ON THE GENERALIZED SCHROEDINGER EQUATION

L. de la Peña-Auerbach[*]

Instituto de Física, Universidad Nacional de México

L.S. García-Colín

Facultad de Ciencias, Universidad Nacional de México

e Instituto Mexicano del Petróleo

(Recibido: 3 noviembre 1967)

SUMMARY

In this paper a derivation of a generalized Schroedinger equation valid for velocity dependent forces, is presented. The physical assumptions involved rest on the hypothesis that a more general quantum mechanical description of a particle may be constructed if we start from a Markoff process as described by Fokker-Planck's equation in phase space. It is also shown that the asymptotic limit leads to the ordinary formalism of quantum mechanics in configuration space together with a new equation in velocity space which allows us to generate quantum numbers which are tentatively associated with some of those characteristic of the particle. This formulation leads to a first estimate of the parameter β introduced in previous papers.

[*]Consultant, Comisión Nacional de Energía Nuclear (México).

RESUMEN

En este trabajo se deduce una ecuación de Schroedinger generalizada, válida para fuerzas dependientes de la velocidad. Las suposiciones de carácter físico empleadas, se basan en la hipótesis de que una descripción cuántica más general de una partícula puede construirse si partimos de un proceso Markoviano descrito por una ecuación de Fokker-Planck en el espacio fase. Se demuestra también que en el límite asintótico se recupera el formalismo usual de la mecánica cuántica en el espacio de configuración y además se obtiene una nueva ecuación en el espacio de velocidades, la cual permite generar números cuánticos que, tentativamente, pueden ser asociados a algunas de las características de la partícula. Esta formulación, permite obtener una primera estimación del parámetro β introducido en trabajos anteriores.

I. INTRODUCTION

This paper is a continuation of a series [1,2,3,4,5] * devoted to a discussion of the possibility of interpreting ordinary quantum mechanics as a stochastic process. In papers I, II, III, and V the connection between the theory of Markoff processes and quantum mechanics, as characterized by Schroedinger's equation together with Heisenberg's principle, was discussed. On the other hand IV was devoted to explore the possibility of constructing a generalized Schroedinger equation from the complete Fokker-Planck equation. Since this latter one defines a Markoff process in the whole phase space the former one would also be valid in such phase-space. Assuming the validity of this generalized equation, it will hold true for the complete time interval (from zero to infinity) thus leading to a violation of Heisenberg's time-energy uncertainty relation [3,4]. Likewise, one can recover the ordinary quantum mechanics [4] by taking the limit $\beta t \rightarrow \infty$ which corresponds, physically, to the achievement of an equilibrium condition in velocity space, whereas mathematically establishes the condition whereby one is allowed to substitute Fokker-Planck's equation by the diffusion or Smoluchowski equation in order to describe the process.

---

*These papers will be hereafter denoted by I, II, III, IV and V respectively.

In this paper we want to consider once more these last questions but from a slightly more general point of view as that used in our previous work. In fact, in IV we considered only the case in which the external force K does not depend on the velocity u. Clearly, this is sufficient if we want to study the motion of a Brownian particle, but it is not so when dealing with the general quantum-mechanical problem.

Section II will be thus devoted to the derivation of the generalized Schroedinger equation without the restriction on K mentioned above. In section III we shall discuss the method followed to recover in this case the ordinary quantum mechanics and as it will turn out it is of a more general nature. Indeed Schroedinger's equation will appear to hold in configuration space but a similar equation will stem out in velocity space, and will be completely independent from the former one. This equation will yield a set of quantum numbers which may be tentatively associated to those characterizing the quantum particle. We propose to identify here some of these quantum numbers with those defining the spin for the case of mesons[*]. With this identification it seems possible to perform a first estimate of the order of magnitude of the parameter β.

II. THE GENERALIZED SCHROEDINGER EQUATION

In order to derive the generalized Schroedinger equation we shall follow a method which is a straightforward extension of that used in I and IV. We start from our fundamental equation, i.e., Fokker-Planck's equation[6], namely[+],

$$\frac{\partial w}{\partial t} = - \partial_\mu A_\mu w + \frac{1}{2} \partial_\mu \partial_\nu B_{\mu\nu} w \qquad (1)$$

where μ refers to all coordinates r_i and u_i of configuration and velocity spaces, respectively. Furthermore, we can write that[7]

---

[*] The theory in its present form can generate only quantum numbers that are integers.
[+] Summation over repeated indices is understood.

$$A_i = u_i \qquad \text{if } \mu \text{ belongs to } r\text{-space}$$

$$A_{u_i} = K_i - \beta u_i \quad \text{if } \mu \text{ belongs to } u\text{-space} \tag{2}$$

Since in Eq. (1), w stands for a probability density it is always positive definite and may cast into the form:

$$w = \exp 2R \tag{3}$$

where R is a real function of its arguments. For the sake of simplicity we shall consider here only the isotropic case, namely, that for which

$$B_{\mu\nu} = 2q\, \delta_{\mu\nu} \qquad \text{if } \mu \text{ belongs to } u\text{-space}$$

$$= 0 \qquad\qquad \text{otherwise} \tag{4}$$

With these considerations, Eq. (1) takes the form:

$$\frac{\partial w}{\partial t} = -\partial_\mu [(A_\mu - B_{\mu\nu}\partial_\nu R)\, w] = -\partial_\mu v_\mu w \tag{5}$$

where we have defined

$$v_\mu = A_\mu - B_{\mu\nu}\partial_\nu R \tag{6}$$

We now restrict ourselves to the case where v_μ may be written in the following way, namely,

$$v_\mu = \lambda_{\mu\nu}\partial_\nu \mathscr{S} \tag{7}$$

where $\mathscr{S}$ is a real function of u, r and t and $\lambda_{\mu\nu}$ are the elements of a symmetric matrix and they will be real and in general time dependent.

224

Let us now define an amplitude Ψ, where

$$\Psi = \exp(R + i\mathscr{S}) \tag{8}$$

in terms of which $w = \Psi^*\Psi$. Our task is to construct the differential equation satisfied by Ψ and this is easily accomplished in the following way: Rewriting of Eq. (5) in terms of R and $\mathscr{S}$ with the aid of Eqs. (3) and (7), yields

$$\frac{\partial R}{\partial t} = -\frac{1}{2}\lambda_{\mu\nu}\partial_{\mu}\partial_{\nu}\mathscr{S} - \lambda_{\mu\nu}\partial_{\mu}R\partial_{\nu}\mathscr{S} \tag{9}$$

If we now multiply Eq. (9) by Ψ, we express the derivatives of R and $\mathscr{S}$ through Eq. (8) and we notice that $\lambda_{\mu\nu}\partial_{\nu}R\partial_{\mu}\mathscr{S} = \lambda_{\mu\nu}\partial_{\mu}R\partial_{\nu}\mathscr{S}$, both being equal to $v_{\mu}\partial_{\mu}R$, we get that

$$i\frac{\partial \Psi}{\partial t} = -\frac{1}{2}\lambda_{\mu\nu}\partial_{\mu}\partial_{\nu}\Psi + \Omega\Psi \tag{10}$$

where Ω is a real function of r, u and t defined by

$$\Omega = -\frac{\partial \mathscr{S}}{\partial t} + \frac{1}{2}\lambda_{\mu\nu}[\partial_{\mu}\partial_{\nu}R + \partial_{\mu}R\partial_{\nu}R - \partial_{\mu}\mathscr{S}\partial_{\nu}\mathscr{S}] \tag{11}$$

Eq. (10) is the generalized Schroedinger equation, written in phase space and which Ψ must satisfy. The function Ω should be thought off as the one playing a role analogous to that assumed by the potential V in the ordinary case, but stress should be made on the point that this analogy is solely formal. Indeed, in the next section we shall see that, at least in some cases, Ω may be related to the Hamiltonian of the system but the general discussion of this difficult question will be deferred to a future paper. (For a discussion of the meaning of Ω in a very particular representation of λ, we refer the reader to paper IV). For the time being we may look into this question as follows: Suppose we are given a priori the concrete, but consistent form of $\lambda_{\mu\nu}(t)$ and $\Omega(u, r; t)$ for some specific problem.

225

Then, we must solve Eq. (10) for Ψ, the probability amplitude whose modulus squared satisfies Eq. (1). If we now substitute the values of $\hat{R}$ and $\hat{\delta}$ found from Ψ, back into Eq. (11) we would recover the starting function Ω^*. Clearly, the knowledge of Ψ allows us to calculate A_μ and in particular the force K which we must ad-scribe to the quantum particle when treated clasically, subjected also to the action of a stochastic force.

III. THE REDUCTION TO USUAL QUANTUM MECHANICS

In order to investigate the conditions under which one can recover the con-ventional quantum mechanical theory, let us write

$$\Psi = U(u, t) \varphi(r, t) \chi(u, r; t) \tag{12}$$

and

$$\Omega = F(u, t) + G(r, t) + E(u, r; t) \tag{13}$$

where it must be assumed that, in the general case, χ is a non-separable, non-trivial function of all phase space coordinates and also, that E is the only term in Ω containing the contributions in which both the r and u spaces are mixed.

Substitution of Eqs. (12) and (13) into Eq. (10) yields a differential equation which in turn may be separated into the following set of equations, namely,

$$i \frac{\partial \varphi}{\partial t} = -\frac{1}{2} \lambda_{ij} \partial_i \partial_j \varphi + G\varphi \tag{14}$$

$$i \frac{\partial U}{\partial t} = -\frac{1}{2} \lambda_{u_i u_j} \partial_i \partial_j U + FU \tag{15}$$

$$i \frac{\partial \chi}{\partial t} = -\frac{1}{2} \lambda_{ij} \partial_i \partial_j \chi - \frac{1}{2} \lambda_{u_i u_j} \partial_{u_i} \partial_{u_j} \chi - \lambda_{iu_j} \partial_i \partial_{u_j} \chi -$$

---

*It is a well known fact that in ordinary quantum mechanics one can express the potential energy V explicitly in terms of the wave function [c.f. L. Landau & E. Lifshitz, Quantum Mechanics, Addison-Wesley Publ. Co. 1958 Chap. III].

$$-\left[\lambda_{ij}\frac{\partial_i\varphi}{\varphi}\partial_j\chi+\lambda_{u_iu_j}\frac{\partial_jU}{U}\partial_{u_i}\chi+\lambda_{iu_j}\frac{\partial_i\varphi}{\varphi}\partial_{u_j}\chi\right.$$

$$\left.+\lambda_{iu_j}\frac{\partial_jU}{U}\partial_i\chi\right]-\left[\lambda_{iu_j}\frac{\partial_i\varphi}{\varphi}\frac{\partial_jU}{U}-E\right]\chi$$

(16)

In these equations, the matrix λ has been explicitly separated into the submatrices λ_{ij}, $\lambda_{u_iu_j}$ and λ_{iu_j} whose elements belong to the configuration space, velocity space and the mixed components, respectively. Also, since U does not depend on the configuration space coordinates we have written $\partial_{u_i}U\equiv\partial_iU$.

To accomplish the recovery of conventional quantum mechanics, which is possible only in the limit $\beta t>>1$[3,4] we must induce the separability of the configuration and velocity spaces, thus requiring that in this limit χ equals a constant. According to Eq.(16) this is achieved through the condition

$$\lim_{\beta t\to\infty}\left[\lambda_{iu_j}\frac{\partial_i\varphi}{\varphi}\frac{\partial_jU}{U}-E(u,r,t)\right]\to 0$$

(17)

Under the assumption that Eq.(17) is fulfilled, χ may be taken equal to one in Eq.(12). Therefore, the amplitude Ψ will be equal to the product $U\varphi$ where these two functions are independent and satisfy Eqs.(14) and (15). Furthermore since the matrix λ is entirely at our disposal we shall choose it in such a way that the three submatrices $\|\lambda_{ij}\|$, $\|\lambda_{u_iu_j}\|$ and $\|\lambda_{iu_j}\|=\|\lambda_{u_ji}\|$, are each one diagonal. It is now evident that Eq.(14) will reduce to Schroedinger's equation provided that the elements λ_{ij}, which may be in general time dependent, reduce in this limit to the value $2D\delta_{ij}$ with $D=\hbar/2m$. Thus, we may identify the function $\hbar G$,

227

in this limit, with the potential V and obtain that

$$i \frac{\partial \varphi}{\partial t} = - \frac{\hbar}{2m} \nabla^2 \varphi + \frac{V}{\hbar} \varphi \tag{18}$$

Here, the "diffusion coefficient" D is related, as usual,[7] to the independent parameters q and β through the relation $D\beta^2 = q$ and its value $\hbar/2m$ has already been obtained in previous work[3,4].

To round up our recovery of ordinary quantum mechanics let us introduce the macroscopic local velocity $c(r, t)^*$, in this asymptotic limit, through the ordinary averaging process familiar in statistical mechanics, namely,

$$c(r, t) = \frac{\int u w \, du}{\int w \, du} \tag{19}$$

On the other hand, from Eqs. (2), (6) and (7) we have that

$$u = 2D \nabla \mathcal{S} + 2D\beta \lambda \nabla_u \mathcal{S} \tag{20}$$

where use has been made of the fact that $\| \lambda_{iu_j} \|$ is diagonal with elements given by $\lambda_{iu_j} = 2D\beta\lambda\delta_{ij}$, λ being a real, arbitrary function of time. Substitution of Eq. (20) back into Eq. (19) yields the result that

$$c(r, t) = 2D\nabla S \tag{21}$$

where we have written $\mathcal{S} = S(r, t) + S_u(u, t)$ and made use of the fact that the contribution from the term proportional to $\nabla_u S_u$ is equal to zero as it will be shown later on [c.f. Eq. (23)]

Eq. (21) gives, indeed, the local macroscopic velocity in the ordinary quantum mechanical theory[1,2,3]. Following now the argument given in II and III

---

*This quantity has been denoted as $v(r, t)$ in previous papers.

we may assure ourselves that we have in fact recovered not only Schroedinger's equation but furthermore the uncertainty relations together with all their implications

Let us now make a brief discussion of Eqs. (15) and (17). Recalling that the matrix $\| \lambda_{u_i u_j} \|$ has been chosen to be diagonal, we take its elements to be equal to a constant in this asymptotic limit and hence write that $\lambda_{u_i u_j} = 2q \, \delta_{ij}$. Furthermore $F(u, t)$ is the function which in Eq. (15) plays a role analogous to the role assumed by G in Eq. (14). Indeed, since this latter function expresses the energy in configuration space we can extrapolate to the velocity space and assume that F expresses the energy of the particle in this space and thus identify it with the kinetic energy. Hence,

$$\hbar F = \frac{1}{2} m u^2 = \frac{u^2}{4D} \, \hbar$$

which in turn, when plugged into Eq. (15), yield:

$$i \, \frac{\partial U}{\partial t} = - q \nabla^2 U + \frac{u^2}{4D} \, U \tag{22}$$

With these assumptions, we may give to $\hbar \Omega$ a very simple physical content, namely,

$$\hbar \Omega = \frac{1}{2} \, m u^2 + V + E(u, r, t)$$

where the function $E(u, r; t)$ represents the contribution to Ω arising from the velocity dependent force. This may be seen directly from Eq. (17) if we realize that the asymptotic value of the term $\dfrac{\nabla_u U}{U}$ is given by $\nabla_u R_u$ which is equal to $(K_u - \beta u)/2q$ as it may be easily verified using Eqs. (2), (6) and (7), and furthermore $\dfrac{\nabla \varphi}{\varphi}$ is essentially a measure of the local macroscopic velocity according to ordinary quantum mechanics. Also, from the fact that $\overline{\nabla_u R} = 0$[4] the average value of E is equal to zero, and hence $\hbar \Omega$ is, in the mean, equal to the total energy of the system.

229

Finally, we observe that Eq. (22) is universal in the sense that once the separability of the two spaces is achieved, this same equation holds for all problems in configuration space. Moreover, since the probability density must be integrable and everywhere finite we must take for the solution of Eq. (22) the only one consistent with these conditions, namely, that of the isotropic harmonic oscillator. Hence

$$U = N_{\nu\lambda} \{ \exp [- i\varepsilon t / h - \xi^2 / 2] \} \xi^{\lambda} F(-\nu, \lambda + 3/2 ; \xi^2) Y_{\lambda\mu}(\theta_{u}, \varphi_{u})$$

(23)

where $\vec{\xi} = au$ with $a^2 = \beta/2q$ and the quantum numbers λ, μ, ν belonging to velocity space, being integers. We recall that ν and λ are intepndent whereas $|\mu| < \lambda$. The ground state $\nu = \lambda = 0$ corresponds to a gaussian distribution with a density equal to $\exp(-\beta u^2/2q)$ and has been discussed in a previous paper (IV).

However, in the general case we have many more possibilities, at least in principle, and thus the question arises as to what is the physical interpretation of the U function, which as it is seen from Eq. (12) multiplies the usual Schroedinger's wave function. We propose here a tentative scheme along the following lines. Recalling once more the fact that we are already discussing the asymptotic limit and hence the function U is entirely independent of the configuration space coordinates, we assume that the quantum numbers ν, λ and μ are related to the particle's own internal quantum numbers. In particular, and as a first proposition, one could associate λ and μ with the particle's spin in which case, with the aid of Eq. (23) the construction of spin-wave functions would follow for the case of mesons, which consistently with ordinary quantum mechanics appear as factors of the Schroedinger's amplitude. This proposal thus identifies the usual spin space with the velocity space when this latter one is orthogonal to the configuration space, which we emphasize is the only one having a meaningful physical content. Further clarification of this point seems pertinent at this stage. The physical picture of the present scheme consists of an initial stage in the motion of the particle where both r- and u- spaces are intimately related and thus contain all the relevant physical

information. However as soon as a time of order β^{-1} is reached a separation is achieved whereby the two spaces become orthogonal to each other and from there on all information concerning the system is accumulated in the configuration space through the usual Schroedinger equation. The velocity space becomes a formal one and thus consistently with the procedure followed in ordinary quantum mechanics it may be used to construct the spin-wave amplitude Ψ.

Pursuing the ideas set forth in the scheme described above it is important to realize that during the initial stage $(\beta t \stackrel{<}{\sim} 1)$ of the particle's motion the fact that the configuration and velocity spaces are interelated implies that an interaction between the spin of the particle and its angular momentum could occur, which according to the interpretation given to the quantum number λ would imply in turn the possibility of a decay. Taking into account the fact that the mean life times for mesonic resonances are of the order of 10^{-22} to 10^{-23} sec. a rough estimate of β is obtained namely, $\beta \sim 10^{22}$ to 10^{23} sec^{-1}. This value may be estimated independently if we recall that from Eq. (23) the particle's energy in velocity space is

$$\varepsilon = \hbar\beta(2\nu + \lambda)$$

up to an additive constant. This energy being an intrinsic characteristic of the particle may be consistently associated with a mass term. This inference corresponds only qualitatively to experiment but once more, may serve to get a rough estimate of β. Indeed, comparing masses between mesons with the same quantum numbers but whose spin differs by unity, the product $\hbar\beta$ turns out to be of the order of 300 to 400 Mev which corresponds to the order of magnitude of β indicated above.

As a concluding remark we would like to indicate that the connection between this work with von Neumman's theorem on hidden variables follows again within the same lines of thought advanced in a previous paper (IV) to which the interested reader is referred.

REFERENCES

1. L. de la Peña-Auerbach, Phys. Letters **24A**, 603, (1967).

2. L. de la Peña-Auerbach, E. Braun and L.S. García-Colín, J. of Math. Phys. (In press).

3. L. de la Peña-Auerbach and L.S. García-Colín; J. of Math. Phys. (In press).

4. L. de la Peña-Auerbach and L.S. García-Colín; J. of Math. Phys. (In press).

5. L. de la Peña-Auerbach and Ana Ma. Cetto (to be published).

6. Ming Chen Wang & G.E. Uhlenbeck, in Selected Papers on Noise and Stochastic Processes, Ed. by N. Wax Dover Publ. Inc. 1954, or Rev. Mod. Phys. **17**, 325, (1945).

7. S. Chandrasekhar, same book as ref. 6 or Rev. Mod. Phys. **15**, 1 (1943).