SOME MATRIX ELEMENTS AND NORMALISATION COEFFICIENTS IN SU_n^*

T.A. Brody**, M. Moshinsky** and I. Renero**
Instituto de Física, Universidad Nacional de México

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ABSTRACT

A previous paper by the authors described the derivation of recursion relations for the Wigner coefficients of SU_n, based on the concept of auxiliary Wigner coefficient, for which the multiplicity problem does not arise. The coefficients in the recursion relations are explicitly obtained here, and the polynomials in the creation operators which constitute the basis for the irreducible representation of SU_n are normalised, after a preliminary result concerning the expansion of determinants in the creation operators is obtained.

RESUMEN

Un trabajo anterior de los autores describió la derivación de relaciones de

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** Asesor Técnico, Comisión Nacional de Energía Nuclear, México.

recurrencia para los coeficientes de Wigner de SU_n, basada en el concepto de coeficiente auxiliar de Wigner, para el cual no hay el problema de la multiplicidad. Aquí se derivan explícitamente los coeficientes en las relaciones de recurrencia y se normalizan los polinomios en los operadores de creación que constituyen la base para una representación irreducible de SU_n, después de obtener un resultado preliminar sobre el desarrollo de determinantes en estos operadores de creación.

1. INTRODUCTION

In a preceding paper (Brody 1965, here to be quoted as II), the authors developed recursion relations for the Wigner coefficients of unitary groups; the final formulae were given in explicit form for the case of the SU_3 group, though the arguments given in the paper allow the straightforward derivation of similar recursion relations for other unitary groups. However, a number of important intermediate steps were only mentioned in very brief outline in II, because of lack ot space. Since several of these intermediate results are of considerable usefulness in calculations of this type, their derivation has been collected here.

The argument of II is concerned with polynomials in the components of n-dimensional vectors, $a^{+}_{\mu s}$; here $\mu = 1 \ldots n$ is the component index and $s = 1 \ldots r$ is the vector index. As was shown in another paper (Moshinsky 1963, here cited as I), such polynomials can be constructed to form bases for the irreducible representations of SU_n. For this purpose, the $a^{+}_{\mu s}$ are considered to be Bose creation operators, and their properties will be found in I. The corresponding annihilation operator will be written as $a^{\mu s}$ and as was shown in I, if only polynomials are considered,

$$a^{\mu s} = \frac{\partial}{\partial a^{+}_{\mu s}} \tag{1.1}$$

The polynomials forming a basis for an irreducible representation are most easily written in terms of determinants formed from the $a^{+}_{\mu s}$; such determinants will be denoted by

$$\Delta^{s_1 s_2 \dots s_j}_{\mu_1 \mu_2 \dots \mu_j} = \sum_{\mathfrak{p}} (-1)^{\mathfrak{p}} \, \mathfrak{p} \, a^+_{\mu_1 s_1} \, a^+_{\mu_2 s_2} \dots a^+_{\mu_j s_j} \tag{1.2}$$

where $\mathfrak{p}$ is a permutation of $s_1 \dots s_j$. If the s-indices run from 1 to j, the determinant may be written

$$\Delta^{1 \dots j}_{\mu_1 \dots \mu_j} = \sum_{s_i = 1}^{j} \epsilon_{s_1 s_2 \dots s_j} \, a^+_{s_1 \mu_1} \, a^+_{s_2 \mu_2} \dots a^+_{s_j \mu_j} \tag{1.3}$$

or if the μ-indices run from 1 to j, it may be written

$$\Delta^{s_1 \dots s_j}_{1 \dots j} = \sum_{\mu_i = 1}^{j} \epsilon_{\mu_1 \mu_2 \dots \mu_j} \, a^+_{s_1 \mu_1} \, a^+_{s_2 \mu_2} \dots a^+_{s_j \mu_j} \tag{1.4}$$

In these two equations, the ϵ are the usual completely antisymmetric tensors. It is, similarly, possible to define determinants $\left(\Delta^{s_1 \dots s_j}_{\mu_1 \dots \mu_j}\right)^+$ composed in an entirely analogous fashion from the annihilation operators $a^{\mu s}$.

The creation and annihilation operators obey the commutation rule

$$[a^{\nu t}, a^+_{\mu s}] = \delta^t_s \, \delta^\nu_\mu \tag{1.5}$$

from which the properties of the determinants may be derived.

From these two kinds of operators it is possible to construct generators of unitary groups. Three groups are of importance in connection with the present work: they are $\mathbf{U}_{nr}$, U_n and $\mathfrak{U}_r$; their generators are, respectively,

$$\mathbf{C}^{\nu t}_{\mu s} = a^{+}_{\mu s}\, a^{\nu t} \tag{1.6}$$

$$C^{t}_{s} = \sum_{\mu} \mathbf{C}^{\mu t}_{\mu s} = \sum_{\mu} a^{+}_{\mu s}\, a^{\mu t} \tag{1.7}$$

$$\mathcal{C}^{\nu}_{\mu} = \sum_{s} \mathbf{C}^{\nu s}_{\mu s} = \sum_{s} a^{+}_{\mu s}\, a^{\nu s} \tag{1.8}$$

The $a^{+}_{\mu s}$ constitute the components of a single vector of dimension nr belonging to the basis of an irreducible representation of the unitary group which in II is denoted by $\mathbf{U}_{nr}$; hence the set of all linearly independent homogenous polynomials of degree N in the $a^{+}_{\mu s}$ forms the basis of the completely symmetric representation of this group, characterised by the partition $[N]$. This set is a basis for a, in general, reducible representation of $U_r \times \mathcal{U}_n$ of $\mathbf{U}_{nr}$. It is from the homogenous polynomials of this set which are of highest weight in the two groups U_r and $\mathcal{U}_n$ that the scalar products are built up which were shown in I to be the Wigner coefficients looked for. However, in II it was shown that the scalar product (2.17) of that paper, there called an auxiliary Wigner coefficient, may be obtained through the recursion formulae derived there in a much more convenient form. The auxiliary Wigner coefficient contains the polynomials $P_{\max}$, $P_{\min}$ defined by II, eq.(2.10) and (2.14), and as is shown at the beginning of section 4 of II, it is sufficient to have available the forms which are, respectively, of highest weight and lowest weight both in U_{2n-2} and $\mathcal{U}_{n-1}$. Such a polynomial will take the form

$$P^{h_1 \dots h_n}_{q_1 \dots q_{n-1}} = A\begin{pmatrix} h_1 \dots h_n \\ q_1 \dots q_{n-1} \end{pmatrix} (\Delta^{1}_{1})^{q_1 - h_2} (\Delta^{1}_{n})^{h_1 - q_1} (\Delta^{12}_{12})^{q_2 - h_3} (\Delta^{12}_{1n})^{h_2 - q_2} \cdots \left(\Delta^{12 \dots n}_{12 \dots n}\right)^{h_n} \tag{1.9}$$

as was shown in I, eq. (4.14). (Here, for convenience, n is written instead of

$2j+1$). The normalisation coefficient $A\begin{pmatrix} h_1 \dots h_n \\ q_1 \dots q_{n-1} \end{pmatrix}$ for this polynomial will be derived in section 3, after a preliminary result is obtained in section 2 of this paper.

The last section will describe the manner in which the matrix elements of the $a^{+}_{\mu s}$ can be obtained and how these can be combined and antisymmetrised so as to obtain the matrix elements (4.5) of **II**.

2. A PRELIMINARY RESULT

Let Q be a polynomial of highest weight in U_r, i.e., one which is a solution of

$$C_s^t\, Q = 0,\; t > s = 1 \dots r-1;\; C_s^s\, Q = k_s\, Q,\; s = 1 \dots r. \tag{2.1}$$

The result to be obtained in this section is

$$E = (\Delta^{1 \dots r}_{1 \dots r})^{+}\; \Delta^{1 \dots r}_{1 \dots r}\; Q|\,0> \tag{2.2}$$

A joint expansion of the two determinants in (2.2) may be obtained in the form

$$(\Delta^{1 \dots r}_{1 \dots r})^{+}\; \Delta^{1 \dots r}_{1 \dots r} = \sum_{\mu_t s_t = 1}^{r} \epsilon_{s_1 s_2 \dots s_r}\; (\Delta^{1}_{\mu_1})^{+} \dots (\Delta^{r}_{\mu_r})^{+}\; \Delta^{s_1}_{\mu_1} \dots \Delta^{s_r}_{\mu_r} \tag{2.3}$$

This expansion is seen to be valid through two considerations: in the first place, the total number of terms is evidently the correct one, $(r!)^2$ which would be obtained by multiplying out the separate expansions of the type (1.3); in the second place, the sign is given correctly to each term by the antisymmetric tensor $\epsilon_{s_1 \dots s_r}$: if the factors belonging to the second determinant are re-ordered (they commute with each other) so that the upper indices have the natural order, the sign is not changed, but is now given by the permutations of the two sets of lower indices.

Applying now the commutation rule (1.5) to the innermost pair in (2.3), there results

$$E = \sum_{\mu_i s_i} (\Delta^1_{\mu_1})^+ \dots (\Delta^{r-1}_{\mu_{r-1}})^+ \epsilon_{s_1 \dots s_r} \left(\delta^r_{s_1} \delta^{\mu_r}_{\mu_1} + \Delta^{s_1}_{\mu_1} (\Delta^r_{\mu_r})^+ \right) \Delta^{s_2}_{\mu_2} \dots \Delta^{s_r}_{\mu_r} Q$$

$$= \sum_{\mu_i s_i} (\Delta^1_{\mu_1})^+ \dots (\Delta^{r-1}_{\mu_{r-1}})^+ \epsilon_{r s_2 \dots s_{r-1} s_r} \Delta^{s_2}_{\mu_2} \dots \Delta^{s_r}_{\mu_1} Q | 0>$$

$$+ \sum_{\mu_i s_i} \epsilon_{s_1 \dots s_r} (\Delta^1_{\mu_1})^+ \dots (\Delta^{r-1}_{\mu_{r-1}})^+ \Delta^{s_1}_{\mu_1} (\Delta^r_{\mu_r})^+ \Delta^{s_2}_{\mu_2} \dots \Delta^{s_r}_{\mu_r} Q | 0 > \tag{2.4}$$

On the right-hand side of (2.4), the indices of the ϵ in the first term may be arranged in the order $s_r s_2 \dots s_{r-1} r$ at the cost of a minus sign, since there must be $r + (r-1)$ interchanges to achieve this order; furthermore, since the indices are dummy variables, s_r may be renamed s_1. The commutation rule (1.5) may now be applied again to the second term of (2.4), and after a similar rearrangement and renaming of the indices, a term exactly equal to the first term in (2.4) is produced. This procedure may be repeated $r-1$ times; the last application of the commutation rule, however, gives

$$\sum_{\mu_i s_i} \epsilon_{s_1 \dots s_r} (\Delta^1_{\mu_1})^+ \dots (\Delta^{r-1}_{\mu_{r-1}})^+ \Delta^{s_1}_{\mu_1} \dots \Delta^{s_{r-1}}_{\mu_{r-1}} \left(\delta^r_{s_r} \delta^{\mu_r}_{\mu_r} + \Delta^{s_r}_{\mu_r} (\Delta^r_{\mu_r})^+ \right) Q | 0 > \tag{2.5}$$

The first term in the parenthesis gives rise to a term similar to those obtained previously, but multiplied by a factor n due to the sum over μ_r; the second term will be seen from (1.7) to be $C_{s_r}^{r}$, and because of the fact that the polynomial Q has been taken to be of highest weight, (2.1) shows that all except C_r^r give a zero result. Hence, collecting terms,

$$E = \sum_{\mu_i s_i} \epsilon_{s_1 \ldots s_{r-1}} (\Lambda^1_{\mu_1})^+ \ldots (\Lambda^{r-1}_{\mu_{r-1}})^+ \Delta^{s_1}_{\mu_1} \ldots \Lambda^{s_{r-1}}_{\mu_{r-1}} (1 + C_r^r)\, Q \,|\, 0 > \tag{2.6}$$

This process may be repeated; in successive applications, the commutation rule will be applied $r-1$, $r-2$, ... times before the last exchange gives rise to terms like (2.5); hence the numbers added to C_r^r, C_{r-1}^{r-1}, ... will be 1, 2, ... The final result will be

$$E = (r + C_1^1)(r - 1 + C_2^2) \ldots (1 + C_r^r)\; Q \,|\, 0 > \tag{2.7}$$

If now R is a polynomial of highest weight in $\mathfrak{U}_n$, i.e. one which is a solution of

$$C_\mu^\nu R = 0,\; \nu > \mu = 1 \ldots n-1; \quad C_\mu^\mu R = h_\mu R,\; \mu = 1 \ldots n \tag{2.8}$$

then the simplified expression for

$$E = (\Lambda^{1 \ldots n}_{1 \ldots n})^+ \Delta^{1 \ldots n}_{1 \ldots n}\; R \,|\, 0 > \tag{2.9}$$

may be obtained by means of the joint expansion of the two determinants carried out in the lower indices

$$(\Delta_{1\ldots n}^{1\ldots n})^{+}\Delta_{1\ldots n}^{1\ldots n} = \sum_{\mu_i s_i=1}^{n} \epsilon^{\mu_1\mu_2\ldots\mu_n} (\Delta_1^{s_1})^{+}(\Delta_2^{s_2})^{+}\ldots(\Delta_n^{s_n})^{+}\Delta_{\mu_1}^{s_1}\Delta_{\mu_2}^{s_2}\ldots\Delta_{\mu_n}^{s_n}$$

(2.10)

and the result will be

$$E = (n + C_1^1)(n-1 + C_2^2)\ldots(1 + C_n^n)\ R\,|\,0> \qquad (2.11)$$

3. THE NORMALISATION OF BASIS POLYNOMIALS

As was mentioned in the introductory section, the typical polynomial to be normalised takes the form

$$P_{q_1\ldots q_{n-1}}^{b_1\ldots b_n} = A\binom{b_1\ldots b_n}{q_1\ldots q_{n-1}}(\Delta_1^1)^{q_1-b_2}(\Delta_n^1)^{b_1-q_1}(\Delta_{12}^{12})^{q_2-b_3}(\Delta_{1n}^{12})^{b_2-q_2}\ldots(\Delta_{1\ldots n-1}^{1\ldots n-1})^{q_{n-1}-b_n}$$
$$\times(\Delta_{1\ldots n}^{1\ldots n-1})^{b_{n-1}-q_{n-1}}\quad(\Delta_{1\ldots n}^{1\ldots n})^{b_n}$$

(3.1)

so that the normality condition is

$$(P_{q_1\ldots q_{n-1}}^{b_1\ldots b_n}, P_{q_1\ldots q_{n-1}}^{b_1\ldots b_n}) \equiv <0|\,(P_{q_1\ldots q_{n-1}}^{b_1\ldots b_n})^{+}\,P_{q_1\ldots q_{n-1}}^{b_1\ldots b_n}|\,0> = 1$$

(3.2)

Here $|\,0>$ represents the vacuum, defined by

$$a^{\mu s}|\,0> = 0 \quad \forall\,\mu, s.$$

Writing now

$$P_{q_1\ldots q_{n-1}}^{b_1\ldots b_n} = A\binom{b_1\ldots b_n}{q_1\ldots q_{n-1}}(\Delta_{1\ldots n}^{1\ldots n})^{b_n}\ P' \qquad (3.3)$$

the normalisation coefficient becomes

$$A\binom{b_1 \ldots b_n}{q_1 \ldots q_{n-1}} = B^{-\frac{1}{2}} \qquad (3.4)$$

where

$$B \equiv \left(P'\left(\Delta^{1\ldots n}_{1\ldots n}\right)^{b_n},\; P'\left(\Delta^{1\ldots n}_{1\ldots n}\right)^{b_n}\right) =$$

$$\left(P'\left(\Delta^{1\ldots n}_{1\ldots n}\right)^{b_n-1},\; \left(\Delta^{1\ldots n}_{1\ldots n}\right)^{+} \Delta^{1\ldots n}_{1\ldots n}\, P'\left(\Delta^{1\ldots n}_{1\ldots n}\right)^{b_n-1}\right)$$

$$(3.5)$$

It will be seen that P' does not contain n among the upper indices. Hence (2.7) is applicable, and since the effect of the operators C^i_i is to count the number of occurrences of i among the upper indices,

$$B = b_n\,(b_{n-1}+1)(b_{n-2}+2)\ldots(b_1+n-1)\left(P'\left(\Delta^{1\ldots n}_{1\ldots n}\right)^{b_n-1},\; P'\left(\Delta^{1\ldots n}_{1\ldots n}\right)^{b_n-1}\right)$$

$$(3.6)$$

The steps leading to (3.6) can be repeated, giving finally

$$B = \prod_{p=1}^{n} \frac{(b_p+n-p)\,!}{(b_p-b_n+n-p)\,!}\,(P',\; P') \qquad (3.7)$$

It is possible to write (3.7) in this form, since it is well known (Moshinsky 1962) that

$$b_p \geqslant b_q \quad \text{if} \quad p < q\,.$$

Another factor may now be removed from P' :

$$P' = P''\left(\Delta^{1\ldots n-1}_{1\ldots n-1}\right)^{q_{n-1}-b_n}$$

Now the value $n-1$ no longer occurs in P'', but among the lower indices. Hence the alternative form (2.11) is used and leads to

$$B = \prod_{p=1}^{n} \frac{(b_p + n - p)!}{(b_p - b_n + n - p)!} (q_{n-1} - b_n)(q_{n-2} - b_n + 1)\ldots(q_1 - b_n + n - 2) \times$$

$$\left(P''\left(\Delta^{1\ldots n-1}_{1\ldots n-1}\right)^{q_{n-1}-b_n-1}, P''\left(\Delta^{1\ldots n-1}_{1\ldots n-1}\right)^{q_{n-1}-b_n-1}\right)$$

Repeating this step, one finally obtains

$$B = \prod_{p=1}^{n} \frac{(b_p + n - p)!}{(b_p - b_n + n - p)!} \prod_{l=1}^{n-1} \frac{(q_l - b_n + n - l - 1)!}{(q_l - q_{n-1} + n - l - 1)!} (P'', P'') \tag{3.8}$$

If $n - 1$ is substituted for n in the lower indices of P'' (which does not contain $n - 1$ any longer), it will be seen to have exactly the form (3.1) for the group $\mathcal{U}_{n-1}$ if one uses

$$b_i' = b_i - q_{n-1} \quad , \quad q_i' = q_i - q_{n-1}$$

for the exponents of the determinants. The substitution of $n-1$ for n does not, of course, affect the value of the scalar product (P'', P''), which may now be evaluated by repeating the entire procedure described so far until only $\langle 0|0\rangle = 1$ is left. This leads to

$$B = \prod_{p_n=1}^{n} \frac{(b_{p_n} + n - p_n)\,!}{(b_{p_n} - b_n + n - p_n)\,!} \prod_{l_n=1}^{n-1} \frac{(q_{l_n} - b_n + n - l_n - 1)\,!}{(q_{l_n} - q_{n-1} + n - l_n - 1)\,!} \times$$

$$\times \prod_{p_{n-1}=1}^{n-1} \frac{(b_{p_{n-1}} - q_{n-1} + n - 1 - p_{n-1})\,!}{(b_{p_{n-1}} - b_{n-1} + n - 1 - p_{n-1})\,!} \prod_{l_{n-1}=1}^{n-2} \frac{(q_{l_{n-1}} - b_{n-1} + n - l_{n-1} - 2)\,!}{(q_{l_{n-1}} - q_{n-2} + n - l_{n-1} - 2)\,!} \tag{3.9}$$

If one adopts the convention

$$q_n = b_{n+1} \equiv 0 \tag{3.10}$$

the products in (3.9) may be conjoined into a double product; hence, from (3.4),

$$A\begin{pmatrix} b_1 \ldots b_n \\ q_1 \ldots q_{n-1} \end{pmatrix} = \left[\prod_{j=1}^{n} \prod_{i=1}^{j} \frac{(b_i - b_j + j - i)\,!(q_i - q_j + j - i)\,!}{(b_i - q_j + j - i)\,!\,(q_i - b_{j+1} + j - i)\,!} \right]^{\frac{1}{2}} \tag{3.11}$$

This is the normalisation coefficient given in II, eq.(4.4).

4. THE MATRIX ELEMENT OF Δ_μ^s

It was shown in II that the general polynomial P in the $a^+_{\mu s}$ belonging to the basis for an irreducible representation of the group $\mathbf{U}_{nr}$ may be described by its simultaneous classification in the chains

$$\mathsf{U}_{nr} \supset U_r \otimes \mathfrak{U}_n \,, \quad \begin{array}{l} U_r \supset U_{r-1} \supset \cdots \\ \mathfrak{U}_n \supset \mathfrak{U}_{n-1} \supset \cdots \end{array} \tag{4.1}$$

Using the notation b_{pq}, $p = 1 \ldots q$ to represent the row lengths of the Young diagrammes in the subgroup $\mathfrak{U}_q$ and similarly k_{uv}, $u = 1 \ldots v$ for U_v, P may be described by the double Gel'fand pattern

$$P\,|\,0\rangle = \left|\begin{array}{cccc} b_{1n} \; b_{2n} \cdots\cdots b_{nn} & & k_{1r} \; k_{2r} \cdots\cdots k_{rr} & \\ b_{1n-1} & b_{n-1\,n-1} & k_{1r-1} \cdots\cdots k_{r-1\,r-1} & \\ & & & \\ b_{11} & & k_{11} & \end{array}\right\rangle \equiv |\, b_{pq}\,, k_{uv} \rangle$$

It was shown in II that the polynomials required for the recursion relations of the auxiliary Wigner coefficients obey the condition

$$k_{in} = \begin{cases} b_{in} \,, & i = 1 \ldots\ldots n \\ 0 \,, & i = n+1 \ldots r \end{cases} \tag{4.2}$$

where $n < r$; in fact, $r = 2n - 2$ is used for the calculations in II. It will be seen that the polynomials whose normalisation was discussed in the previous section are the particular case

$$b_{in} = b_i \;;\quad b_{in-1} = q_i \;;\quad b_{ij} = q_i \;;\quad j = 1 \ldots n-2$$

In this section the matrix elements

$$< b'_{pq} \, , \, k'_{uv} | \; a^+_{\mu s} \; | \; b_{pq} \, , \, k_{uv} >$$

of $\Delta^s_\mu \equiv a^+_{\mu s}$ will be determined. This quantity will evidently be zero unless $b'_{pq} = b_{pq} + 1$ for some one value of p in every row q with $q \geqslant \mu$, and similarly for the k'_{uv}, i.e. for

$$b'_{pq} = b_{pq} + \delta_{pl_q} \sum_{\rho = \mu}^{n} \delta_{q\rho} \quad ; \quad k'_{uv} = k_{uv} + \delta_{ul_v} \sum_{\rho = s}^{r} \delta_{v\rho}$$

while all the other quantum numbers are the same on both sides. The Wigner-Eckart theorem (Flach 1964, p.210) may then be applied twice, in U_n and in U_r, to give

$$< b'_{pq} \, , \, k'_{uv} | \; a^+_{\mu s} \; | \; b_{pq} \, , \, k_{uv} > \; = \; < b'_{pn} || \; a^+ || \; b_{pn} > \; \times$$

$$\left\langle \begin{matrix} b'_{pn} \;\; 0 \\ b'_{pq} \end{matrix} \middle| \; C^{n+1}_{\mu} \; \middle| \begin{matrix} b'_{pn} \;\; 0 \\ b_{pq} \end{matrix} \right\rangle \left\langle \begin{matrix} b'_{pn} \;\; 0 \ldots 0 \\ k'_{uv} \end{matrix} \middle| \; C^{r+1}_{s} \; \middle| \begin{matrix} b'_{pn} \;\; 0 \ldots 0 \\ k_{uv} \end{matrix} \right\rangle$$

The form taken by the last two factors in (4.3) is seen to be justified since, in the subgroup U_n of U_{n+1}, C^{n+1}_{μ} transforms exactly as $a^+_{\mu s}$ does, and similarly with C^{r+1}_s in U_r. The condition (4.2) is, of course, taken into account.

If the three factors in (4.3) can be determined, the whole matrix element is determined. This is what will be carried out below, in two steps.

(i) The matrix elements of the operators C_{μ}^{n+1} and C_s^{r+1} which belong to the set of generators of $\mathfrak{U}_{n+1}$ and U_{r+1} respectively, have been obtained by Gel'fand and Zetlin (1950) and rederived by others (Nagel 1964, and references given there). They are

$$< h_{pq} + \delta_{pl_q} \sum_{p=m}^{n} \delta_{qp} \mid \mathrm{C}_m^{n+1} \mid h_{pq} > =$$

$$= \prod_{r=m+1}^{n} S(l_{r-1} - l_r) \left[(h_{l_r r} - h_{l_{r-1} r-1} + l_{r-1} - l_r)(h_{l_r r} - h_{l_{r-1} r-1} + l_{r-1} - l_r + 1) \right]^{-\frac{1}{2}} \times$$

$$\prod_{r=m+1}^{n+1} < h_{pq} + \delta_{pl_q} \delta_{q\, r-1} \mid \mathrm{C}_{r-1}^{r} \mid h_{pq} > \tag{4.4}$$

where

$$S(x) = \begin{cases} +1 & \text{for } x \geqslant 0 \\ -1 & \text{for } x < 0 \end{cases} \tag{4.5}$$

and similarly for C_s^{r+1}.

(ii) The reduced matrix element which is the first factor in (4.3) can be found if a particular case of (4.3) can be evaluated; for this purpose a_{ll}^{+} is chosen, since then the polynomials $\mathfrak{P}$ and $\overline{\mathfrak{P}}$ in

$$< h'_{pq}, k'_{uv} \mid a_{ll}^{+} \mid h_{pq}, k_{uv} > = < 0 \mid \overline{\mathfrak{P}}^{+} \Delta_l^{l} \mathfrak{P} \mid 0 > \tag{4.6}$$

can be of highest weight and then take the simple form given by eq. (3.23) of I:

$$\mathfrak{P} = A \begin{pmatrix} h_p \\ h_p \end{pmatrix} P = A \begin{pmatrix} h_p \\ h_p \end{pmatrix} (\Delta_1^1)^{h_1 - h_2} (\Delta_{12}^{12})^{h_2 - h_3} \ldots (\Delta_{1\ldots n}^{1\ldots n})^{h_n} \tag{4.7}$$

$$\mathfrak{P} = A \begin{smallmatrix} b_p + \delta_{pl} \\ b_p + \delta_{pl} \end{smallmatrix} \quad \overline{P} = A \begin{smallmatrix} b_p + \delta_{pl} \\ b_p + \delta_{pl} \end{smallmatrix} \; (\Delta_1^1)^{b_1 - b_2} \cdots (\Delta_{1 \ldots l-1}^{1 \ldots l-1})^{b_{l-1} - b_l - 1} (\Delta_{1 \ldots l}^{1 \ldots l})^{b_l - b_{l+1} + 1} \cdots (\Delta_{1 \ldots n}^{1 \ldots n})^{b_n}$$

Rather than evaluate (4.7), however, its complex conjugate is determined, whose value is the same since both are real; this is evident from the fact that the value of the matrix element is built up from the commutation relation (1.5). Using the complex conjugate of (4.7) has the advantage that the result (2.7) developed in section II is applicable to $< 0 | P^+ \, (\Delta_l^l)^+ \, \overline{P} | 0 >$. Thus, calling this matrix element M, (note that it does not contain the two normalisation coefficients in (4.7)) and writing

$$P = P' (\Delta_{1 \ldots n}^{1 \ldots n})^{b_n} \quad , \quad \overline{P} = \overline{P}' (\Delta_{1 \ldots n}^{1 \ldots n})^{b_n}$$

one obtains by repeated application of (2.7)

$$M = \frac{b_l + n - l + 1}{b_l - b_n + n - l + 1} \prod_{p=1}^{n} \frac{(b_p + n - p)\,!}{(b_p - b_n + n - p)\,!} \quad < 0 | P'^{\,+} \, (\Delta_l^l)^+ \, \overline{P}' | 0 >$$

Continuing in this fashion, with

$$P = P'' \, (\Delta_{1 \ldots l+1}^{1 \ldots l+1})^{b_{l+1} - b_{l+2}} \cdots (\Delta_{1 \ldots n}^{1 \ldots n})^{b_n}$$

and analogously for $\overline{P}''$, one obtains after simplifying the factorials

$$M = \prod_{p=l+1}^{n} \frac{b_l - b_{p+1} + p - l + 1}{b_l - b_p + p - l + 1} \; \frac{\prod_{p=1}^{n} (b_p + n - p)\,! \, < 0 | P''^{\,+} \, (\Delta_l^l)^+ \, \overline{P}'' | 0 >}{\prod_{p=1}^{l} (b_p - b_{l+1} + l - p + 1)\,! \; \prod_{q=l+2}^{n} \; \prod_{p=1}^{q-1} (b_p - b_q + q - p)}$$

(4.8)

Now the last two factors in $\overline{P}''$ are

$$(\Delta^{1\ldots l-1}_{1\ldots l-1})^{h_{l-1}-h_l-1} \qquad (\Delta^{1\ldots l}_{1\ldots l})^{h_l-h_{l+1}+1}$$

$(\Delta^l_l)^+$ acts only on the second of these to yield

$$(h_l - h_{l+1}+1)\,(\Delta^{1\ldots l}_{1\ldots l})^{h_l-h_{l+1}} \qquad (\Delta^{1\ldots l-1}_{1\ldots l-1})$$

and thus

$$(\Delta^l_l)^+\ \overline{P}'' = (h_l - h_{l+1}+1)\ P''$$

so that

$$<0\,|\,P''^{+}(\Delta^l_l)^+\ \overline{P}''\,|\,0> = \frac{(h_l - h_{l+1}+1)}{\left[A\binom{h_1-h_{l+1}\ldots h_l-h_{l+1}}{h_1-h_{l+1}\ldots h_l-h_{l+1}}\right]^2} \qquad (4.9)$$

where $A\binom{h_1-h_{l+1}\ldots h_l-h_{l+1}}{h_1-h_{l+1}\ldots h_l-h_{l+1}}$ is the coefficient which normalises P''. In this argument, l was taken $l<n$, if not, the last part is applied directly from the beginning.

The three normalisation coefficients needed to complete the calculation of (4.7) are immediately obtained from (3.11), since the polynomials to which they correspond are, in fact, of the form (3.1) with $q_i = h_i$; hence

$$A\binom{h_p}{h_p} = A\binom{h_1\ldots h_n}{h_1\ldots h_n} = \prod_{1\leq i<j\leq n}(h_i-h_j+j-i)\Big/\prod_{j=1}^{n}(h_j+n-j)\,! \qquad (4.10)$$

$$A\binom{h_p+\delta_{pl}}{h_p+\delta_{pl}} = \prod_{1\leq i<j\leq n}(h_i - h_j + j - i + \delta_{il} - \delta_{jl}) \Big/ \prod_{j=1}^{n}(h_j + n - j + \delta_{jl})\,!$$

$$A\binom{h_1-h_{l+1}\ldots h_l-h_{l+1}}{h_1-h_{l+1}\ldots h_l-h_{l+1}} = \prod_{1\leq i<j\leq l}(h_i - h_j + j - i) \Big/ \prod_{j=1}^{l}(h_j - h_{l+1} + l - j)\,!$$

Substituting (4.8), (4.9) and (4.10) in (4.7) there results, after some simplification,

$$<0|\overline{\mathfrak{P}}^{+}\Lambda_l^l\,\mathfrak{P}|0> = \tag{4.11}$$

$$= \left[\frac{h_l - h_{l+1} + 1}{h_l + n - l + 1}\prod_{1\leq i<j\leq n}\frac{(h_i - h_j + j - i + \delta_{il} - \delta_{jl})}{(h_i - h_j + j - i)}\right]^{1/2}\prod_{j=l+1}^{n}\frac{h_l - h_{j+1} + j - l + 1}{h_l - h_j + j - l + 1}\ .$$

This is a particular case of (4.3); (4.5) will now give the matrix elements of C_l^{n+1} and C_l^{r+1} in (4.3), which for this case take the form

$$\left\langle \begin{matrix} h_p+\delta_{pl} & 0 \\ h_p+\delta_{pl} & \end{matrix}\right| C_l^{n+1} \left| \begin{matrix} h_p+\delta_{pl} & 0 \\ h_p & \end{matrix}\right\rangle = \left[\prod_{j=l+1}^{n}\frac{h_l - h_{j+1} + j - l + 1}{h_l - h_j + j - l + 1}\right]^{1/2}$$

and

$$\left\langle \begin{matrix} h_p+\delta_{pl} & 0\ldots 0 \\ h_p+\delta_{pl} & 0\ldots 0 \end{matrix}\right| C_l^{r+1} \left| \begin{matrix} h_p+\delta_{pl} & 0\ldots 0 \\ h_p & 0\ldots 0 \end{matrix}\right\rangle = \left[\prod_{j=l}^{n}\frac{h_l - h_{j+1} + j - l + 1}{h_l - h_j + j - l + 1}\right]^{1/2}$$

since $h_{n+1} = \ldots = h_{r+1} = 0$

Substituting these two results in (4.11), one obtains straightforwardly

$$[illegible] = \left[\frac{\prod_{1 \leq i < j \leq n} (h_i - h_j + j - i + \delta_{il} - \delta_{jl}}{(h_l + n - l + 1) \prod_{1 \leq i < j \leq n} (h_i - h_j + j - i)} \right]^{1/2}$$

$$= \left[\frac{1}{h_l + n - l + 1} \cdot \frac{\dim (h_p + \delta_{pl})}{\dim h_p} \right]^{1/2} \qquad (4.12)$$

where

$$\dim h_p = \prod_{1 \leq i < j \leq n} \frac{h_i - h_j + j - i}{j - i} \qquad (4.13)$$

is the dimensionality of the representation $[h_p]$ (Boerner 1955).

On substituting this result in (4.3), the use of (4.5) and of the similar equation for the matrix elements of C_s^{r+1} makes all the matrix elements of Δ^s_{μ} available.

From these the matrix elements (4.5) of **II**, which yield the recursion coefficients for the Wigner coefficients of the unitary groups, are derived by expanding the Λ matrices in terms of the $\Delta^s_{\mu} \equiv a^+_{\mu s}$. Thus, for instance,

$$< h'_{pq}, k'_{uv} | \Lambda^{st}_{\mu\nu} | h_{pq}, k_{uv} > =$$

$$= \sum_{h''_{pq}, k''_{uv}} < h'_{pq}, k'_{uv} | \Delta^s_{\mu} | h''_{pq}, k''_{uv} > < h''_{pq}, k''_{uv} | \Delta^t_{\nu} | h_{pq}, k_{uv} > -$$

$$- \sum_{h'''_{pq}, k'''_{uv}} < h'_{pq}, k'_{uv} | \Lambda^s_{\nu} | h'''_{pq}, k'''_{uv} > < h'''_{pq}, k'''_{uv} | \Delta^t_{\mu} | h_{pq}, k_{uv} > . \qquad (4.14)$$

The sums in (4.14) extend over all compatible values of the h''_{pq}, k''_{uv}, h'''_{pq}, k'''_{uv}; however, if the conditions (4.2) are to be fulfilled, it is clear that the sums reduce to single terms for all those values of q (or v) in which h'_{pq} and h_{pq} (or k'_{uv} and k_{uv}) differ only for a single value of q (or v). This fact reduces considerably the somewhat laborious task of simplifying the expressions of the type of (4.14).

For the particular case of SU_3, the results were given in the Appendix of II.

5. ORTHONORMALITY OF THE AW_3

In reference II, the authors gave an erroneous expression for the orthonormality relations of auxiliary Wigner coefficients AW_3 of SU_3. In fact, there can be no sum over the b_i'', since both are determined by the sum rules

$$b_1' + b_2' + b_1'' + b_2'' = b_1 + b_2 + b_3$$

$$b_1' + b_2' + b_2'' = u_1 + u_2 + u_3$$

A correct orthonormality relation may be obtained from the fact that the ordinary Wigner coefficients, expressed in terms of the polynomials (2.15) of II, are orthonormal. The AW_3 may be obtained from them by means of a transformation bracket:

$$\left\langle \begin{matrix} b_1' \, b_2' & & b_1'' \, b_2'' \\ & , & \\ q_1' \, q_2' & & q_1'' \, q_2'' \end{matrix} \left| \begin{matrix} b_1 \, b_2 \, b_3 \\ \\ q_1 \, q_2 \end{matrix} \right. ; \; u_1 \, u_2 \, u_3 \right\rangle$$

$$= \sum_{\chi} \left\langle \begin{matrix} b_1' \, b_2' & & b_1'' \, b_2'' \\ & , & \\ q_1' \, q_2' & & q_1'' \, q_2'' \end{matrix} \left| \begin{matrix} b_1 \, b_2 \, b_3 \\ \\ q_1 \, q_2 \end{matrix} \right. \chi \right\rangle \left\langle \begin{matrix} b_1 \, b_2 \, b_3 \\ b_1' \, b_2' \, b_1'' \, b_2'' ; \chi \\ b_1' \qquad b_2'' \end{matrix} \left| \begin{matrix} b_1 \, b_2 \, b_3 \\ b_1' \, b_2' ; \; u_1 \, u_2 \, u_3 \\ b_1' \end{matrix} \right. \right\rangle \qquad (5.1)$$

The polynomials (2.15) of II giverise to the complete Wigner coefficients, whereas the dependence on SU_2 of the AW_3 has been factored out, as is described in the argument leading up to (3.5) of II. In order to remove this dependence from the complete Wigner coefficients, use is made of the Wigner-Eckart theorem; a factor depending on the q_i', q_i'', and q_i remains, due to the choice (3.3) made in II for the values of b_{11}' and b_{11}''.

The result of these considerations is the orthonormality relation

$$\sum_{q_i'\, q_i''} \left[\frac{(q_1' + q_1'' - q_2)!\,(q_2 - q_2' - q_2'')!}{(q_1 - q_2 + 1)(q_1' - q_2')!\,(q_1'' - q_2'')!} \right]^{1/2} \left\langle \begin{matrix} b_1'\ b_2' & & b_1''\ b_2'' \\ & , & \\ q_1'\ q_2' & & q_1''\ q_2'' \end{matrix} \middle| \begin{matrix} b_1\ b_2\ b_3 & \\ & ,\ u_1\ u_2\ u_3 \\ q_1\ q_2 & \end{matrix} \right\rangle$$

$$\times \left[\frac{(q_1' + q_1'' - \bar{q}_2)!\,(\bar{q}_2 - q_2' - q_2'')!}{(\bar{q}_1 - \bar{q}_2 + 1)(q_1' - q_2')!\,(q_1'' - q_2'')!} \right]^{1/2} \left\langle \begin{matrix} b_1'\ b_2' & & b_1''\ b_2'' \\ & , & \\ q_1'\ q_2' & & q_1''\ q_2'' \end{matrix} \middle| \begin{matrix} \bar{b}_1\ \bar{b}_2\ \bar{b}_3 & \\ & ,\ \bar{u}_1\ \bar{u}_2\ \bar{u}_3 \\ \bar{q}_1\ \bar{q}_2 & \end{matrix} \right\rangle$$

$$= \delta_{b_1 \bar{b}_1}\, \delta_{b_2 \bar{b}_2}\, \delta_{b_3 \bar{b}_3}\, \delta_{q_1 \bar{q}_1}\, \delta_{q_2 \bar{q}_2}\, F \qquad (5.2)$$

where

$$F = \sum_{\chi} \left\langle \begin{matrix} b_1\ b_2\ b_3 \\ b_1'\ b_2'\ b_1''\ b_2''\ \chi \\ b_1' \quad b_2'' \end{matrix} \middle| \begin{matrix} b_1\ b_2\ b_3 \\ b_1'\ b_2' \ ,\ u_1\ u_2\ u_3 \\ b_1' \end{matrix} \right\rangle \left\langle \begin{matrix} \bar{b}_1\ \bar{b}_2\ \bar{b}_3 \\ b_1'\ b_2'\ b_1''\ b_2''\ \chi \\ b_1' \quad b_2'' \end{matrix} \middle| \begin{matrix} \bar{b}_1\ \bar{b}_2\ \bar{b}_3 \\ b_1'\ b_2' \ ,\ \bar{u}_1\ \bar{u}_2\ \bar{u}_3 \\ b_1' \end{matrix} \right\rangle \qquad (5.3)$$

(5.3)

is a function which no longer depends on any of the q's; its value might be determined, though in view of the rather particular nature of the operator whose eigenvalue is χ (see I) its usefulness is limited.

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