{t_1} determined by an endomorph $t'A$. Now, take each member of the partition and enclose the states it contains in a closed curve drawn on the kinematic diagram. The diagram will thus be divided in regions. These regions are the states of $t'A$ and the arrows interconnecting them show their sequence relations. An ascending path in $\mathcal{L}(A)$ represents a process whereby one divides the kinematic diagram in covariant regions, then these regions are included in larger covariant regions, etc., until finally the whole kinematic diagram is enclosed in a curve. The region corresponding to this last step represents the one-state endomorph. We might thus think of the processes of abstraction, description, etc., as occurring within the automaton.

Nothing prevents us, of course, from talking of a separate automaton B say, which behaves in every respect as $t'A$. Such a B, however, would be called a ***homomorph*** but not an ***endomorph*** of A. In fact, B is ***isomorphic*** to an endomorph of A. This terminology corresponds to the common usage in mathematical language.

5. The Output Relation.

This far we have considered only the inner behaviour of A as described by the sequence relations of its inner states. Each such state contains a certain amount of information on the external world of A. This we might call ***internal information.***

Since we want A to communicate all or part of this information to the external world, we must provide an output channel and an output alphabet $\mathfrak{Z} = \{z\}$ of symbols $z_1, z_2, \dots z_p$.

Then we fix the information to be expressed, as well as the manner of expression, by choosing an output function

$$z = g(y) \tag{52}$$

of which we make the following assumptions:

i) $g(y)$ is defined for all states y, i.e., assigns an output to each y,

ii) $g(y)$ is uniform, i.e., it assigns one and only one output to each state.

We can use 'z_j' as a name for a predicate, introducing the formula '$z_j(y)$', meaning "y has the output z_j" thereby treating the output as a property of the state. Consequently, we can use 'z_j' as a name for the abstract

$$z_j = \hat{y}\, z_j(y) \qquad (53)$$

namely, the class of states having the output z_j.

Assumptions i) and ii) concerning (52) are rendered in this language by

$$\text{i)}\quad z_1 \cup z_2 \cup \cdots\cdots \cup z_p = \mathcal{V} \qquad (54a)$$

$$\text{ii)}\quad z_i \cup z_j = \vartheta\ (\text{if } i \neq j) \qquad (54b)$$

(54) shows that assignation of outputs in this manner effects a partition of $\mathcal{V}$ in classes of states having the same output.

From each z_j a partnership relation $\dot{z}_j$ can be formed, defined through

$$\dot{z}_j(y_1, y_2) \equiv .\ z_j(y_1) \,.\, z_j(y_2) \qquad (55)$$

Two states y_1 and y_2 bear to each other the relation $\dot{z}_j$ if and only if both have the output z_j. From (55),

$$\dot{z}_j(y, y) \equiv z_j(y) \qquad (55a)$$

y is $\dot{z}_j$ – related to itself, if and only if it has the output z_j.

It is easy to see** that $\dot{z}_j$ has the properties

**See Appendix II for detailed proofs.

$$\left.\begin{array}{lll} \text{reflexive} & z_j = \breve{\dot{z}}_j & \\ & \text{transitive} & \dot{z}_j^2 \subseteq \dot{z}_j \end{array}\right\} \qquad (56)$$

and further,

$$\left.\begin{array}{l} \dot{z}_i \, \dot{z}_j = \theta \quad \text{if} \quad i \neq j \\ \dot{z}_j^2 = \dot{z}_j \end{array}\right\} \qquad (57)$$

Form now the relation

$$\dot{z} = \dot{z}_1 \cup \dot{z}_2 \cup \ldots\ldots \cup \dot{z}_j \qquad (58)$$

which we call the *output relation*. From its definition it is clear that two states bear relation $\dot{z}$ to each other, if and only if they have the same output, irrespective of what the output might be.

$\dot{z}$ can be easily shown** to be reflexive, symmetric and transitive, its equivalence classes being precisely classes of states having the same output.

Let now t be an endomorphism of A. We say that with respect to the overt behaviour of A, t is *acceptable*, if it preserves outputs. t must be such that the t-image η of a state y, has the same output as y:

$$t^{*} z_j \subseteq z_j \qquad (59)$$

Classes z_j must be closed under t. In other words, acceptable endomorphisms must propagate each property z_j, which in turn will be hereditary under t.

**See Appendix II for detailed proofs.

(59) implies** that t commutes with every $\dot{z}_j$:

$$[t, \dot{z}_j] = \mathcal{O} \tag{60}$$

and hence, with the output relation:

$$[t, \dot{z}] = \mathcal{O} \tag{61}$$

(61) is, however, weaker than (60). On the other hand from (59) or (60) one can derive the equivalent condition

$$t \subseteq \dot{z} \tag{62}$$

Taking inverses in (62) remembering the symmetric character of $\dot{z}$, one obtains

$$\breve{t} \subseteq \dot{z}$$

which, multiplied orderly by (62) yields

$$\breve{t}t \subseteq \dot{z}^2 = \dot{z}$$

Since t is an endomorphism, one has

$$\mathcal{I} \subseteq \breve{t}t \subseteq \dot{z} \tag{63}$$

for all acceptable endomorphisms t.

**See Appendix II for detailed proofs.

Since (63) can also be written as

$$\breve{\mathcal{I}}\,\mathcal{I} \subseteq \breve{t}\,t \subseteq \breve{\dot{z}}\,\dot{z}$$

by virtue of (43) one may write

$$\mathcal{I} \mathrel{\dot{\subseteq}} t \mathrel{\dot{\subseteq}} \dot{z} \tag{64}$$

where lattice inclusion has been used.

Acceptable endomorphisms can be defined through (64).

It is easily shown** that if t_1 and t_2 are acceptable, so will be $t_1 \mathbin{\dot{\cap}} t_2$ and $t_1 \mathbin{\dot{\cup}} t_2$. Hence, the set of acceptable endomorphisms, $\mathcal{L}_2(A)$, is a lattice, this being a sub-lattice of $\mathcal{L}(A)$.

Because of (64), $\mathcal{L}_2(A)$ is bounded by $\mathcal{I}$ and $\dot{z}$. It can be shown** that $\mathcal{L}_2(A)$ contains an absolute upper bound t_0, such that

$$\mathcal{I} \mathrel{\dot{\subseteq}} t_0 \mathrel{\dot{\subseteq}} \dot{z} \tag{65}$$

and that for every t, fulfilling (64),

$$t \mathrel{\dot{\subseteq}} t_0 \tag{65a}$$

t_0 is maximal in $\dot{z}$, and unique. It is the highest level endomorphism that can be lattice-included in $\dot{z}$. The automaton $t_0'A$ is then the endomorphic image of A having the least number of states, whose overt behaviour is the same as that of A.

If $t'A$ is an acceptable endomorph of A with $t \neq \mathcal{I}$, an automaton B iso-

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**See Appendix II.

morphic to $t'A$ can be considered as a *reduction* of A. If B_0 is isomorphic to $t_0'A$ then B_0 is the *minimal form* of A. All states of A and its reduced forms B, not appearing in B_0, are considered as *redundant*.

If the reader recalls Huffman's reduction procedure[2] under the light of these results, as well as those of Sec. 2, he will realize that Huffman's procedure is a method of reduction that leads directly to the obtention of the minimal form of A.

A particular case of interest arises when each state has a different output. Then, except for an unessential coding, one can take (52) as

$$z = g(y) = y \tag{66}$$

Then, all the z_j's are unitary classes;

$$z_j = \lfloor y_j \tag{67}$$

partnership relations are unitary;

$$\dot{z}_j = \lfloor (y_j ; y_j) \tag{68}$$

and the corresponding output relation is the identical;

$$\dot{z} = \mathcal{I} \tag{69}$$

then, from (64) one sees that the only acceptable endomorphism is

$$t = \mathcal{I} \tag{70}$$

hence $t_0 = \mathcal{I}$ and A itself is the only acceptable endomorph.

Hence, in discussing the internal information of A, there is no sense in talking about redundancies or reductions.

The results of previous sections are applicable even in case outputs are considered, since they refer to the inner behaviour of A. It is only when one wishes to carry on a reduction, that acceptability has to be taken into account.

As a final example to illustrate the point, as well as some other features of our lattices, consider the neuron net shown in Fig. 9, having three neurons, p, q, r.

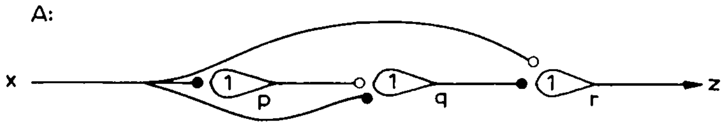

Fig.9

The states of the net are the following combinations of values of p, q, r:

State	a	b	c	d
p q r	000	110	001	100

for quiescent initial conditions, all other combinations are excluded.

The kinematic diagram of A is shown in Fig. 9a, the lattice of endomorphs on the left of Fig. 9b and the event version of the lattice, or *program*, at the right of Fig. 9b.

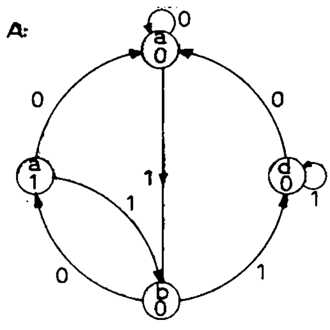

Fig.9a

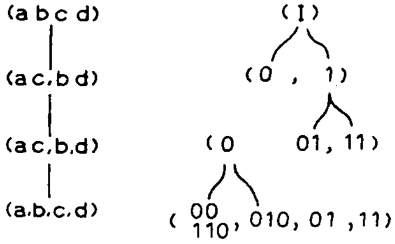

Fig.9b

The purpose of A is evidently to recognize the event 010, corresponding to state c. Fig. 9b shows how this is done. First A can respond to stimuli: endomorph (I). Then, it distinguishes a 0 from a 1, this being the function of neuron p. Then, this information is used on a lower level to separate a 01 from a 11 and from a 0. This is a result of the combined action of neurons p and q. Finally, on the bottom level, the available information: 0, 01, 11, is used to separate a 0 preceded by 01, i.e., 010 (state c) from a 0 preceded by either of the other two known elements, 0 and 11, i.e., 00 or 110 (state a). 01 (state b) and 11 (state d) are merely passed down to this level. This last step is a result of the combined actions of p, q and r.

Observe the strategy followed by A to do its job, as shown by the program of Fig. 9b. Its purpose is to recognize 010. First, it observes 0's and 1's. Then, separates 01's from 11's, and puts them aside of 0's. Finally, when a 0 comes after 01, it recognizes 010 and fires z.

Consider now net $\bar{A}$ shown on Fig. 10, which has the same purpose. It also has three neurons, p, n, s. Its states, however are:

State	a	b	c	d	e
p n s	000	100	011	110	010

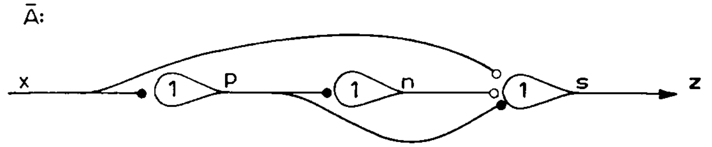

Fig.10

Kinematic diagram and lattice are shown in Figs. 10a and 10b.

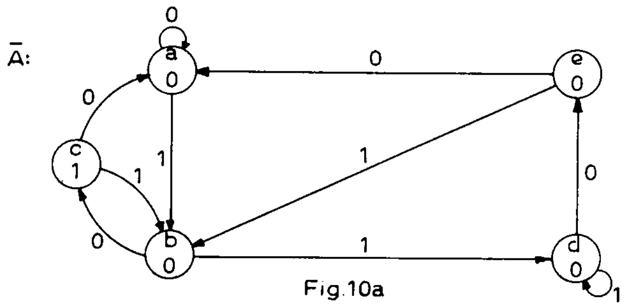

Fig.10a

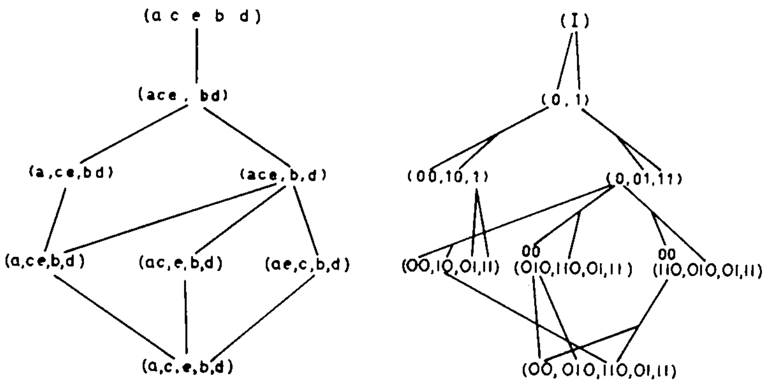

Fig.10b

The reader will immediately notice that, in spite of an identical overt behaviour: z fires with 010, notwithstanding the fact that both A and $\bar{A}$, have ***the same number of neurons, the same number of synapses and terminals,*** inner behaviours are quite different.

A is the practical sort of fellow that, in order to do his job, collects just the required information and goes directly to his end. Moreover, he is always prepared having the right information at the right time, so that when the expected event appears, with a very simple final action reaches his goal.

$\bar{A}$ on the other hand, decidedly appears as the contemplative type that, in order to tell about a fact (010), gathers all sorts of information, whether pertinent to his purpose or not.

This difference of strategy shows itself in the different manner of recognizing the eve- .

Assume both, A and $\bar{A}$, start in state a with x = 0. Now, if a 1 comes, both go to state b, corresponding to 01. Finally, if a 0 arrives, both go to state c and z fires.

This looks quite similar. But, what happens to neurons? Using the tables of states, let us compare both actions.

	1st.	2nd.	3rd.	
A	000	000	110	001
$\bar{A}$	000	000	100	011
appears	0	1	0	

Prevision observed in A's program shows on its actions: after a 01 has appeared, the possibilities for the occurrence of the desired event have increased. A in state 110 shows its maximum tension: 2 neurons fired which from the table of states the reader can verify to be the most A reaches, whereas $\bar{A}$ appears to have scarcely noticed the situation: state 100. Finally, when the decisive moment arrives, A, in passing from 110 to 001 seems rather to relax, using just the energy required to fire z. $\bar{A}$ on the contrary, seems to have been caught by surprise: from 100 goes to 011, a state of maximum tension. The total number of firings is the same, three in both cases, but they are differently used.

The writer wished to ask the kind reader's forgiveness for this little piece of fancy, having as only excuses for the dramatization of the tale, his desire to fade somewhat the sour flavour of a rather long and boresome paper overloaded with unimportant trivialities as mathematical equations and his desire to emphasize certain serious questions.

In case of $\bar{A}$, states a, b, d, e have output 0, c has output 1. The partition corresponding to $\dot{z}$ is (abde, c). From the lattice of Fig. 10b, one sees that the lattice of acceptable endomorphs is

$$\begin{array}{c}(ae, c, b, d)\\ | \\ (a, e, c, b, d)\end{array}$$

Thus, one might say that states a and e are redundant. Indeed, combining states a and e in Fig. 10a, one obtains precisely the kinematic diagram of A.

Now, as we already know, in the first place, elimination of the redundancy does not cause any lessening of the number of neurons.

Next, a given structure of given logical elements, be they real or imaginary determines a lattice of endomorphs and a corresponding program closely related to the actual function of the elements.

Then, in the so-called logic nets, there is plenty of more logic to be considered, than that contained in propositional equations.

Finally, since the structure of the lattice is determined by the logical properties of the elements of the net and their interconnections, the program displayed by the lattice must reflect, in some way or other, the actual functions of the said elements.

In this respect a concept derived from lattice considerations. that of *strategy of operation* or *program*, might prove very useful for comparing and judging practical nets. Admitted that the net's operation might be satisfactory under the usual points of view, one still may consider *the manner* in which the operation is effected. This is what "program" means. As it may be apparent from our example, in carrying on such an analysis one may find striking surprises.

Certain words as "prevision", "preparedness", "readyness", "surprise", etc., that were used in the comparison of A and $\overline{A}$ are very far, indeed, from mere jokes. There are certain features in the operation of a net that, related or not to any lattice, have a definite significance. For example, it is not solely the total energy consumed in an operation, the only thing that matters. Every engineer will agree that the way in which this energy is administered by the system during the process, has a great deal of importance. Again, there is nothing wrong in saying, for example, that a net devised for the recognition of a certain event acts with prevision, if it spends little energy in working on observations which very unlikely lead to a completion of the event, distributes adequately its energy content increasing reasonably its strength when incoming data are such so as to increase the probability of completion of the event, relaxing when this is completed, in order to wait in a low energy level for the appearance of new significative data. The opposite behaviour would undoubtedly be senseless.

It seems that in case of finite, discrete automata, these ideas, intuitively used by designers, are displayed in lattices.

APPENDIX I

THE LATTICE OF ENDOMORPHS

The identical transformation $\mathcal{J}$ sending each state into its equal, is, trivially, a transformation. But the identity relation commutes with every other relation.

$$[x_1\ \mathcal{J}] = 0 \tag{A.1}$$

hence, $\mathcal{J}$ is an endomorphism.

The transformation sending every state y onto a single state $I = \mathbb{O}'y$ will sent the x-sequent of y, $x'y$, necessarily onto the same state I. Hence

$$I = \mathbb{O}'x'y = \mathbb{O}x'y \quad \ldots. (a)$$

Now the x-sequent of I can not be anything but I itself:

$$I = x'I = x'\mathbb{O}'y = x\,\mathbb{O}'y \ \ldots. (b)$$

from (a) and (b) we see that for any y

$$x\,\mathbb{O}'y = \mathbb{O}x'y$$

whence

$$\left.\begin{array}{rl} x\,\mathbb{O} & = \mathbb{O}x \\ \text{or} \quad [x_1\ \mathbb{O}] & = 0 \end{array}\right\} \tag{A.2}$$

and $\mathbb{O}$ is an endomorphism.

Assume now that t_1 and t_2 are two endomorphisms of A $\Pi_{t_1} = [\breve{t}_1" \lfloor \eta]$ and $\Pi_{t_2} = [\breve{t}_2" \lfloor \xi]$ being the associated partitions, $t_1'A$ and $t_2'A$ the corresponding endomorphs. Assume further, that

$$\breve{t}_1 \, t_1 \subseteq \breve{t}_2 \, t_2 \tag{A.3}$$

The members of Π_{t_1} are t_1-equivalence classes. If $y_1, y_2 \in \breve{t}_1" \lfloor \eta$, both are related by $\breve{t}_1 \, t_1$. But (A.3) requires that both be related by $\breve{t}_2 \, t_2$, and thus, belong to some member, $\breve{t}_2" \lfloor \xi$ of Π_{t_2}. Hence, for any member $\breve{t}_1" \lfloor \eta$ of Π_{t_1}, there exists a member $\breve{t}_2" \lfloor \xi$ of Π_{t_2} such that $\breve{t}_1" \lfloor \eta \subseteq \breve{t}_2" \lfloor \xi$. Formally

$$(\eta)(\exists \xi)(\breve{t}_1" \lfloor \eta \subseteq \breve{t}_2" \lfloor \xi) \tag{A.4}$$

which means that

$$\Pi_{t_1} \subseteq \Pi_{t_2} \tag{A.5}$$

Moreover, from (A.4) one sees that each class $\breve{t}_2" \lfloor \xi$ must contain an integral number (possibly 1) of classes $\breve{t}_1" \lfloor \eta$. Let then τ be the transformation which sends each η for which $\breve{t}_1" \lfloor \eta \subseteq \breve{t}_2" \lfloor \xi$ to the same ξ. τ then maps each state η of $t_1'A$ onto some state ξ of $t_2'A$. We can look at τ as a relation $\tau(\xi, \eta)$ where $\xi = \tau' \eta$. Besides, τ effects a partition Π_τ of the states of $t_1'A$ in τ-equivalence classes $\breve{\tau}" \lfloor \xi$, one for each state ξ of $t_2'A$. The retrojection of $\breve{\tau}" \lfloor \xi$ by t_1, namely, $\breve{t}" \breve{\tau}" \lfloor \xi = \breve{t}_1 \, \breve{\tau}" \lfloor \xi$ is obviously the class $\breve{t}_2" \lfloor \xi$, that is,

$$\breve{t}_2" \lfloor \xi = \breve{t}_1 \, \breve{\tau}" \lfloor \xi \tag{A.6}$$

which means that $(\xi)(y)\Big(t_2(\xi, y) \equiv \tau t_1(\xi, y)\Big)$
hence

$$t_2 = \tau t_1 \tag{A.7}$$

The definition of τ translated to formal language is

$$\tau(\xi, \eta) \equiv . \ \breve{t}_1" \lfloor \eta \subseteq \breve{t}_2" \lfloor \xi$$

or

$$\tau(\xi, \eta) \equiv (y)\Big(\breve{t}_1(y, \eta) \supset \breve{t}_2(y \xi)\Big) \tag{A.8}$$

however, since $t_1'' \lfloor \eta = 0$, that is, since

$$(\exists y) \ \breve{t}_1(y, \eta)$$

is true for any η, by a well known theorem of quantification we obtain from (A.8) the consequence

$$(\exists y)\Big(\breve{t}_1(y, \eta) . \ t_2(\xi, y)\Big)$$

which amounts to

$$t_2 \breve{t}_1(\xi, \eta)$$

Consequently

$$(\xi)(\eta)\Big(\tau(\xi, \eta) \supset t_2 \breve{t}_1(\xi, \eta)\Big)$$

or

$$\tau \subseteq t_2 \overset{\cup}{t}_1 \tag{A.9}$$

Let now y be any state of A, and let

$$\eta = t_1' y \qquad \xi = t_2' y \tag{a}$$

by (A.7), for all y,

$$t_2' y = \tau t_1' y \tag{b}$$

and by virtue of (a),

$$\xi = \tau' \eta \tag{c}$$

Now, since t_1 and t_2 are endomorphisms, from (a) we get

$$x' \eta = x t_1' y = t_1 x' y \tag{d}$$

and

$$x' \xi = x t_2' y = t_2 x' y \tag{e}$$

(b) however, being true for any y, will be true for $x'y$.
Hence

$$t_2 x' y = \tau t_1 x' y \tag{f}$$

Substituting in this expression $t_2 x' y$ as given by (e) and $t_1' x' y$ as given by (d), one obtains

$$x'\xi = \tau x'\eta \qquad \text{(g)}$$

finally, substituting in (g) the expression (c) for ξ, one sees that

$$x\tau'\eta = \tau x'\eta \qquad \text{(h)}$$

for every η. Hence

$$x\tau = \tau x$$

for any x. So

$$[x, \tau] = 0 \qquad \text{(A.10)}$$

and τ is an endomorphism.

Conversely, if $t_1'A = A_1$ is an endomorph of A and $\tau' A_1 = A_2$ is an endomorph of A_1, each state ξ of A_2 is a τ of a state η of A_1, which in turn is a t_1 of a state y of A. Thus, $\xi = \tau t_1' y = t_2' y$ where $t_2 = \tau t_1$ is a many-to-one mapping of A onto A_2. Moreover, since τ and t_1 are endomorphisms,

$$x t_2 = x\tau t_1 = \tau x t_1 = \tau t_1 x = t_2 x$$

so t_2 is an endomorphism and $A_2 = t_2' A$.

Moreover, τ being an endomorphism, fulfills the condition $\mathcal{I} \subseteq \breve{\tau}\tau$, from which we obtain

$$\breve{t}_1 t_1 \subseteq \breve{t}_1 \breve{\tau}\tau t_1 = (\tau t_1)^{\smile} \tau t_1 = \breve{t}_2 t_2$$

that is

$$\breve{t}_1 t_1 \subseteq \breve{t}_2 t_2$$

which allows the conclusions

$$\Pi_{t_1} \subseteq \Pi_{t_2}$$

and

$$t_1'A \subseteq t_2'A$$

Hence, a necessary and sufficient condition for $t_2'A$ to be an endomorph of $t_1'A$, i.e., for the existence of an endomorphism τ of $t_1'A$ such that $\tau t_1 = t_2$, is that $\breve{t}_1 t_1 \subseteq \breve{t}_2 t_2$.

Moreover, we have also proved that an endomorph of an endomorph is in turn an endomorph, namely that $\dot{\subseteq}$ as a relation between automata is transitive. Since it is trivially reflexive, it provides a partial ordering among automata.

The total endomorphism $\mathbb{O}$ maps every state y onto a single state. Hence $\breve{\mathbb{O}}\mathbb{O}$ consists of one member, this member containing all pairs of states. Then, any endomorphism t must fulfill

$$\breve{t}t \subseteq \breve{\mathbb{O}}\mathbb{O}$$

We showed (eq. 17.- of the text) that also

$$\breve{\mathcal{I}}\mathcal{I} \subseteq \breve{t}t$$

combining these results one obtains

$$\breve{\mathcal{I}}\mathcal{I} \subseteq \breve{t}t \subseteq \breve{\mathbb{O}}\mathbb{O} \qquad \text{(A.11)}$$

thereby establishing the existence of universal bounds.

In order to obtain the remainding results of Sec. 3, assume t_1 and t_2 are two endomorphisms of A, and let for the moment

$$r = \breve{t}_1 t_1 \qquad\qquad s = \breve{t}_2 t_2$$

r and s are reflexive, symmetric and transitive.

We want to prove first, that $r \cap s$ and $\#(r \cup s)$ are reflexive, symmetric and transitive.

For the proof consider that reflexivity of r and s means that $\mathcal{I} \subseteq r$ and $\mathcal{I} \subseteq s$, so that

$$\mathcal{I} \subseteq r \cap s \qquad \text{(a)}$$

shows $r \cap s$ to be reflexive.

Multiply now $\mathcal{I} \subseteq r$ and $\mathcal{I} \subseteq s$ by s and r respectively, first from the left, then from the right. One gets

$$s \subseteq sr \qquad\qquad s \subseteq rs$$

$$r \subseteq rs \qquad\qquad r \subseteq sr$$

so

$$r \subseteq rs \cup sr, \qquad\qquad s \subseteq rs \cup sr$$

and

$$r \cap s \subseteq rs \cup sr \qquad \text{(b)}$$

From (b)

$$r \cap s \cap rs \cap sr = r \cap s \qquad \text{(c)}$$

Since r and s are transitive,

$$r^2 \subseteq r \quad \text{and} \quad s^2 \subseteq s$$

then $(r \cap s)^2 \subseteq r^2 \cap s^2 \cap rs \cap sr \subseteq r \cap s \cap rs \cap sr = r \cap s$ by (c). Hence,

$r \cap s$ fulfills

$$(r \cap s)^2 \subseteq r \cap s \qquad \text{(d)}$$

and is transitive.

Finally, since $r = \breve{r}$ and $s = \breve{s}$ are symmetric,

$$(r \cap s)^{\smile} = \breve{r} \cap \breve{s} = r \cap s \qquad \text{(e)}$$

is in turn symmetric.

As to $\#(r \cup s)$, by the very definition of ancestral immediately follows that it is reflexive.

$$\mathcal{I} \subseteq \#(r \cup s) \qquad \text{(f)}$$

transitive

$$\left(\#(r \cup s)\right)^2 = \#(r \cup s) \qquad \text{(g)}$$

and further

$$\left(\#(r \cup s)\right)^{\smile} = \#(r \cup s)^{\smile} = \#(\breve{r} \cup \breve{s}) = \#(r \cup s) \qquad \text{(h)}$$

is symmetric.

Hence, $r \cap s = \breve{t}_1\, t_1 \cap \breve{t}_2\, t_2$

and $\#(r \cup s) = \#(\breve{t}_1\, t \cup \breve{t}_2\, t_2)$

determine partitions of the elements of $\mathcal{Y}$, in equivalence classes.

Let $\Pi_{t_1} = [\breve{t}_1'' \lfloor \eta_1]$ and $\Pi_{t_2} = [\breve{t}_2'' \lfloor \eta_2]$ be the partitions corresponding to $\breve{t}_1\, t_1$ and $\breve{t}_2\, t_2$ resp., η_1 and η_2 being the states of $t_1' A$ and $t_2' A$.

For any state $y \in \mathcal{Y}$, there exist η_1 and η_2 such that $y \in \breve{t}_1'' \lfloor \eta_1$ and $y \in \breve{t}_2'' \lfloor \eta_2$. These classes contain the t_1-partners and t_2-partners of y.

Now, y must fall in one of the equivalence classes of $\breve{t}_1 t_1 \cap \breve{t}_2 t_2$ having as partners those states simultaneously being t_1-partners and t_2-partners of y, therefore the states contained in $\breve{t}_1'' \lfloor \eta_1 \cap \breve{t}_2'' \lfloor \eta_2$.

Hence, each member of the partition determined by $\breve{t}_1 t_1 \cap \breve{t}_2 t_2$ is an intersection of one member of Π_{t_1} with one member of Π_{t_2}. Thus, the equivalence relation $\breve{t}_1 t_1 \cap \breve{t}_2 t_2$ determines the partition $\Pi_{t_1} \cap \Pi_{t_2}$.

From each member of $\Pi_{t_1} \cap \Pi_{t_2}$ select one state η, and let t be the transformation that sends all states y of each member to state η. Clearly t is a many-to-one mapping of $\mathcal{Y}$ onto the subset of η's, the members of $\Pi_{t_1} \cap \Pi_{t_2}$ are the classes $\breve{t}'' \lfloor \eta$ of t-partners related by $\breve{t}t$.

Hence

$$\breve{t}t = \breve{t}_1 t_1 \cap \breve{t}_2 t_2 \qquad \text{(A.12)}$$

and

$$\Pi_t = \Pi_{t_1} \cap \Pi_{t_2} \qquad \text{(A.13)}$$

Let now

$$\eta_1 = t_1' y \qquad \eta_2 = t_2' y \qquad \tau = t' y \qquad \text{(a)}$$

where, by construction,

$$\breve{t}'' \lfloor \eta = \breve{t}_1'' \lfloor \eta_1 \cap \breve{t}_2'' \lfloor \eta_2 \qquad \text{(b)}$$

Since t_1 and t_2 are endomorphisms,

$$x' \eta_1 = x t_1' y = t_1 x' y \qquad \text{(c)}$$

and $$x'\eta_2 = x t_2' y = t_2 x' y \qquad \text{(d)}$$

(c) and (d) mean, however that

$$x'y \in \breve{t}_1'' \lfloor (x'\eta_1)$$

and $$x'y \in \breve{t}_2'' \lfloor (x'\eta_2)$$

therefore

$$x'y \in \breve{t}_1'' \lfloor (x'\eta_1) \cap \breve{t}_2'' \lfloor (x'\eta_2) \qquad \text{(e)}$$

By construction, however, η itself fulfills (a) since it is one of the states of class (b). Hence, it also fulfills (e);

$$x'\eta \in \breve{t}_1'' \lfloor (x'\eta_1) \cap \breve{t}_2'' \lfloor (x'\eta_2) \qquad \text{(f)}$$

We can choose the state $x'\eta$ as the representative η for the class appearing on (e) and (f). Then, according to (b)

$$\breve{t}'' \lfloor (x'\eta) = \breve{t}_1'' \lfloor (x'\eta_1) \cap \breve{t}_2'' \lfloor (x'\eta_2) \qquad \text{(g)}$$

whence (e) becomes

$$x'y \in \breve{t}'' \lfloor (x'\eta)$$

which amounts to

$$x'\eta = t x' y \qquad \text{(h)}$$

Substituting the value of η from (a), we get

$$xt'y = tx'y$$

and since this holds for every y, must be

$$xt = tx$$

or

$$[x, t] = 0 \qquad \text{(A.14)}$$

so t is an endomorphism. The endomorph $t'A$ will be denoted bv

$$t'A = t_1'A \cap t_2'A \qquad \text{(A.15)}$$

From (A.12) one sees that

$$\breve{t}t \subseteq \breve{t}_1 t \qquad \text{and} \qquad \breve{t}t \subseteq \breve{t}_2 t$$

which according to previous results implies

$$t'A \subseteq t_1'A, \qquad t'A \subseteq t_2'A \qquad \text{(A.16)}$$

showing that $t'A$ is a l.b. of $t_1'A$ and $t_2'A$.

But (A.12) shows $\breve{t}t$ to be the G.L.B. of $\breve{t}_1 t_1$ and $\breve{t}_2 t_2$. The corresponding property for $t'A$ follows immediately, from which we conclude that every pair of endomorphs have a G.L.B.

Consider now the ancestral $\#(\breve{t}_1 t_1 \cup \breve{t}_2 t_2)$. Being, as shown before, an equivalence relation, it effects a partition of the states of $\mathcal{Y}$ in equivalence classes.

In order to determine what these classes might be, take a state $y \in \mathcal{Y}$. Its partners are the states

$$\#(\breve{t}_1 t_1 \cup \breve{t}_2 t_2)" \lfloor y .$$

However, by definition of ancestral, this class is the smallest class that contains y and is closed under the relation $\breve{t}_1 t_1 \cup \breve{t}_2 t_2$.

Hence, besides y, the class must contain the t_1-partners of y, namely the whole member of Π_{t_1} to which y belongs, b_1, say. But then, it must contain the t_2-partners of y, i.e., the whole member of Π_{t_2} to which y belongs, say k_1. Since both, b_1 and k_1 contain y, $b_1 \cap k_1 \neq \Lambda$, are conjoint. Now, if $b_1 = k_1$, the class $l_1 = b_1 = k_1$ is closed under $\breve{t}_1 t_1 \cup \breve{t}_2 t_2$ and is one of the equivalence classes we are looking for. But if $b_1 \neq k_1$, the class $k_1 \cup b_1$ will not be closed due to the presence of elements in $k_1 \cap b_1'$ or in $b_1 \cap k_1'$ or in both. Hence we have to add all t_1-partners of the elements of $b_1 \cap k_1'$, that is all members of Π_{t_1} conjoint to b_1, and all t_2-partners of the elements of $k_1 \cap b_1'$, that is, all members of Π_{t_2} conjoint to k_1.

If the class so obtained consists of a whole number of members of Π_{t_1} and a whole number of members of Π_{t_2}, it will be closed and thus, one of the equivalence classes l_1, say, we are looking for. If not, there will be some t_1-partners lacking t_2-partners, and/or conversely. The class must be enlarged by adding all t_2-partners of the lone t_1-partners and all t_1-partners of the lone t_2-partners, etc. The process stops when the class contains a whole number of t_2-members.

Each equivalence class will be then of the form

$$l = b_{i_1} \cup b_{i_2} \cup \cdots\cdots \cup b_{i_p} = k_{j_1} \cup k_{j_2} \cup \cdots\cdots \cup k_{j_q}$$

with all its component members chain-connected.

The partition corresponding to $\#(\breve{t}_1 t_1 \cup \breve{t}_2 t_2)$ is then $\Pi_{t_1} \cup \Pi_{t_2}$.

Out of each member of this partition, select a state η and call t the transformation sending each state y of the member to this η. The members of $\Pi_{t_1} \cup \Pi_{t_2}$ are then describable as classes $\breve{t}" \lfloor \eta$ of t-partners related by $\mathring{t}t$, so that

$$\breve{t}t = \#(\breve{t}_1 t_1 \cup \breve{t}_2 t_2) \tag{A.17}$$

and

$$\Pi_t = \Pi_{t_1} \cup \Pi_{t_2} \tag{A.18}$$

Now, as it was shown under (A.13), if a state y lays in the intersection of a t_1-class and a t_2-class, namely if

$$y \in \mathring{t}_1" \lfloor \eta_1 \cap \mathring{t}_2" \lfloor \eta_2$$

then

$$x'y \in \breve{t}_1" \lfloor (x'\eta_1) \cap \breve{t}_2" \lfloor (x'\eta_2)$$

Hence, if a t_1-class and a t_2-class are conjoint, their x-transforms are again conjoint. Therefore, a set of chain-connected t_1- and t_2-classes, transforms under x is a set of the same type. Hence, if

$$\eta = t'y \tag{a}$$

the set $\breve{t}" \lfloor \eta$ transforms under x into a set $\breve{t}" \lfloor \bar{\eta}$ say, where $\bar{\eta}$ is again one of the η's. Then

$$\bar{\eta} = tx'y \tag{b}$$

But since η itself belongs to $\mathring{t}'' \lfloor \eta$, there follows that $x'\eta \in \breve{t}'' \lfloor \bar{\eta}$. Hence, we can choose $\bar{\eta}$ as $x'\eta$,

$$\bar{\eta} = x'\eta \qquad \text{(c)}$$

Substituting in (b) we get

$$x'\eta = tx'y$$

and using (a),

$$xt'y = tx'y$$

for all y, whence

$$xt = tx$$

or

$$[x, t] = 0 \qquad \text{(A.19)}$$

t is then an endomorphism. The endomorph $t'A$ will be denoted by

$$t'A = t_1'A \cup t_2'A \qquad \text{(A.20)}$$

From (A.17) one sees that

$$\breve{t}_1 t_1 \subseteq \breve{t} t \qquad \text{and} \qquad \mathring{t}_2 t_2 \subseteq \mathring{t} t$$

and this implies that

$$t_1'A \subseteq t'A \qquad t_2'A \subseteq t'A \qquad \text{(A.21)}$$

showing that $t'A$ is an u.b. of $t_1'A$ and $t_2'A$.

Any other endomorphism τ such that $\breve{t}_1 t_1 \subseteq \breve{\tau}\tau$ and $\breve{t}_2 t_2 \subseteq \breve{\tau}\tau$ yields $\breve{t}_1 t_1 \cup \breve{t}_2 t_2 \subseteq \breve{\tau}\tau$. Then any class closed under τ must be closed under $\breve{t}_1 t_1 \cup \breve{t}_2 t_2$ which means that $\#(\breve{t}_1 t_1 \cup \breve{t}_2 t_2) \subseteq \breve{\tau}\tau$, i.e., $\breve{t} t \subseteq \breve{\tau}\tau$, so $t'A \subseteq \tau'A$. $t'A$ is then seen to be the l.u.b. of $t_1'A$ and $t_2'A$.

Every pair of endomorphs has then a G.L.B. and a l.u.b. The set $\mathcal{E}(A)$ of endomorphs of A is a lattice which, by virtue of (A.11) is bounded.

This completes the proof of the statements asserted on Sec. 3.

APPENDIX II

Reflexivity of $\dot{z}_j$ is immediate from the definition, Eq. (55) of the text. Transitivity, rather obvious, is however formally proved as follows:

$$\dot{z}_j^2(y_1, y_2) \equiv (\exists y)\Big(\dot{z}_j(y_1, y) \cdot \dot{z}_j(y, y_2)\Big) \equiv$$

$$\equiv (\exists y)\Big(z_j(y) \cdot z_j(y_1) \cdot z_j(y_2)\Big) \equiv$$

$$\equiv \dot{z}_j(y_1, y_2) \cdot (\exists y)\, z_j(y) \equiv$$

$$\equiv \dot{z}_j(y_1, y_2)$$

So $\dot{z}_j^2 = \dot{z}_j$ is idempotent and hence, transitive.

For $i \neq j$

$$\dot{z}_i \dot{z}_j(y_1, y_2) \equiv (\exists y)\Big(\dot{z}_i(y_1, y) \cdot \dot{z}_j(y, y_2)\Big) \equiv$$

$$\equiv (\exists y)\Big(z_i(y_1) \cdot z_j(y_2) \cdot z_i(y) \cdot z_j(y)\Big) \equiv$$

$$\equiv z_i(y_1) \cdot z_j(y_2) \cdot (\exists y)\Big(z_i(y) \cdot z_j(y)\Big)$$

Since for $i \neq j$ there is no such y, the proposition is false for every pair $y_1 ; y_2$ and

$$\dot{z}_i \dot{z}_j = \mathfrak{N} \text{ if } i \neq j$$

As to $\dot{z}$, Eq. (58) since $\dot{z} = \bigcup \dot{z}_j$, then $\breve{\dot{z}} = \bigcup \breve{\dot{z}}_j = \bigcup \dot{z}_j = z$ is symmetric.

$$\dot{z}^2 = \bigcup_{ij} \dot{z}_i \dot{z}_j = \bigcup_i \dot{z}_i^2 = \bigcup_i \dot{z}_i = \dot{z}$$

is idempotent and transitive.

Then, from (54a) and (55a) of the text,

$$\dot{z}(y, y) = \sum_j \dot{z}_j(y, y) = \sum_j z_j(y)$$

is always true. Hence, so it is

$$(y)\Big(\dot{z}(y, y) \supset \mathcal{I}(y, y)\Big)$$

whence $\mathcal{I} \subseteq \dot{z}$, and $\dot{z}$ is transitive. In short,

$$\left.\begin{array}{l} \breve{\dot{z}} = \dot{z} \\ \dot{z}^2 = \dot{z} \\ \mathcal{I} \subseteq \dot{z} \end{array}\right\} \qquad \text{(A.22)}$$

Let now y be any state, having the t-image

$$\eta = t' y \qquad \text{(a)}$$

there exists a j such that $y \in z_j$. Then, by virtue of Eq. (59) of the text, $\eta \in z_j$. Since both belong to z_j,

$$\dot{z}_j'' \,\lfloor\, \eta = \dot{z}_j'' \,\lfloor\, y \qquad \text{(b)}$$

From (a)

$$\lfloor \eta = t'' \lfloor y \qquad \text{(c)}$$

From (b) and because of closure,

$$z_j'' \lfloor \eta = t z_j'' \lfloor y \qquad \text{(d)}$$

Substituting (c):

$$z_j \, t'' \lfloor y = t z_j'' \lfloor y$$

whence $z_j t = t z_j$, from which Eq. (60) follows. (61) results then readily from (58) and (60).

Again according to (59),

$$t(\eta, y) \cdot z_j(y) \cdot \supset z_j(\eta),$$

take conjunction of both sides with $z_j(y)$ using (55);

$$t(\eta, y) \cdot z_j(y) \cdot \supset \dot{z}_j(\eta, y)$$

take alternation with respect to j remembering (54a) and (58);

$$t(\eta, y) \supset \dot{z}(\eta, y)$$

for all η and y. Hence

$$t \subseteq \dot{z}$$

which is Eq.(62).

Assume now that t_1 and t_2 are two acceptable endomorphisms. Then, by (63),

$$\breve{t}_1 t_1 \subseteq \dot{z} \qquad \text{and} \qquad \breve{t}_2 t_2 \subseteq \dot{z} \quad \text{(a)}$$

But then $\breve{t}_1 t_1 \cap \breve{t}_2 t_2 \subseteq \dot{z}$, and according to (44),

$$t_1 \dot{\cap} t_2 \subseteq \dot{z} \qquad \text{(b)}$$

but also

$$r = \breve{t}_1 t_1 \cup \breve{t}_2 t_2 \subseteq \dot{z} \qquad \text{(c)}$$

however if $r \subseteq \dot{z}$ then by induction one proves that $r^n \subseteq \dot{z}$. For, assuming $r^k \subseteq \dot{z}$, multiplying orderly by $r \subseteq \dot{z}$, get $r^{k+1} \subseteq \dot{z}^2 = \dot{z}$ so the induction is complete. Next, from (A.22), $\mathcal{I} \subseteq \dot{z}$.

If $\mathcal{I} \subseteq \dot{z}$ and $r^n \subseteq \dot{z}$ for all n, then

$$\# r = \mathcal{I} \cup r \cup r^2 \cup r^3 \cup \ldots \subseteq \dot{z}$$

from (c)

$$\#(\breve{t}_1 t_1 \cup \breve{t}_2 t_2) \subseteq \dot{z}$$

by (45) this yields

$$t_1 \dot{\cup} t_2 \subseteq \dot{z} \qquad \text{(d)}$$

(b) and (d) show the set of acceptable endomorphisms to be closed under $\dot{\cup}$ and $\dot{\cap}$, and hence, to be a lattice, $\mathcal{L}_2(A)$.

We show that $\mathcal{L}_2$ contains an upper bound t_0. Let $t \in \mathcal{L}_2$. If there is no $t_1 \in \mathcal{L}_2$ such that $t \subseteq t_1$, $t = t_0$ is an upper bound. If there is, take t_1 and repeat the argument. Since $\mathcal{L}_2$ is finite, one must attain an upper bound to fulfilling (65) and (65a).

Now, this t_0 must be unique. For if there existed another upper bound $\bar{t}_0 \in \mathcal{L}_2$, then $t_0 \dot{\cup} \bar{t}_0 \in \mathcal{L}_2$ and $\bar{t}_0 \dot{\subseteq} t_0 \dot{\cup} \bar{t}_0$ contradicting the assumption. Hence, t_0 is unique.

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