A FEW COMMENTS ON THE REPRESENTATIONS OF SU(3).

M. Resnikoff*

The Latin American School of Physics

Instituto de Física, Universidad Nacional de México

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ABSTRACT

Using Schur's lemma, the representations $\mathfrak{D}^{\lambda\mu}$ *are shown to be irreducible. An alternate derivation of Weyl's character formula is obtained by summing the diagonal representatives of* $\mathfrak{D}^{\lambda\mu}$. *Finally, the normalized differential volume element dV for the Murnaghan parameterization of SU(3) is included.*

RESUMEN

Usando el lema de Schur, se muestra que las representaciones $\mathfrak{D}^{\lambda\mu}$ *son irreducibles. Una derivación alternativa de la fórmula de caracteres de Weyl se obtiene sumando los representativos diagonales de* $\mathfrak{D}^{\lambda\mu}$. *Finalmente se incluye*

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*Present address: Department of Physics and Astronomy, University of Maryland, College Park, Maryland.

†Present address: Department of Physics and Astronomy, University of Maryland, College Park, Maryland.

el elemento diferencial de volumen normalizado dV para la parametrización de Murnaghan del SU_3 .

1. INTRODUCTION

This paper discusses several properties of the representations of the group $SU(3)$. The proof of the irreducibility of the representations $\mathcal{D}^{\lambda\mu}(u)$ is similar in form to one given by Wigner[1] for the group $SU(2)$. Given the parameterization of an $SU(3)$ matrix U and the $SU(3)$ base vector $|\lambda\mu; \alpha>$, the necessary representation coefficients for the proof are determined in section 3. The diagonal representative, in terms of the two class parameters δ_1, δ_2, is given in section 4; upon summation, the expression is exactly Weyl's character formula for $SU(3)$[2].

2. PARAMETERIZATION

The parameterization of a general $SU(3)$ transformation U has been aiven by Murnaghan[3]:

$$U = D(\delta_1, \delta_2, \delta_3)\, U_{23}(\phi_2, \sigma_3)\, U_{12}(\theta_1, \sigma_2)\, U_{13}(\phi_1, \sigma_1) \tag{2.1}$$

where $D(\delta_1, \delta_2, \delta_3)$ is the diagonal matrix

$$D(\delta) \equiv D(\delta_1, \delta_2, \delta_3) = \begin{bmatrix} e^{i\delta_1} & 0 & 0 \\ 0 & e^{i\delta_2} & 0 \\ 0 & 0 & e^{i\delta_3} \end{bmatrix} \tag{2.2}$$

and U_{23}, U_{12}, U_{13} are $SU(2)$ transformations in the (23), (12), (13), planes, respectively. E.g.,

$$U_{12}(\theta_1, \sigma_2) = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 e^{-i\sigma_2} & 0 \\ \sin\theta_1 e^{i\sigma_2} & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The ranges of the parameters are

$$-\Pi/2 \leqslant \delta_1, \delta_2, \sigma_1, \sigma_2, \sigma_3, \theta_1 < \Pi/2 \qquad -\Pi \leqslant \phi_1, \phi_2 < \Pi \tag{2.3}$$

Given the above parameterization, Eqs. (2.1), (2.3), the normalized diffe ential volume element (Jacobian of the transformation) may be calculated:

$$dV = (16\Pi^5)^{-1} \left| \sin 2\phi_1 \, \sin 2\phi_2 \, \sin 2\theta_1 \, \cos 2\theta_1 \right| dx \tag{2.4a}$$

where $dx = d\theta_1 \, d\phi_1 \, d\phi_2 \, d\sigma_1 \, d\sigma_2 \, d\sigma_3 \, d\delta_1 \, d\delta_2$ (2.4b)

3. THE REPRESENTATION $\mathcal{D}^{\lambda\mu}(u)$

The base vectors $|\lambda\mu; a>$, where $[\lambda\mu]$ are the partition numbers, and $a \equiv (y, t, t_0)$, appear in many places in the literature[4]. If the variables a_i of the base vector undergo a unitary transformation T_U:

$$a_i' = \sum U_{\alpha\, i} a_\alpha$$

the unitary representations are defined:

$$T_U |\lambda\mu; a'> = \sum_{a''} \mathcal{D}^{\lambda\mu}_{a''a'}(u) |\lambda\mu; a''> \tag{3.1}$$

or,

$$\mathfrak{D}^{\lambda\mu}_{aa'}(u) = \left(|\lambda\mu ; a>, \dot{T}_u \, |\lambda\mu ; a'> \right)$$

since the base vectors are orthonormal. Let the general $SU(3)$ matrix U be written $U = \| a_{ij} \|$, where the explicit functional dependence in terms of the eight parameters is given by Eq. (2.1). The special representation coefficient $\mathfrak{D}^{\lambda\mu}_{aa_m}$ is then

$$\mathfrak{D}^{\lambda\mu}_{aa_m}(u) = N(\lambda\mu ; a) \left[N(\lambda\mu ; a_m) \right]^{-1} (-1)^q (a_{33})^{\lambda - p} (\bar{a}_{13})^q$$

$$\times \sum \binom{r}{k} \frac{(\mu - q)! \; p!}{(\mu - q - k)! \, [p - (r - k)]!} \, f_k \tag{3.2}$$

where

$$a_m \equiv (y_{\min}, t_{\min}, t_0 = -t_{\min}) \quad (p, q, r = 0) \tag{3.3}$$

and

$$f_k = (a_{31})^{p - (r - k)} (a_{32})^{r - k} (\bar{a}_{11})^k (-\bar{a}_{12})^{\mu - q - k}$$

The row labels y, t, t_0 are related to the numbers p, q, r:

$$y = -(2\lambda + \mu) + 3(p + q), \quad t = \frac{1}{2}\mu + \frac{1}{2}(p - q), \quad t_0 = t - r \tag{3.4}$$

where $0 \leqslant p \leqslant \lambda$, $0 \leqslant q \leqslant \mu$, $r = 0, 1, \ldots, 2t$

For the particular transformation $U = D(\delta)$, the representation coefficient is also diagonal,

$$\mathfrak{D}^{\lambda\mu}_{\alpha\alpha'}(D(\delta)) = (e^{i\delta_1})^{\mu+p-r}(e^{i\delta_2})^{q+r}(e^{i\delta_3})^{\lambda+\mu-(p+q)}\,\delta_{\alpha\alpha'} \tag{3.5}$$

4. IRREDUCIBILITY

The proof of the irreducibility of the representations $\mathfrak{D}^{\lambda\mu}$ may be done in analogy to the proof given by Wigner[1] for the group $SU(2)$[5]; it is necessary to prove that the only matrix M which commutes with $\mathfrak{D}^{\lambda\mu}(u)$, for all U, is a multiple of the unit matrix.[6]

Since $\mathfrak{D}^{\lambda\mu}_{\alpha\alpha'}(D(\delta))$ is diagonal, by Eq. (3.5), M must be a diagonal matrix also. Thus,

$$M\,\mathfrak{D}^{\lambda\mu}(u) = \mathfrak{D}^{\lambda\mu}(u)\,M$$

becomes

$$M_{\alpha\alpha}\,\mathfrak{D}^{\lambda\mu}_{\alpha\alpha'}(u) = \mathfrak{D}^{\lambda\mu}_{\alpha\alpha'}\,M_{\alpha'\alpha'}$$

or, in particular,

$$M_{\alpha\alpha}\,\mathfrak{D}^{\lambda\mu}_{\alpha\alpha_m}(u) = \mathfrak{D}^{\lambda\mu}_{\alpha\alpha_m}(u)\,M_{\alpha_m\alpha_m} \tag{4.1}$$

If $\mathfrak{D}^{\lambda\mu}_{\alpha\alpha_m}$ is not identically equal to zero, then

$$M_{\alpha\alpha} = M_{\alpha_m\alpha_m} \quad \text{for all } \alpha$$

and M is then a multiple of the unit matrix.

The explicit functional dependence of the u_{ij}, is

$$a_{31} = e^{-i(\delta_1 + \delta_2 - \sigma_1)} (\sin\theta_1 \cos\phi_1 \sin\phi_2 \, e^{i\phi} + \sin\phi_1 \cos\phi_2)$$

$$a_{32} = e^{-i(\delta_1 + \delta_2 - \sigma_3)} \cos\theta_1 \sin\phi_2$$

$$\bar{a}_{11} = \cos\theta_1 \cos\phi_1 \, e^{-i\delta_1}$$

$$-\bar{a}_{12} = \sin\theta_1 \, e^{i(\sigma_2 - \delta_1)} \qquad \phi = \sigma_2 + \sigma_3 - \sigma_1$$

It is clear that for $\sigma_1, \sigma_2, \sigma_3 = 0$ and $0 < \theta_1, \phi_1, \phi_2 < \Pi/2$, f_k [see Eq. (3.3)] is a product of positive terms. Further, since

$$\bar{a}_{13} = -\cos\theta_1 \sin\phi_1 \, e^{i(\sigma_1 - \theta_1)}$$

$$a_{33} = (-\sin\theta_1 \sin\phi_1 \sin\phi_2 \, e^{i\phi} + \cos\phi_1 \cos\phi_2) \, e^{-i(\delta_1 + \delta_2)}$$

the factor $(\bar{a}_{13})^q (a_{33})^{\lambda - p}$ is not identically equal to zero. Hence, the representations $\mathfrak{D}^{\lambda\mu}(u)$ are irreducible.

5. CHARACTER FORMULA

Weyl[2] has shown that every primitive characteristic of $SU(3)$ must have the form

$$\chi^{\lambda\mu}(\delta_1, \delta_2) = \xi^{\lambda\mu}(\delta_1, \delta_2) \, [\xi^{00}(\delta_1, \delta_2)]^{-1} \qquad (5.1)$$

where $\xi^{\lambda\mu}(\delta_1, \delta_2)$ is the determinant

$$\xi^{\lambda\mu}(\delta_1, \delta_2) = \begin{vmatrix} 1 & 1 & 1 \\ e^{ib_2 \delta_1} & e^{ib_2 \delta_2} & e^{ib_2 \delta_3} \\ e^{ib_1 \delta_1} & e^{ib_1 \delta_2} & e^{ib_1 \delta_3} \end{vmatrix} \tag{5.2}$$

and $b_2 = \mu + 1$, $b_1 = \lambda + \mu + 2$. If δ_3 is chosen $\delta_3 = -(\delta_1 + \delta_2)$ then Eq. (3.5) becomes

$$\mathfrak{D}^{\lambda\mu}_{aa'}(D(\delta)) = (e^{i\delta_1})^{y/2 - t_0} (e^{i\delta_2})^{y/2 + t_0} \delta_{aa'}$$

and the character is

$$\chi^{\lambda\mu}(\delta_1, \delta_2) = \sum_{y,\, t,\, t_0} (e^{i\delta_1})^{y/2 - t_0} (e^{i\delta_2})^{y/2 + t_0} \tag{5.3}$$

where the limits of the sum are given by Eq. (3.4). The finite sums over y, t, t_0 may be explicitly carried out since they are geometric progressions.

$$\chi^{\lambda\mu}(\delta_1, \delta_2) = \sum_{y,\, t} \left(e^{i(\delta_1 + \delta_2)y/2}\right) \sum_{t_0 = -t}^{t} \left(e^{-i(\delta_1 - \delta_2)t_0}\right) \tag{5.4}$$

The second sum is

$$\sum_{t_0} \left(e^{-i(\delta_1 - \delta_2)}\right)^{t_0} = \left(1 - e^{-i(\delta_1 - \delta_2)}\right)^{-1} \left(e^{i(\delta_1 - \delta_2)t} - e^{-i(\delta_1 - \delta_2)t}\right)$$

This may be put into Eq. (5.4),

$$\left(1-e^{-i(\delta_1-\delta_2)}\right)^{-1}\left[\sum_{p,q}(e^{-i\delta_1})^{\lambda}(e^{-i\delta_2})^{\lambda+\mu}(e^{i\delta_1})^{2p+q}(e^{i\delta_2})^{p+2q}-\right.$$

$$\left.-\sum_{p,q}\begin{pmatrix}\text{exchange}\\ \delta_1\leftrightarrow\delta_2\end{pmatrix}\right]$$

and the remaining sums carried out over p, q:

$$\frac{\left[-(e^{-i\delta_1})^{\lambda}(e^{-i\delta_2})^{\lambda+\mu}\left(1-e^{i(2\delta_1+\delta_2)\lambda}\right)\left(1-e^{i(\delta_1+2\delta_2)\mu}\right)+(\delta_1\leftrightarrow\delta_2)\right]}{\left[e^{i(\delta_1+2\delta_2)}\xi^{00}(\delta_1,\delta_2)\right]}$$

The result may be shown to be exactly Weyl's formula, Eqs. (5.1), (5.2). $\chi^{\lambda\mu}(\delta_1, \delta_2)$, as given by Eq. (5.3), is the exact form of the finite series obtained when the denominator of Eq. (5.1), $\xi^{00}(\delta_1, \delta_2)$, is divided into the numerator, $\xi^{\lambda\mu}(\delta_1, \delta_2)$, and the summation represents an alternate derivation of Weyl's result.

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2. H. Weyl, The Theory of Groups and Quantum Mechanics (Dover Publications, Inc., New York, 1931), pp. 377.
3. F. D. Murnaghan, The Unitary and Rotation Groups (Spartan Books, Washington, D.C., 1962). Murnaghan has also given the procedure for constructing a general unitary matrix of n dimensions as a product of $SU(2)$ transformations in the $\frac{1}{2}n(n-1)$ planes.
4. See, e.g., M. Moshinsky, Rev. Mod. Phys. **34**, 813 (1962), or, M. Resnikoff, J. Math. Phys. (to be published).
5. Moshinsky has sketched this method of proof in his paper (reference 4).
6. This proof uses the properties of the whole group. Proofs employing the infinitesimal properties of the group are given in reference 4.

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