

EXACT DETERMINATION OF THE GAMOW-TELLER MATRIX ELEMENTS  
FOR SUPERMULTIPLY STATES\*

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ABSTRACT

*The explicit expression of the Gamow-Teller operator  $\sigma_q \tau_{\pm}$  in terms of the generators of the unitary group of 4 dimensions  $U_4$  associated with spin-isospin was given by Moshinsky and Nagel. We can therefore determine the matrix elements of the generators of  $U_4$  with respect to states*

$$| \{v_1 v_2 v_3 v_4\} \beta STM_S M_T \rangle \quad (1)$$

*where  $\{v_1 v_2 v_3 v_4\}$  is the irreducible representation (IR) of  $U_4$ ,  $STM_S M_T$  are, re-*

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spectively, the spin, isospin and their projections, and  $\beta$  indicates some extra quantum numbers. The latter matrix elements can be obtained using the result of Gelfand and Zetlin for the matrix elements of the generators of  $U_4$  for states characterized by a canonical chain of subgroups, and then obtaining the transformation brackets between the canonical chain and the states 1. In this article we carry the last step and so have an exact procedure for determining the matrix elements of the Gamow-Teller operator with respect to supermultiplet states. An application to  $F^{18} \rightarrow O^{18}$  was given in another publication.

#### RESUMEN

La expresión explícita del operador Gamow-Teller  $\sigma_q \tau_{\pm}$  en términos de los generadores de un grupo unitario de 4 dimensiones asociados con el espacio de espin-isoespin fueron dados por Moshinsky y Nagel. Podemos por consiguiente determinar los elementos de matriz del operador Gamow-Teller si conocemos los elementos de matriz de los generadores de  $U_4$  con respecto a los estados

$$| \{v_1 v_2 v_3 v_4\} \beta STM_S M_T \rangle \quad (1)$$

donde  $\{v_1 v_2 v_3 v_4\}$  es la representación irreducible de  $U_4(RI)$ ,  $STM_S M_T$  son respectivamente, el espin, isoespin y sus proyecciones y  $\beta$  indica algunos números cuánticos adicionales. Estos últimos elementos de matriz pueden ser obtenidos utilizando los resultados de Gelfand y Zetlin para los elementos de matriz de los generadores de  $U_4$  para estados caracterizados por una cadena canónica de subgrupos, y a continuación obteniendo los paréntesis de transformación entre la cadena canónica y los estados 1. Hemos llevado a cabo este último paso y por lo tanto tenemos un procedimiento exacto para la determinación de los elementos de matriz del operador Gamow-Teller con respecto a estados de supermultiplete. En una publicación anterior se aplicaron estos resultados a la transición  $F^{18} \rightarrow O^{18}$ .

## 1. INTRODUCTION

In recent publications<sup>1,2,3</sup> the supermultiplet classification of the states has come back into favor as an intermediate step in the  $SU_3$  scheme. In this classification the  $n$  particle states could be denoted by the ket<sup>2,3</sup>

$$|\alpha L, \{v_1 v_2 v_3 v_4\} \beta ST, JM\rangle \quad (1.1)$$

where  $\{v_1 v_2 v_3 v_4\}$  is the partition of  $n$  characterizing the irreducible representation (IR) of the  $U_4$  group associated with the four single particle spin isospin states,  $L, J, S, T$  are respectively the orbital and total angular momentum, spin and isospin,  $\alpha, \beta$  are additional quantum numbers<sup>2,3</sup>

## 2. MATRIX ELEMENTS OF ALLOWED BETA OPERATORS

We are interested in determining the matrix elements of allowed Beta operators with respect to the states (1.1). This is trivial for allowed Fermi transitions since the one body operator  $\sum_{i=1}^n t_{\bar{q}}^{(i)}, \bar{q} = \pm 1$ , where  $t_{\bar{q}}^{(i)}$  is the  $\bar{q}$  component of the isospin of particle  $i$ , can be expressed in terms of the total isotopic spin which is an integral of motion of (1.1).

For the Gamow-Teller allowed transitions the situation is not so simple. The one body operator is given by,

$$\sum_{i=1}^n t_{\bar{q}}^{(i)} S_q^{(i)} \equiv R_{\bar{q}q} \quad (2.1)$$

where  $S_q^{(i)}$  is the  $q$  component of the spin of the particle  $i$ . The matrix element of (2.1) with respect to states (1.1) can be reduced by standard Racah algebra<sup>4</sup> to the matrix element

$$\langle \{v_1 v_2 v_3 v_4\} \beta' S' T' M'_S M'_T | R_{qq}^- | \{v_1 v_2 v_3 v_4\} \beta S T M_S M_T \rangle \quad (2.2)$$

We note that  $R_{qq}^-$  is a Racah tensor of order 1 both in spin and isospin space. Applying Racah's<sup>4</sup> result to our matrix element we obtain the following expression,

$$\begin{aligned} & \langle \alpha' L', \{v_1 v_2 v_3 v_4\} \beta' S' T' M'_S M'_T; J' M' | R_{qq}^{(1)(1)} | \alpha L, \{v_1 v_2 v_3 v_4\} \beta S T M_S M_T; J M \rangle = \\ & = (-)^{S'-J-2J'+L+1} \left[ \frac{2J+1}{2T'+1} \right]^{\frac{1}{2}} W(S' J' S J; L 1) \langle J 1 M q | J' M' \rangle \langle T 1 M_T \bar{q} | T' M'_T \rangle \times \end{aligned}$$

$$\langle \{v_1 v_2 v_3 v_4\} \beta' S' T' ||| R^{(1)(1)} ||| \{v_1 v_2 v_3 v_4\} \beta S T \rangle \quad (2.3)$$

where

$$\begin{aligned} & \langle \{v_1 v_2 v_3 v_4\} \beta' S' T' M'_S M'_T | R_{qq}^{(1)(1)} | \{v_1 v_2 v_3 v_4\} \beta S T M_S M_T \rangle = \\ & = \frac{1}{\sqrt{(2S'+1)(2T'+1)}} \langle T 1 M_T \bar{q} | T' M'_T \rangle \\ & \langle S 1 M_S q | S' M'_S \rangle \langle \{v_1 v_2 v_3 v_4\} \beta' S' T' ||| R^{(1)(1)} ||| \{v_1 v_2 v_3 v_4\} \beta S T \rangle \quad (2.4) \end{aligned}$$

As we can see from the last expression all the orbital part is contained in the Racah W and Clebsch-Gordan coefficient  $\langle | \rangle$  and we obtain a matrix element which depends only upon states in the supermultiplet classification.

The operators  $R_{qq}^-$  are linear combinations<sup>2</sup> of the generators  $C_r^{r'}$ ,  $r, r' = 1, 2, 3, 4$  of the  $U_4$  group and so (2.2) could be determined if we in turn could obtain the matrix elements of these generators. To achieve this purpose we note that Gelfand and Zetlin<sup>5</sup> determined explicitly the matrix elements of the generators of  $U_4$  with respect to the states characterized by the IR of the chain of

Groups  $U_4 \supset U_3 \supset U_2 \supset U_1$  i.e., the Gelfand states

$$|b_{kl}\rangle; 1 \ll k \ll l \ll 4 \quad (2.5)$$

where  $[b_{1l} \dots b_{ll}]$  is the irreducible representation of the  $U_l$  group  $l = 1, 2, 3, 4$ .

The states in (2.2) are on the other hand classified by the IR of the chain of Groups  $U_4 \supset SU_2^{(\sigma)} \times SU_2^{(\tau)}$  where  $SU_2^{(\sigma, \tau)}$  are the two dimensional unitary unimodular groups associated with the spin and isospin spaces separately. To calculate (3) we need then besides the Gelfand and Zetlin result<sup>6</sup> the transformation bracket,

$$\langle b_{kl} | \{v_1 v_2 v_3 v_4\} \beta_{STM_S M_T} \rangle \quad (2.6)$$

where  $v_i = b_{i4}$ ,  $i = 1, 2, 3, 4$ .

As the ket in (2.6) is an eigenstate of  $S^2, T^2, S_0, T_0$  it could be a question of first determining the matrix elements of these operators with respect to the Gelfand states (2.5). As  $S_q, T_q$  are in turn linear combinations<sup>2</sup> of the generators  $C_r^{\pm}$  which are given by<sup>2</sup>

$$S_1 = -\frac{1}{\sqrt{2}} (C_1^3 + C_2^4) \quad T_1 = -\frac{1}{\sqrt{2}} (C_1^2 + C_3^4)$$

$$S_0 = \frac{1}{2} (C_1^1 + C_2^2 - C_3^3 - C_4^4) \quad T_0 = \frac{1}{2} (C_1^1 + C_3^3 - C_2^2 - C_4^4)$$

$$S_{-1} = \frac{1}{\sqrt{2}} (C_3^1 + C_4^2) \quad T_{-1} = \frac{1}{\sqrt{2}} (C_2^1 + C_4^3)$$

$S^2$  and  $T^2$  are given by the following expressions in terms of these operators,

$$S^2 = -2S_{-1}S_1 + S_0(S_0 + 1)$$

$$T^2 = -2T_{-1}T_1 + T_0(T_0 + 1)$$

Using the hermicity property of the generators of  $U_4$ ,  $(C_S^S)^\dagger = C_S^S$ , will obtain for the matrix elements of  $S^2$  and  $T^2$  with respect to Gelfand states,

$$\begin{aligned} \langle b'_{pq} | S^2 | b_{pq} \rangle &= \sum_{b''_{pq}} \langle b''_{pq} | (C_1^3 + C_2^4) | b'_{pq} \rangle \langle b''_{pq} | (C_1^3 + C_2^4) | b_{pq} \rangle \\ &+ M_S(M_S + 1) \delta_{b'_{pq} b_{pq}} \end{aligned} \quad (2.7a)$$

and

$$\begin{aligned} \langle b'_{pq} | T^2 | b_{pq} \rangle &= \sum_{b''_{pq}} \langle b''_{pq} | (C_1^2 + C_3^4) | b'_{pq} \rangle \langle b''_{pq} | (C_1^2 + C_3^4) | b_{pq} \rangle \\ &+ M_T(M_T + 1) \delta_{b'_{pq} b_{pq}} \end{aligned} \quad (2.7b)$$

where  $b''_{pq}$  is an intermediate state and,

$$M_S = \frac{1}{2} \langle b'_{pq} | (C_1^1 + C_2^2 - C_3^3 - C_4^4) | b_{pq} \rangle$$

$$M_T = \frac{1}{2} \langle b'_{pq} | (C_1^1 + C_3^3 - C_2^2 - C_4^4) | b_{pq} \rangle$$

As we can see from the last expression the matrix elements of  $S_0$  and  $T_0$  are diagonal with respect to Gelfand states, since they are only given in terms of weight generators, so we will have to consider only  $S^2$  and  $T^2$ .



Using the closed expression obtained by Nagel and Moshinsky<sup>7</sup> for the matrix elements of the  $U_4$  generators with respect to Gelfand states which is equivalent to the one obtained by Gelfand and Zetlin we arrive to the expressions for the matrix elements of these generators given in table I, the rest of the matrix elements of our generators are given in reference [2].

Using these results the matrix elements (2.7a) and (2.7b) are finally written as in table II and III.

Once we have the commuting matrices  $S^2$  and  $T^2$  with respect to states (2.5) characterized by a definite IR of  $\{v_1 v_2 v_3 v_4\}$  of  $U_4$ , the unitary matrices that diagonalize them simultaneously are precisely the coefficients (2.6). In this way we have an exact and explicit procedure for calculating allowed Gamow-Teller matrix elements with respect to states (1.1) in the supermultiplet classification. An application of this method was given<sup>8</sup> for the transition between the ground states of  $F^{18}$  and  $O^{18}$ , though due to an error in the calculation the right theoretical  $ft$  value is 2366seg. A program for the diagonalization of  $S^2$  and  $T^2$  is in progress.

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$$\left\langle \begin{matrix} h_{13} & h_{23} & h_{33} \\ h_{12} & h_{22} & h_{32} \\ h_{11} & h_{21} & h_{31} \end{matrix} \right| \begin{matrix} \rightarrow \\ 4 \end{matrix} \left| \begin{matrix} h_{13} & h_{23} & h_{33} \\ h_{12} & h_{22} & h_{32} \\ h_{11} & h_{21} & h_{31} \end{matrix} \right\rangle = \frac{(h_{12}-h_{22})(h_{13}-h_{23}+1)(h_{12}-h_{22}+1)(h_{13}-h_{23}+2)(h_{12}-h_{22}+1)(h_{13}-h_{23}+1)}{(h_{12}-h_{22})(h_{13}-h_{23}+1)(h_{12}-h_{22}+1)(h_{13}-h_{23}+2)(h_{12}-h_{22}+1)(h_{13}-h_{23}+1)}$$

$$\left\langle \begin{matrix} h_{13} & h_{23} & h_{33} \\ h_{12} & h_{22} & h_{32} \\ h_{11} & h_{21} & h_{31} \end{matrix} \right| \begin{matrix} \rightarrow \\ 4 \end{matrix} \left| \begin{matrix} h_{13} & h_{23} & h_{33} \\ h_{12} & h_{22} & h_{32} \\ h_{11} & h_{21} & h_{31} \end{matrix} \right\rangle = \frac{(h_{12}-h_{22}+1)(h_{13}-h_{23}+1)(h_{12}-h_{22}+1)(h_{13}-h_{23}+2)(h_{12}-h_{22}+1)(h_{13}-h_{23}+1)}{(h_{12}-h_{22})(h_{13}-h_{23}+1)(h_{12}-h_{22}+1)(h_{13}-h_{23}+2)(h_{12}-h_{22}+1)(h_{13}-h_{23}+1)}$$

$$\left\langle \begin{matrix} h_{13} & h_{23} & h_{33} \\ h_{12} & h_{22} & h_{32} \\ h_{11} & h_{21} & h_{31} \end{matrix} \right| \begin{matrix} \rightarrow \\ 4 \end{matrix} \left| \begin{matrix} h_{13} & h_{23} & h_{33} \\ h_{12} & h_{22} & h_{32} \\ h_{11} & h_{21} & h_{31} \end{matrix} \right\rangle = \frac{(h_{12}-h_{22})(h_{13}-h_{23}+1)(h_{12}-h_{22}+1)(h_{13}-h_{23}+2)(h_{12}-h_{22}+1)(h_{13}-h_{23}+1)}{(h_{12}-h_{22})(h_{13}-h_{23}+1)(h_{12}-h_{22}+1)(h_{13}-h_{23}+2)(h_{12}-h_{22}+1)(h_{13}-h_{23}+1)}$$



$$\begin{array}{c}
 \left\langle \begin{array}{c} h_{13}+1 \quad h_{23} \quad h_{33} \\ h_{12}+1 \quad h_{22} \\ h_{11}+1 \end{array} \right\rangle \\
 \begin{array}{c} \text{C} \\ \text{1} \end{array} \\
 \left| \begin{array}{c} h_{13} \quad h_{23} \quad h_{33} \\ h_{12} \quad h_{22} \\ h_{11} \end{array} \right\rangle
 \end{array}
 =
 \frac{(h_{11}-h_{12}+1)(h_{12}-h_{23}+1)(h_{22}-h_{33}+2)(h_{23}-h_{33}+2)(h_{22}-h_{23}+1)(h_{23}-h_{33}+1)(h_{23}-h_{33}+2)(h_{23}-h_{33}+3)}{(h_{12}-h_{22}+2)(h_{12}-h_{22}+1)(h_{13}-h_{23}+2)(h_{23}-h_{23}+1)(h_{23}-h_{33}+2)(h_{23}-h_{33}+3)}$$

$$\begin{array}{c}
 \left\langle \begin{array}{c} h_{13} \quad h_{23} \quad h_{33} \\ h_{12} \quad h_{22} \\ h_{11} \end{array} \right\rangle \\
 \begin{array}{c} \text{C} \\ \text{1} \end{array} \\
 \left| \begin{array}{c} h_{13} \quad h_{23} \quad h_{33} \\ h_{12} \quad h_{22} \\ h_{11}+1 \end{array} \right\rangle
 \end{array}
 =
 \frac{(h_{12}-h_{11})(h_{23}-h_{22})(h_{22}-h_{33}+1)(h_{23}-h_{12}+1)(h_{13}-h_{23})(h_{23}-h_{33}+1)(h_{23}-h_{33}+2)(h_{23}-h_{33}+3)}{(h_{12}-h_{22})(h_{12}-h_{22}+1)(h_{13}-h_{23}+2)(h_{23}-h_{33}+3)(h_{23}-h_{33}+1)(h_{23}-h_{33}+2)}$$

$$\begin{array}{c}
 \left\langle \begin{array}{c} h_{13}+1 \quad h_{23} \quad h_{33} \\ h_{12}+1 \quad h_{22} \\ h_{11} \end{array} \right\rangle \\
 \begin{array}{c} \text{C} \\ \text{1} \end{array} \\
 \left| \begin{array}{c} h_{13} \quad h_{23} \quad h_{33} \\ h_{12} \quad h_{22} \\ h_{11} \end{array} \right\rangle
 \end{array}
 =
 \frac{(h_{11}-h_{12}+1)(h_{13}-h_{12})(h_{12}-h_{33}+2)(h_{23}-h_{22}+1)(h_{23}-h_{33}+1)(h_{23}-h_{33}+2)(h_{23}-h_{33}+3)}{(h_{12}-h_{22}+2)(h_{12}-h_{22}+1)(h_{13}-h_{23})(h_{23}-h_{33}+2)(h_{23}-h_{33}+1)(h_{23}-h_{33}+3)}$$

$$\left\langle \begin{array}{c} h_{13}+1 \quad h_{23} \quad h_{33} \\ h_{12}+1 \quad h_{22} \\ h_{11} \end{array} \right\rangle \begin{array}{c} C_4 \\ C_2 \end{array} \left\langle \begin{array}{c} h_{13} \quad h_{23} \quad h_{33} \\ h_{12} \quad h_{22} \\ h_{11} \end{array} \right\rangle = \frac{(h_{12}-h_{11}+1)(h_{12}-h_{23}+1)(h_{12}-h_{33}+2)(h_{13}-h_{22}+2)(h_{14}-h_{13})(h_{13}-h_{24}+1)(h_{13}-h_{34}+2)(h_{15}-h_{14}+3)}{(h_{12}-h_{22}+2)(h_{12}-h_{22}+1)(h_{13}-h_{23}+2)(h_{13}-h_{23}+1)(h_{13}-h_{33}+2)(h_{13}-h_{33}+1)}$$

$$\left\langle \begin{array}{c} h_{13}+1 \quad h_{23} \quad h_{33} \\ h_{12} \quad h_{22}+1 \\ h_{11} \end{array} \right\rangle \begin{array}{c} C_4 \\ C_2 \end{array} \left\langle \begin{array}{c} h_{11} \quad h_{22} \quad h_{33} \\ h_{12} \quad h_{22} \\ h_{11} \end{array} \right\rangle = \frac{(h_{11}-h_{12})(h_{12}-h_{22})(h_{22}-h_{33}+1)(h_{13}-h_{22}+1)(h_{14}-h_{23})(h_{15}-h_{24}+1)(h_{13}-h_{34}+2)(h_{15}-h_{14}+3)}{(h_{12}-h_{22})(h_{12}-h_{22}+1)(h_{13}-h_{23}+2)(h_{13}-h_{23}+1)(h_{13}-h_{33}+2)(h_{13}-h_{33}+1)}$$

$$\left\langle \begin{array}{c} h_{13} \quad h_{23} \quad h_{33} \\ h_{12}+1 \quad h_{22} \\ h_{11} \end{array} \right\rangle \begin{array}{c} C_4 \\ C_2 \end{array} \left\langle \begin{array}{c} h_{13} \quad h_{23} \quad h_{33} \\ h_{12} \quad h_{22} \\ h_{11} \end{array} \right\rangle = \frac{(h_{12}-h_{11}+1)(h_{13}-h_{12})(h_{12}-h_{33}+2)(h_{13}-h_{22}+1)(h_{14}-h_{23}+1)(h_{14}-h_{23}+1)(h_{15}-h_{14}+3)}{(h_{12}-h_{22}+2)(h_{12}-h_{22}+1)(h_{13}-h_{23})(h_{13}-h_{23}+2)(h_{13}-h_{33}+1)(h_{13}-h_{33}+2)}$$



$$\begin{pmatrix} h_{13} & h_{23} & h_{33} \\ h_{12} & h_{22} & h_{11} \end{pmatrix} \begin{matrix} C_4 \\ C_3 \end{matrix} = \frac{(h_{13}-h_{12}+1)(h_{13}-h_{22}+2)(h_{14}-h_{13})(h_{13}-h_{24}+1)(h_{13}-h_{24}+2)(h_{13}-h_{24}+3)}{(h_{13}-h_{23}+2)(h_{13}-h_{23}+3)(h_{13}-h_{23}+1)(h_{13}-h_{23}+2)}$$

$$\begin{pmatrix} h_{13} & h_{23} & h_{33} \\ h_{12} & h_{22} & h_{11} \end{pmatrix} \begin{matrix} C_4 \\ C_3 \end{matrix} = \frac{(h_{12}-h_{23})(h_{23}-h_{22}+1)(h_{14}-h_{23}+1)(h_{14}-h_{23})(h_{23}-h_{24}+1)(h_{23}-h_{24}+2)}{(h_{13}-h_{23})(h_{13}-h_{23}+2)(h_{13}-h_{23}+1)(h_{13}-h_{23}+1)}$$

$$\begin{pmatrix} h_{13} & h_{23} & h_{33} \\ h_{12} & h_{22} & h_{11} \end{pmatrix} \begin{matrix} C_4 \\ C_3 \end{matrix} = \frac{(h_{12}-h_{33}+1)(h_{22}-h_{33})(h_{14}-h_{33}+2)(h_{24}-h_{33}+1)(h_{24}-h_{33})(h_{33}-h_{24}+1)}{(h_{13}-h_{23}+1)(h_{23}-h_{33})(h_{13}-h_{33}+2)(h_{23}-h_{33}+1)}$$



































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$$\begin{aligned}
 & (h_{14} - h_{13} + 1)(h_{14} - h_{13} + 2)(h_{13} - h_{124}) (h_{14} - h_{13} + 1)(h_{13} - h_{124} + 1)(h_{14} - h_{124} + 2)(h_{13} - h_{124} + 1)(h_{13} - h_{124} + 1)(h_{12} - h_{13} + 1)(h_{12} - h_{13}) \\
 & (h_{13} - h_{23} + 1)(h_{13} - h_{133} + 1)(h_{13} - h_{123})(h_{13} - h_{133})(h_{13} - h_{133} + 2)(h_{13} - h_{133} + 1)(h_{13} - h_{133} + 1)(h_{13} - h_{133})
 \end{aligned}$$

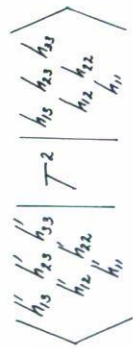
$$\sum_{h_{13} h_{14}} \delta''_{h_{23} h_{13}} \delta''_{h_{14} h_{13}} \delta''_{h_{12} h_{12}} \delta''_{h_{11} h_{11}} +$$

$$\begin{aligned}
 & (h_{14} - h_{13})(h_{14} - h_{13} + 2)(h_{13} - h_{124} + 1)(h_{13} - h_{124} + 1)(h_{13} - h_{124} + 2)(h_{13} - h_{124} + 1)(h_{13} - h_{124} + 1)(h_{12} - h_{13} + 2)(h_{12} - h_{13} + 1)(h_{12} - h_{13}) \\
 & (h_{13} - h_{123} + 2)(h_{13} - h_{123} + 1)(h_{13} - h_{133} + 3)(h_{13} - h_{133} + 2)(h_{13} - h_{133} + 1)(h_{13} - h_{133})
 \end{aligned}$$

$$\delta''_{h_{13} h_{13}} \delta''_{h_{23} h_{13}} \delta''_{h_{14} h_{12}} \delta''_{h_{12} h_{12}} \delta''_{h_{11} h_{11}} +$$

$$\begin{aligned}
 & (h_{14} - h_{13} + 2)(h_{14} - h_{13} + 1)(h_{14} - h_{123} + 1)(h_{14} - h_{123} + 1)(h_{14} - h_{123} + 2)(h_{13} - h_{123} + 1)(h_{13} - h_{123} + 1)(h_{12} - h_{13} + 1)(h_{12} - h_{13}) \\
 & (h_{13} - h_{133} + 1)(h_{13} - h_{133} + 2)(h_{13} - h_{133} + 1)(h_{13} - h_{133} + 1)(h_{13} - h_{133} + 2)(h_{13} - h_{133} + 1)(h_{13} - h_{133} + 1)(h_{13} - h_{133})
 \end{aligned}$$





$$= \left\{ (h'_{12} - h''_{11})(h'_{11} - h'_{22} + 1) + \frac{1}{2} \left[ 2h'_{11} - 2(h'_{12} + h'_{22}) + 2(h'_{13} + h'_{23} + h'_{33}) - (h'_{14} + h'_{24} + h'_{34} + h'_{44}) - (h'_{14} + h'_{24} + h'_{34} + h'_{44}) \right] \right\} \frac{1}{2} \left[ 2h'_{11} - 2(h'_{12} + h'_{22}) + 2(h'_{13} + h'_{23} + h'_{33}) - \right.$$

$$\left. - (h'_{14} + h'_{24} + h'_{34} + h'_{44}) \right\} + \frac{(h'_{14} - h'_{13})(h'_{13} - h'_{24} + 1)(h'_{23} - h'_{34} + 2)(h'_{13} - h'_{12} + 1)(h'_{13} - h'_{22} + 2)}{(h'_{13} - h'_{23} + 2)(h'_{13} - h'_{23} + 1)(h'_{13} - h'_{33} + 3)(h'_{13} - h'_{33} + 2)}$$

$$+ \frac{(h'_{14} - h'_{23} + 1)(h'_{24} - h'_{23})(h'_{23} - h'_{34} + 1)(h'_{23} - h'_{44} + 2)(h'_{23} - h'_{22} + 1)(h'_{12} - h'_{23})}{(h'_{13} - h'_{23})(h'_{13} - h'_{33} + 1)(h'_{23} - h'_{33} + 2)(h'_{23} - h'_{33} + 1)}$$

$$+ \left. \frac{(h'_{14} - h'_{33} + 2)(h'_{24} - h'_{23} + 1)(h'_{34} - h'_{33})(h'_{23} - h'_{44} + 1)(h'_{12} - h'_{33} + 1)(h'_{22} - h'_{33})}{(h'_{13} - h'_{33} + 1)(h'_{13} - h'_{33} + 2)(h'_{23} - h'_{33} + 1)(h'_{23} - h'_{33})} \right\} \sum_{h'_{13}} \delta'_{11} h'_{13} \sum_{h'_{23}} \delta'_{11} h'_{23} \sum_{h'_{33}} \delta'_{11} h'_{33} \sum_{h'_{12}} \delta'_{11} h'_{12} \sum_{h'_{22}} \delta'_{11} h'_{22} \sum_{h'_{11}} \delta'_{11} h'_{11} +$$

$$+ \sqrt{\frac{(h'_{14} - h'_{13})(h'_{13} - h'_{24} + 1)(h'_{13} - h'_{34} + 2)(h'_{13} - h'_{44} + 3)(h'_{13} - h'_{12} + 1)(h'_{13} - h'_{22} + 2)(h'_{12} - h'_{11} + 1)(h'_{11} - h'_{22})}{(h'_{13} - h'_{23} + 2)(h'_{13} - h'_{23} + 1)(h'_{13} - h'_{33} + 3)(h'_{13} - h'_{33} + 2)}} \sum_{h'_{13}} \delta'_{11} h'_{13} \sum_{h'_{23}} \delta'_{11} h'_{23} \sum_{h'_{33}} \delta'_{11} h'_{33} \sum_{h'_{12}} \delta'_{11} h'_{12} \sum_{h'_{22}} \delta'_{11} h'_{22} \sum_{h'_{11}} \delta'_{11} h'_{11} +$$

$$\begin{aligned}
 & \frac{(h_{14} - h_{23} + 1)(h_{14} - h_{33})(h_{23} - h_{34} + 1)(h_{23} - h_{34} + 2)(h_{12} - h_{22} + 1)(h_{12} - h_{22})(h_{12} - h_{10} + 1)(h_{10} - h_{22})}{(h_{13} - h_{23})(h_{13} - h_{23} + 1)(h_{23} - h_{33} + 2)(h_{23} - h_{33} + 1)} \\
 & + \sum_{h_{13}, h_{13}} \delta_{h_{13}, h_{23}} \delta_{h_{23}, h_{33}} \delta_{h_{33}, h_{12}} \delta_{h_{12}, h_{22}} \delta_{h_{22}, h_{10}} \delta_{h_{10}, h_{11}} +
 \end{aligned}$$

$$\begin{aligned}
 & \frac{(h_{14} - h_{33} + 2)(h_{14} - h_{33} + 1)(h_{34} - h_{33})(h_{35} - h_{34} + 1)(h_{12} - h_{33})(h_{12} - h_{10} + 1)(h_{10} - h_{22})}{(h_{13} - h_{33} + 1)(h_{13} - h_{33} + 2)(h_{23} - h_{33} + 1)(h_{23} - h_{33})} \\
 & + \sum_{h_{13}, h_{13}} \delta_{h_{13}, h_{23}} \delta_{h_{23}, h_{33}} \delta_{h_{33}, h_{12}} \delta_{h_{12}, h_{22}} \delta_{h_{22}, h_{10}} \delta_{h_{10}, h_{11}} +
 \end{aligned}$$

$$\begin{aligned}
 & \frac{(h_{14} - h_{33} + 1)(h_{13} - h_{34})(h_{13} - h_{34} + 1)(h_{13} - h_{34} + 2)(h_{13} - h_{22})(h_{13} - h_{10} + 1)(h_{10} - h_{22})}{(h_{13} - h_{23} + 1)(h_{13} - h_{23})(h_{13} - h_{33} + 2)(h_{13} - h_{33} + 1)} \\
 & + \sum_{h_{13}, h_{13}} \delta_{h_{13}, h_{23}} \delta_{h_{23}, h_{33}} \delta_{h_{33}, h_{12}} \delta_{h_{12}, h_{22}} \delta_{h_{22}, h_{10}} \delta_{h_{10}, h_{11}} +
 \end{aligned}$$

$$\frac{(h_{14} - h_{33} + 1)(h_{14} - h_{33} + 1)(h_{13} - h_{34})(h_{13} - h_{34} + 1)(h_{13} - h_{34} + 2)(h_{13} - h_{22})(h_{13} - h_{10} + 1)(h_{10} - h_{22})}{(h_{13} - h_{23})(h_{13} - h_{23} - 1)(h_{13} - h_{33} + 2)(h_{13} - h_{33})(h_{13} - h_{33} + 1)(h_{23} - h_{33} + 2)(h_{23} - h_{33} + 1)}$$

$$\sum_{h_{13}, h_{13}} \delta_{h_{13}, h_{23}} \delta_{h_{23}, h_{33}} \delta_{h_{33}, h_{12}} \delta_{h_{12}, h_{22}} \delta_{h_{22}, h_{10}} +$$

$$\begin{aligned}
 & (h_{14} - h_{13} + 1)(h_{14} - h_{13} + 2)(h_{13} - h_{14}) (h_{14} - h_{13} + 1)(h_{13} - h_{14} + 1)(h_{14} - h_{13} + 2)(h_{13} - h_{14} + 1)(h_{13} - h_{14} + 2)(h_{13} - h_{14} + 1)(h_{13} - h_{14} + 2)(h_{13} - h_{14} + 1)(h_{13} - h_{14} + 2) \\
 & (h_{13} - h_{23} + 1)(h_{13} - h_{23} + 2)(h_{13} - h_{23} + 1)(h_{13} - h_{23} + 2)(h_{13} - h_{23} + 1)(h_{13} - h_{23} + 2)(h_{13} - h_{23} + 1)(h_{13} - h_{23} + 2)(h_{13} - h_{23} + 1)(h_{13} - h_{23} + 2)
 \end{aligned}$$

$$\int_{h_{13}}^{h_{14}} h_{13}^{-1} \int_{h_{23}}^{h_{14}} h_{23}^{-1} h_{13}^{+1} h_{12} \int_{h_{12}}^{h_{14}} h_{12}^{-1} h_{22} \int_{h_{22}}^{h_{11}} h_{11} +$$

$$\begin{aligned}
 & (h_{14} - h_{13})(h_{14} - h_{23} + 2)(h_{13} - h_{14} + 1)(h_{13} - h_{14} + 2)(h_{13} - h_{14} + 1)(h_{13} - h_{14} + 2)(h_{13} - h_{14} + 1)(h_{13} - h_{14} + 2)(h_{13} - h_{14} + 1)(h_{13} - h_{14} + 2) \\
 & (h_{13} - h_{23} + 2)(h_{13} - h_{23} + 1)(h_{13} - h_{23} + 2)(h_{13} - h_{23} + 1)(h_{13} - h_{23} + 2)(h_{13} - h_{23} + 1)(h_{13} - h_{23} + 2)(h_{13} - h_{23} + 1)(h_{13} - h_{23} + 2)
 \end{aligned}$$

$$\int_{h_{13}}^{h_{14}} h_{13}^{+1} \int_{h_{23}}^{h_{14}} h_{23}^{-1} h_{12} \int_{h_{12}}^{h_{14}} h_{12}^{-1} h_{22} \int_{h_{22}}^{h_{11}} h_{11} +$$

$$\begin{aligned}
 & (h_{14} - h_{13} + 2)(h_{14} - h_{23} + 2)(h_{13} - h_{14} + 1)(h_{13} - h_{14} + 2)(h_{13} - h_{14} + 1)(h_{13} - h_{14} + 2)(h_{13} - h_{14} + 1)(h_{13} - h_{14} + 2)(h_{13} - h_{14} + 1)(h_{13} - h_{14} + 2) \\
 & (h_{13} - h_{23} + 1)(h_{13} - h_{23} + 2)(h_{13} - h_{23} + 1)(h_{13} - h_{23} + 2)(h_{13} - h_{23} + 1)(h_{13} - h_{23} + 2)(h_{13} - h_{23} + 1)(h_{13} - h_{23} + 2)(h_{13} - h_{23} + 1)(h_{13} - h_{23} + 2)
 \end{aligned}$$





$$\sum_{h_1, h_2} \delta_{h_1, h_2} \sum_{h_3, h_4} \delta_{h_3, h_4} \sum_{h_5, h_6} \delta_{h_5, h_6} \sum_{h_7, h_8} \delta_{h_7, h_8} +$$

$$\frac{(h_{11} - h_{23} + 2)(h_{24} - h_{23} + 1)(h_{23} - h_{11})(h_{23} - h_{44} + 1)(h_{12} - h_{33})(h_{12} - h_{33} + 1)(h_{11} - h_{11})(h_{11} - h_{22} + 1)}{(h_{13} - h_{23} + 1)(h_{13} - h_{23} + 2)(h_{23} - h_{33} + 1)(h_{23} - h_{33})}$$

$$\sum_{h_1, h_2} \delta_{h_1, h_2} \sum_{h_3, h_4} \delta_{h_3, h_4} \sum_{h_5, h_6} \delta_{h_5, h_6} \sum_{h_7, h_8} \delta_{h_7, h_8} +$$

$$\frac{(h_{14} - h_{33} + 3)(h_{14} - h_{33} + 2)(h_{14} - h_{33} + 1)(h_{13} - h_{44})(h_{12} - h_{33} + 2)(h_{12} - h_{33} + 1)(h_{11} - h_{22} + 1)}{(h_{13} - h_{33} + 3)(h_{13} - h_{33} + 2)(h_{13} - h_{33} + 1)(h_{23} - h_{33})}$$

