HARMONIC-OSCILLATOR STATES FOR THREE-PARTICLE SYSTEMS APPLICATION TO THE FORM FACTOR OF THE PROTON AS A SYSTEM OF THREE QUARKS

V.C. Aguilera Navarro*, M. Moshinsky†

Instituto de Física, Universidad Nacional de México

W.W. Yeh‡

Physics Department, State University of New York, Stony Brook, N.Y.

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ABSTRACT

In this paper we shall construct explicitly three-particle harmonic-oscillator states with the following properties: a) That they are translationally invariant, b) That they have definite total orbital angular momentum, c) That they correspond to definite irreducible representations of the symmetric group of three particles S(3).

* Fellow of the Conselho Nacional de Pesquisas, Brasil. On leave from the Instituto de Física Teórica, São Paulo.

† Asesor Director de la Comisión Nacional de Energía Nuclear, México.

‡ Fellow of the National Science Foundation.

Once we have states with the above properties we proceed to apply them to the discussion of the form factor of the proton as a system of three quarks.

RESUMEN

En este trabajo construiremos explícitamente estados de oscilador armónico para sistemas de tres partículas con las siguientes propiedades: a) Que sean translacionalmente invariantes, b) Que tengan un momento angular orbital total definido c) Que correspondan a una representación definida del grupo simétrico de tres partículas S(3).

Una vez obtenidos los estados con las propiedades mencionadas procederemos a aplicarlos a la discusión del factor de forma del proton como sistema de tres quarks.

I. INTRODUCTION

In this paper we shall be interested in the construction of harmonic-oscillator states of three particles with the following properties: a) That they are translationally invariant, b) That they have definite total orbital angular momentum. c) That they correspond to definite irreducible representations of the symmetric group of three particles $S(3)$.

The restriction to translational invariance implies that the three-particle states will be functions of only two of the relative coordinates, which we could choose as the Jacobi coordinates[1] defined below. We would have then two single-particle harmonic-oscillator states associated with these two coordinates which we could couple to a definite total orbital angular momentum. Thus states having the properties a), b) are trivial to construct. The main problem then, which we discuss in the next section, is to obtain states that also have the property c).

Once we have the states with the above properties we proceed to apply them to the discussion of the form factor of the proton as a system of three quarks.

II. TRANSLATIONALLY INVARIANT THREE-PARTICLE STATES OF DEFINITE PERMUTATIONAL SYMMETRY

Our three particles could be characterized by the ordinary coordinates x^1, x^2, x^3, or by those related to them by the orthogonal Jacobi transformation

$$\dot{x}^1 = \frac{1}{\sqrt{2}} (x^1 - x^2)$$

$$\dot{x}^2 = \frac{1}{\sqrt{6}} (x^1 + x^2 - 2x^3) \tag{1}$$

$$\dot{x}^3 = \frac{1}{\sqrt{3}} (x^1 + x^2 + x^3)$$

The first two of this $\dot{x}^1, \dot{x}^2$ are obviously translationally invariant and so arbitrary translationally invariant three-particles states could be expanded in terms of

$$< \dot{x}^1 | \dot{n}_1 \dot{l}_1 \dot{m}_1 > < \dot{x}^2 | \dot{n}_2 \dot{l}_2 \dot{m}_2 > \tag{2}$$

where $< \dot{x} | nlm >$ are single-particle harmonic-oscillator states in the full Dirac notation, where for convenience in what follows we shall assume that our units are such that $\hbar$, the mass m of the particle, and the frequency ω of the oscillator are 1.

Instead of (2) we could have kets coupled to a total angular momentum Λ and projection M, i.e.

$$| \dot{n}_1 \dot{l}_1, \dot{n}_2 \dot{l}_2, \Lambda M > = [< \dot{x}^1 | \dot{n}_1 \dot{l}_1 > < \dot{x}^2 | \dot{n}_2 \dot{l}_2 >]_{\Lambda M} \tag{3}$$

The ket (3) could also be expressed[1] as a polynomial $\dot{P}$ in the creation operators

$$\dot{\eta}^s = \frac{1}{\sqrt{2}} (\dot{x}^s - i\dot{p}^s), \quad s = 1,2 \tag{4}$$

acting on the ground state $|0>$ i.e.

$$|\dot{n}_1 \dot{l}_1, \dot{n}_2 \dot{l}_2, \Lambda M> = \dot{P}(\dot{n}_1 \dot{l}_1, \dot{n}_2 \dot{l}_2, \Lambda M) | 0 > \tag{5}$$

where, $\dot{P}$ is given by

$$\dot{P}(\dot{n}_1 \dot{l}_1, \dot{n}_2 \dot{l}_2, \Lambda M) =$$

$$= A_{\dot{n}_1 \dot{l}_1} A_{\dot{n}_2 \dot{l}_2} (\dot{\eta}^1 \cdot \dot{\eta}^1)^{\dot{n}_1} (\dot{\eta}^2 \cdot \dot{\eta}^2)^{\dot{n}_2} \left[\mathcal{Y}_{\dot{l}_1}(\dot{\eta}^1) \mathcal{Y}_{\dot{l}_2}(\dot{\eta}^2) \right]_{\Lambda M} \tag{6}$$

and

$$|0> = \pi^{-3/2} \exp\left\{ -\frac{1}{2} \left[(\dot{x}^1)^2 + (\dot{x}^2)^2 \right] \right.$$

$$\left. = \pi^{-3/2} \exp\left\{ -\frac{1}{2} \frac{m\omega}{\hbar} \left[(x^1)^2 + (x^2)^2 + (x^3)^2 - \frac{1}{3}(x^1 + x^2 + x^3)^2 \right] \right\} \tag{7}$$

The ground state $|0>$ is clearly invariant under permutation of the coordinates x^1, x^2, x^3 so that the discussion of the symmetry properties of linear combinations of kets (3) reduces to the analysis of the symmetry properties of the same linear combinations of polynomials $\dot{P}$. For example, if we want to symmetrise the polynomial $\dot{P}(\dot{n}_1 \dot{l}_1, \dot{n}_2 \dot{l}_2, \Lambda M)$ we have to apply all the six permutations of the group $S(3)$ to $\dot{P}$ and then add the resulting polynomials.

All elements of $S(3)$ can be built up from the transposition (1,2) and the cyclic permutation (1,2,3), which from (1), (4) have the following effect on the creation operators

$$(1,2)\begin{pmatrix}\dot{\eta}^1\\ \\ \dot{\eta}^2\end{pmatrix}=\begin{pmatrix}-1 & 0\\ \\ 0 & 1\end{pmatrix}\begin{pmatrix}\dot{\eta}^1\\ \\ \dot{\eta}^2\end{pmatrix};\ (1,2,3)\begin{pmatrix}\dot{\eta}^1\\ \\ \dot{\eta}^2\end{pmatrix}=\begin{pmatrix}-\frac{1}{2} & \frac{\sqrt{3}}{2}\\ \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2}\end{pmatrix}\begin{pmatrix}\dot{\eta}^1\\ \\ \dot{\eta}^2\end{pmatrix} \tag{8}$$

The effect of any permutation of $S(3)$ on $\dot{P}$ can be obtained once we have the effect of (1,2) and (1,2,3) on this polynomial. But from (8) and (6), it is clear that the application of (1,2) to $\dot{P}$ just multiplies it by $(-)^{2\dot{n}_2+\dot{l}_2}$, while the application of (1,2,3) to $\dot{P}$ gives a linear combination of $\dot{P}$ whose coefficients are transformation brackets[2] associated with an angle $\beta/2$ such that

$$\cos\frac{\beta}{2}=-\frac{1}{2}, \qquad \sin\frac{\beta}{2}=\frac{\sqrt{3}}{2} \qquad \text{or } \frac{\beta}{2}=\frac{2\pi}{3}. \tag{9}$$

Using these results we could build up in a straightforward but laborious manner the symmetrical state of the previous paragraph.

We shall show though that a much simpler procedure for constructing the symmetrised state, or for that matter a three-particle translationally invariant harmonic-oscillator state of arbitrary symmetry, can be obtained if we introduce the auxiliary operators

$$\eta^1 \equiv \frac{1}{\sqrt{2}}\left(-i\,\dot{\eta}^1+\dot{\eta}^2\right)$$

$$\eta^2 \equiv \frac{1}{\sqrt{2}}\left(i\,\dot{\eta}^1+\dot{\eta}^2\right). \tag{10}$$

In this paper η^1, η^2 will be given by the definitions (10) and are not to be confused with creation operators associated with the coordinates x^1, x^2. From (8) it is clear that under the generators of $S(3)$, η^1, η^2 transforms as

$$(1,2)\begin{pmatrix}\eta^1\\ \eta^2\end{pmatrix}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\eta^1\\ \eta^2\end{pmatrix}\;;\quad (1,2,3)\begin{pmatrix}\eta^1\\ \eta^2\end{pmatrix}=\begin{pmatrix}e^{-\frac{2\pi i}{3}} & 0\\ 0 & e^{\frac{2\pi i}{3}}\end{pmatrix}\begin{pmatrix}\eta^1\\ \eta^2\end{pmatrix} \tag{11}$$

We now consider polynomials $P(n_1 l_1, n_2 l_2, \Lambda M)$ that are defined in exactly the same way as (6) but only with η^s, n_s, l_s replacing $\dot{\eta}^s$, $\dot{n}_s$, $\dot{l}_s$, $s = 1,2$. The application of (1,2), (1,2,3) to P means carrying out the inverse operation[3] on η^1, η^2 so from (11) and (6) we get

$$(1,2)\; P(n_1 l_1, n_2 l_2, \Lambda M) = (-)^{l_1 + l_2 - \Lambda}\; P(n_2 l_2, n_1 l_1, \Lambda M) \tag{12}$$

$$(1,2,3)\; P(n_1 l_1, n_2 l_2, \Lambda M) = e^{\frac{2\pi i}{3} 2g}\; P(n_1 l_1, n_2 l_2, \Lambda M) \tag{13}$$

where

$$2g \equiv 2n_1 + l_1 - 2n_2 - l_2 \quad . \tag{14}$$

To obtain polynomials of definite permutational symmetry we require appropriate projection operators. We refer the reader to Hamermesh's book[4] where the projection operator $\mathbb{P}^f$ associated with a definite irreducible representation f of a finite group G is given by

$$\mathbb{P}^f \equiv \frac{d_f}{[G]} \sum_p \chi^{f*}(p)\, p \tag{15}$$

where p is an element of the group, $\chi^f(p)$ the character associated with this element for the irreducible representation f, $[G]$ is the order of the group and d_f the dimension of the irreducible representation f.

For the group of permutations $S(3)$ there are three irreducible representations[5] characterised by the partitions $f = \{3\}, \{21\}$ and $\{111\}$ of dimension d_f equal to 1, 2 and 1, respectively. The first and third are the familiar completely symmetric and antisymmetric representations. From the table of characters of $S(3)$ given, for example, in Hamermesh[5], we get for the projection operators of $S(3)$

$$\wp^{[3]} = \frac{1}{6}\left[e + (1,2) + (1,3) + (2,3) + (1,2,3) + (1,3,2)\right] \tag{16}$$

$$\wp^{[21]} = \frac{1}{3}\left[2e - (1,2,3) - (1,3,2)\right] = \frac{1}{3}\left[2e - (1,2,3) - (1,2,3)^{-1}\right] \tag{17}$$

$$\wp^{[111]} = \frac{1}{6}\left[e - (1,2) - (1,3) - (2,3) + (1,2,3) + (1,3,2)\right] \tag{18}$$

where e is the identity element.

We shall first apply $\wp^{[21]}$ to the polynomial $P(n_1 l_1, n_2 l_2, \Lambda M)$ and from (13) get

$$\wp^{[21]} P(n_1 l_1, n_2 l_2, \Lambda M) = \frac{1}{3}\left(2 - e^{\frac{2\pi i}{3} 2g} - e^{-\frac{2\pi i}{3} 2g}\right) P(n_1 l_1, n_2 l_2, \Lambda M)$$

$$= \frac{4}{3}\sin^2\left(\frac{\pi}{3} 2g\right) P(n_1 l_1, n_2 l_2, \Lambda M) = (1 - \delta_{\nu 0})\, P(n_1 l_1, n_2 l_2, \Lambda M) \tag{19}$$

where ν is defined by the congruence relation

$$2g \equiv \nu \pmod 3 \quad . \tag{20}$$

From (19) we see that ν must be either 1 or 2, as when $\nu = 0$ the projected state of partition $\{21\}$ vanishes.

As the dimensionality of the irreducible representation $f = \{21\}$ of $S(3)$ is $d_f = 2$ we have two states in it that are characterised by the Young tableaux[6]

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \rightarrow (211)\ , \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \rightarrow (121) \qquad (21a,b)$$

They could also be characterised by the Yamanouchi symbols $(r_3\, r_2\, r_1)$ which specify the row in which we find each of the numbers 3,2,1 as also indicated in (21).

From the Young tableaux one concludes that the states characterised by the Yamanouchi symbols (211) and (121) are respectively symmetric and antisymmetric under exchange of particles 1 and 2. To get then the states characterised by (211) and (121) we need to apply to the polynomials $P(n_1 l_1, n_2 l_2, \Lambda M)$ with $\nu = 1,2$ the projection operators that give states symmetric or antisymmetric in the first two particles i.e.

$$\wp^{[2]} = \frac{1}{2}\,[e + (1,2)]\ , \quad \wp^{[11]} = \frac{1}{2}\,[e - (1,2)] \qquad (22)$$

From (12) this immediately gives for $\nu \neq 0$

$$\wp^{[2]} P(n_1 l_1, n_2 l_2, \Lambda M) =$$

$$= \frac{1}{2}\Big[P(n_1 l_1, n_2 l_2, \Lambda M) + (-)^{l_1 + l_2 - \Lambda} P(n_2 l_2, n_1 l_1, \Lambda M)\Big]$$

$$= \frac{1}{\sqrt{2}}\,\phi(n_1 l_1, n_2 l_2, \Lambda M; \{21\}\,(211)) \qquad (23)$$

$$\wp^{[11]} P(n_1 l_1, n_2 l_2, \Lambda M) =$$

$$= \frac{1}{2}\left[P(n_1 l_1, n_2 l_2, \Lambda M) - (-)^{l_1 + l_2 - \Lambda} P(n_2 l_2, n_1 l_1, \Lambda M) \right]$$

$$\equiv \frac{i}{\sqrt{2}} (-1)^{\nu} \phi(n_1 l_1, n_2 l_2, \Lambda M; \{21\}\ (121)) \qquad (24)$$

where in (23) and (24) the ϕ are the normalised polynomials [recall that $P(n_1 l_1, n_2 l_2, \Lambda M)$ is never identical to $P(n_2 l_2, n_1 l_1, \Lambda M)$ since $\nu \equiv 2n_1 + l_1 - 2n_2 - l_2 \neq 0$] characterised by the partition $\{21\}$ and the corresponding Yamanouchi symbol as well as by $n_1 l_1, n_2 l_2, \Lambda M$. In (24) the phase factor $i(-1)^{\nu}$ is prescribed by the ladder procedure[7].

To get the symmetric and antisymmetric states we could apply $\wp^{[3]}$ and $\wp^{[111]}$ respectively, remembering that all permutations can be expressed in terms of (1,2) and (1,2,3). A more elegant procedure though is to note that from the analysis carried out for $f = \{21\}$ we conclude that only linear combinations of polynomials $P(n_1 l_1, n_2 l_2, \Lambda M)$ for which $2n_1 + l_1 - 2n_2 - l_2 \equiv 0 \pmod 3$ would be either symmetric or antisymmetric. Those that are symmetric are characterised by the Young tableaux

1	2	3

$\rightarrow$ (111) (25)

and thus would also be symmetric under permutation of particles 1 and 2 as is quite obvious. Those that are antisymmetric are characterised by the Young tableaux

1
2
3

$\rightarrow$ (321) (26)

and thus would be antisymmetric under permutation of particles 1 and 2, as is also obvious. We could then get the $\{3\}$ (111) and $\{111\}$ (321) normalised states by applying respectively the operators $\wp^{[2]}$ and $\wp^{[11]}$ of (22) to the polynomial $P(n_1 l_1, n_2 l_2, \Lambda M)$ i.e.

$$\begin{bmatrix} \wp^{[2]} \\ \wp^{[11]} \end{bmatrix} P(n_1 l_1, n_2 l_2, \Lambda M) =$$

$$= \frac{1}{2}\left[P(n_1 l_1, n_2 l_2, \Lambda M) \pm (-)^{l_1 + l_2 - \Lambda} P(n_2 l_2, n_1 l_1, \Lambda M) \right]$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} \phi(n_1 l_1, n_2 l_2, \Lambda M; \{3\}(111)) \\ \phi(n_1 l_1, n_2 l_2, \Lambda M; \{111\}(321)) \end{bmatrix} . \tag{27}$$

The expression (27) is valid when $2n_1 + l_1 - 2n_2 - l_2 \equiv 0$ but with the pair $(n_1 l_1)$ different from $(n_2 l_2)$. When

$$n_1 = n_2 = n \ , \quad l_1 = l_2 = l \tag{28}$$

we get from (27)

$$P(nl, nl, \Lambda M) = \begin{cases} \phi(nl, nl, \Lambda M; \{3\}(111)) & \text{if } \Lambda \text{ even} \\ \phi(nl, nl, \Lambda M; \{111\}(321)) & \text{if } \Lambda \text{ odd.} \end{cases} \tag{29}$$

Thus we obtained polynomials with definite permutational symmetry for the three-particle translationally invariant problem in terms of the creation operators η^1, η^2 of (10). For the calculation of the matrix elements of the Hamiltonian, as well as for the form factor of the proton to be discussed in the next Section, it is much more convenient to have the polynomials expressed in terms of the creation operators $\dot{\eta}^1$, $\dot{\eta}^2$ i.e. in terms of $\dot{P}(\dot{n}_1 \dot{l}_1, \dot{n}_2 \dot{l}_2, \Lambda M)$. This is easily achieved when we realise that the transformation matrix (10) connecting η^1, η^2 with $\dot{\eta}^1$, $\dot{\eta}^2$ can be decomposed in the following form

$$\begin{pmatrix} -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -i & 0 \\ 0 & 1 \end{pmatrix} \equiv \boldsymbol{ABCD}. \tag{30}$$

The effect of **A** = **C** on the polynomial P is given by (12), (11), while that of ***D*** just multiplies the polynomial by $(-i)^{2n_1+l_1}$. Finally the application of ***B*** to P gives a linear combination of P's whose coefficients are standard transformation brackets (STB). Combining all of these operations we arrive at the result

$$P(n_1 l_1, n_2 l_2, \Lambda M) =$$

$$= \sum_{\substack{\dot{n}_1 \dot{n}_2 \\ \dot{l}_1 \dot{l}_2}} P(\dot{n}_1 \dot{l}_1, \dot{n}_2 \dot{l}_2, \Lambda M)(-1)^{\dot{n}_1 + l_1} i^{\dot{l}_1} \langle \dot{n}_1 \dot{l}_1, \dot{n}_2 \dot{l}_2, \Lambda | n_1 l_1, n_2 l_2, \Lambda \rangle \tag{31}$$

where we also made use of the symmetry relation[8] of the STB.

From (31) and the above discussion we see that

$$| n_1 l_1, n_2 l_2, \Lambda M; f, r \rangle \equiv \phi(n_1 l_1, n_2 l_2, \Lambda M; f, r) | 0 \rangle =$$

$$= A(\nu, f, r) \Big[P(n_1 l_1, n_2 l_2, \Lambda M) \pm (-1)^{l_1 + l_2 - \Lambda} P(n_2 l_2, n_1 l_1, \Lambda M) \Big] | 0 \rangle =$$

$$= \sum_{\substack{\dot{n}_1 \dot{n}_2 \\ \dot{l}_1 \dot{l}_2}} |\dot{n}_1 \dot{l}_1, \dot{n}_2 \dot{l}_2, \Lambda M> \Big\{ A(\nu, f, r)(-1)^{\dot{n}_1 + l_1} i^{\dot{l}_1} \times$$

$$\times \Big[1 \pm (-1)^{l_1 + l_2 + \dot{l}_1} \Big] < \dot{n}_1 \dot{l}_1, \dot{n}_2 \dot{l}_2, \Lambda | n_1 l_1, n_2 l_2, \Lambda > \qquad \equiv$$

$$\equiv \sum_{\substack{\dot{n}_1 \dot{n}_2 \\ \dot{l}_1 \dot{l}_2}} |\dot{n}_1 \dot{l}_1, \dot{n}_2 \dot{l}_2, \Lambda M > < \dot{n}_1 \dot{l}_1, \dot{n}_2 \dot{l}_2, \Lambda | n_1 l_1, n_2 l_2, \Lambda, f, r > \qquad (32)$$

where f and r are short hand notations for the partitions $\{f_1 f_2 f_3\}$ and Yamanouchi symbols $(r_3 r_2 r_1)$ and in deriving (32) we made use of a symmetry property of the transformation brackets[8]. The coefficient $A(\nu, f, r)$ and the value + or - in (32) are specified in (23,24,27). It is clear incidentally that the last transformation bracket in (32) is either real or pure imaginary as the factor $[1 \pm (-1)^{l_1 + l_2 + \dot{l}_1}]$ for fixed l_1, l_2 restricts $\dot{l}_1$ to either even or odd values. We could then give a trivial redefinition of the ket (32) so that the last transformation bracket is always real.

Let us designate by

$$|SM_S\, TM_T, fr> \qquad (33)$$

the three-particle spin-isospin state[1] with $SM_S\,(TM_T)$ being the total spin (isospin) and its projection and f, r having the same meaning as in the previous paragraph. The completely antisymmetric state[1] under exchange of coordinates, spin and isospin is then given by

$$|\eta > \equiv$$

$$\equiv \frac{1}{\sqrt{d_f}} \sum_r (-1)^r \, [\phi(n_1 l_1, n_2 l_2 \Lambda; fr) \, | \, 0 > | \, STM_T ; \tilde{f} \, \tilde{r} >]_{JM} \qquad (34)$$

In (34) $\tilde{f}, \tilde{r}$ refer to partition and Yamanouchi symbol associate to f, r e.g. if $f, r = \{3\}, (111), \tilde{f}, \tilde{r} = \{111\}, (321)$. The phase $(-1)^r$ is defined in such way that $(-1)^r = +1, +1 +1, -1$ for $r = (111), (321), (211), (121)$, respectively. The symbol η stands then for the set of quantum numbers

$$\eta \equiv n_1 l_1, n_2 l_2 \Lambda f, STM_T, JM \; . \qquad (35)$$

We have thus constructed explicitly the translationally invariant three-particle state.

III. FORM FACTOR OF THE PROTON AS A SYSTEM OF THREE QUARKS

It is often useful to describe baryons and mesons as systems of quarks and antiquarks [9]. In particular the proton can be thought of as a system of three quarks. Assuming that the quarks are particles of spin $\frac{1}{2}$ satisfying Fermi statistics, the wave function describing the proton could be developed in terms of three-particle states similar to those discussed in (34). We will show in this section that experimental evidence suggests then that the configuration space part of this three-quark wave function is antisymmetric and of total orbital angular momentum zero. If we know the coefficients $a(\dot{n}_1 \dot{l}_1, \dot{n}_2 \dot{l}_2)$ of the expansion of the configuration space part of the wave function in terms of harmonic-oscillator states (32), and if the quarks are taken as point particles, the form factor of the proton would be given by

$$F(q^2) = \Sigma\, a^*(n_1' l_1', n_2' l_2')\, a(n_1 l_1, n_2 l_2)$$

$$< n_1' l_1', n_2' l_2' \Lambda = 0, f = \{1^3\} \mid \frac{\sin \kappa \dot{x}^2}{\kappa \dot{x}^2} \mid n_1 l_1, n_2 l_2 \Lambda = 0, f = \{1^3\} > \tag{36}$$

as shown in reference[10]. In (36) κ is given by

$$\kappa = \sqrt{\frac{2\hbar}{3m\omega}}\, q \quad \text{or} \quad \kappa^2 = \frac{2\hbar}{3m\omega}\, q^2 \tag{37}$$

where $\hbar q$ is the momentum transfer.

Thus the quark model gives a theoretical prediction for the form factor, about which we have also experimental information[11].

We now proceed to prove the remark made above i.e. that experimental evidence suggests an antisymmetric form for the configuration space part of the three quark state that represent the proton.

We need first of all to describe the states associated with the internal coordinates of the quarks. For this purpose we take as an analogous model the states $|\sigma\tau>$ associated with the internal coordinates of the nucleon, where

$$|\sigma\tau> = |\sigma>|\tau> , \tag{38}$$

and $|\sigma>$, $|\tau>$ being the spin and isospin states with $\sigma = \pm\frac{1}{2}$, $\tau = \pm\frac{1}{2}$. A nucleon then has four spin-isospin states which form a basis for an irreducible representation (IR) of a unitary unimodular group in four dimensions $SU(4)$ characterised by the partition $\{1\}$[12]. Besides the two-component states $|\sigma>$ and $|\tau>$ are independently basis for IR of the unitary unimodular groups in two dimensions $SU(2)^{(\sigma)}$ and $SU(2)^{(\tau)}$, characterised also by $\{1\}$. The state (38) is then completely defined by the IR of the groups in the chain

$$SU(4) \supset SU(2)^{(\sigma)} \times SU(2)^{(\tau)} \tag{39}$$

where x denotes the direct product of the group in question.

The nucleon can be found in two isospin states corresponding to neutron or proton. The quark can be found in three states denoted by p, n, λ which can be characterised by the ket

$$|\rho> \tag{40}$$

where for p, n, λ the quantum number takes respectively the values $\rho = 1,2,3$. Besides as the quark has spin it is described by a state $|\sigma>$, $\sigma = \pm 1/2$ identical to the one appearing in (38). Thus the internal state of the quark is represented by the ket

$$|\rho\sigma> = |\rho>|\sigma> , \quad \rho = 1,2,3 , \quad \sigma = \pm \frac{1}{2} . \tag{41}$$

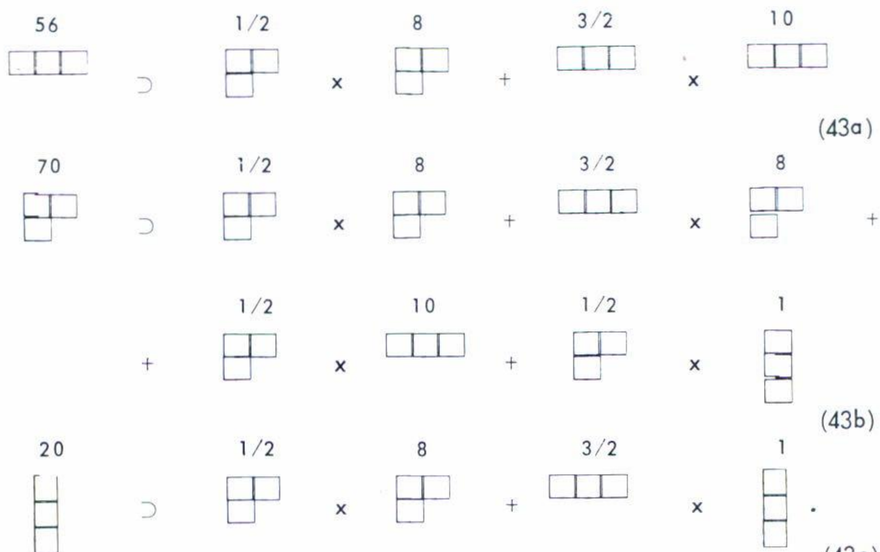
Clearly then this state is completely defined by the IR of the groups in the chain

$$SU(6) \supset SU(2)^{(\sigma)} \times SU(3)^{(\rho)} . \tag{42}$$

In a system of definite number of quarks, a partition $f = \{f_1\, f_2\, f_3\, f_4\, f_5\, f_6\}$ of this number, with a maximum of six rows[13] will define not only the IR of the $U(6)$ group for this system, but also characterise the symmetry of the state under permutation of the internal coordinates[12]. As the quarks obey Fermi statistics this implies that the configuration part of the state has a symmetry related to the associate partition $\tilde{f}$.

The IR of $SU(2)^{(\sigma)}$ and $SU(3)^{(\rho)}$ are also characterised by partitions f' and f'' respectively, of the number of quarks, where f' anf f'' are restricted to a maximum of two and three rows[13] i.e. $f' = \{f_1'\, f_2'\}$, $f'' = \{f_1''\, f_2''\, f_3''\}$. the problem of finding the IR $f' \times f''$ of $SU(2)^{(\sigma)} \times SU(3)^{(\rho)}$ contained in a given IR f of $SU(6)$ is similar in structure to the problem of finding the IR m of the rotation group $R(2)$

contained in a given l of $R(3)$. The solution of the latter problem is given by the inequality $-l \leqslant m \leqslant l$. For the former we need to use the phletysm procedure[14] to obtain for the three-quark system the relations



(43a)

(43b)

(43c)

In (43) we have on the left hand side of $\supset$ the partitions $f = \{3\}, \{21\}, \{1^3\}$ that characterise the IR of $SU(6)$ for the three-quark system. On the right hand side of $\supset$ we have the partitions $f' \times f''$ in Young diagram form that characterise the IR of $SU(2)^{(\sigma)} \times SU(3)^{(\rho)}$. Above the f, f'' of $SU(6), SU(3)^{(\rho)}$ we put the dimension of the IR, while above f' of $SU(2)^{(\sigma)}$ we put the total spin S. The dimension of the latter is of course $2S+1$.

Experimentally we are interested in the IR, $f = \{3\}$ of $SU(6)$ of dimension 56 as this is the one that contains the octet of spin $\frac{1}{2}$ (N, Λ, Σ, Ξ) and the decuplet of spin $\frac{3}{2}$ ($N^*, \Sigma^*, \Xi^*, \Omega^-$) and we thus expect that it represents the lowest lying states of the baryon spectrum. As the proton is one of the members of the octet of spin $\frac{1}{2}$, we see that in this model its state is symmetric under the exchange of the internal coordinates of the three-quark system. This implies then that the configuration space state of the proton must be antisymmetric under exchange of the quark coordinates i.e. belong to the partition $\{1^3\}$. Furthermore experimentally the total angular momentum of the baryons in the octet is $J = \frac{1}{2}$ while from (43a) the total spin of the octet is $S = \frac{1}{2}$. If we want to have just one octet of $J = \frac{1}{2}$ in our lowest lying states we must assume that the total orbital angular momentum Λ of the configuration space three-quark system is $\Lambda = 0$ so that $\mathbf{S} + \Lambda$ when $S = \frac{1}{2}$ give just one $J = \frac{1}{2}$.

We are then interested in three-particle translationally invariant configuration space states that are antisymmetric i.e. $f = \{1^3\}$ and have $\Lambda = 0$. From (27) we see that these states can be expanded in terms of the harmonic-oscillator states

$$|n_1 l, n_2 l, 00; \{111\} (321) > \equiv$$

$$\equiv \frac{1}{\sqrt{2}} \Big[P(n_1 l, n_2 l, 00) - P(n_2 l, n_1 l, 00) \Big] \, | \, 0 > \tag{44}$$

where

$$2(n_1 - n_2) \equiv 0 \pmod 3, \quad \text{and} \quad N = 2(n_1 + n_2 + l) \tag{45}$$

with N being the total number of quanta. From (44) we see that these states vanish if $n_1 = n_2$. Without loss of generality we can then take $n_1 < n_2$ and as 2 is not divisible by 3, we conclude from (45) that $n_1 - n_2$ is divisible by 3. We can then write

$$n_1 \equiv n, \qquad n_2 = n + 3\nu, \qquad N = 2(2n + 3\nu + l) \tag{46}$$

where n can be 0 or positive integer while ν is a positive integer only. Using the expansion (31) we can then write the state (44) as

$$|nl, n+3\nu l, 00; \{1^3\}(321)> =$$

$$= \frac{1}{\sqrt{2}} \sum_{\dot{n}_1 \dot{n}_2 \dot{l}} |\dot{n}_1 \dot{l}, \dot{n}_2 \dot{l}, 00> (-1)^{\dot{n}_1 + l} i^{\dot{l}} [1-(-1)^{\dot{l}}] \times$$

$$\times <\dot{n}_1 \dot{l}, \dot{n}_2 \dot{l}, 0 | nl, n+3\nu l, 0> =$$

$$= i\sqrt{2} \sum_{\dot{n}_1 \dot{n}_2 \dot{l}} |\dot{n}_1 2\dot{l}+1, \dot{n}_2 2\dot{l}+1, 00> \times$$

$$\times (-1)^{\dot{n}_1 + l + \dot{l}} <\dot{n}_1 2\dot{l}+1, \dot{n}_2 2\dot{l}+1, 0 | nl, n+3\nu l, 0> \qquad (47)$$

where, if we wish, we could eliminate the phase factor i and have a real state. In (47) the $<|>$ are standard transformation brackets tabulated in reference[8].

Clearly the state of lowest number of quanta is from (46) given by $n = l = 0$, $\nu = 1$ or $N = 6$ i.e.

$$|00, 30, 00; \{1^3\}(321)> . \qquad (48)$$

We could in a first approximation consider this as the state with whose help we calculate the form factor of the proton, given by (36). Using (47) the calculation is straightforward and we obtain

$$F(\kappa^2) = e^{-\kappa^2/4} \left\{ 1 - \frac{1}{2}\kappa^2 + \frac{17}{160}\kappa^4 - \frac{31}{3360}\kappa^6 + \right.$$

$$\left. + \frac{27}{7168}\kappa^8 - \frac{1}{14336}\kappa^{10} \right\} \tag{49}$$

where κ^2 is related to q^2 by (37).

We could fix $\hbar\omega$ by demanding that at $q^2 = 0$ the tangent to the curve $F(q^2)$ as function of q^2 is equal to the tangent of the experimental curve. The curve $F(q^2)$ with this $\hbar\omega$ and the experimental points are graphed in fig. 1. The curve comes consistently below the experimental points and besides, though this is not shown in the graph, the curve becomes negative at large q^2 in such a way that

$$4\pi \int_0^\infty F(q^2)\, q^2\, dq =$$

$$= \left[\int F(\mathbf{q})\, e^{i\mathbf{q}\cdot\mathbf{r}}\, d\mathbf{q} \right]_{r=0} = \rho(0) = 0. \tag{50}$$

We can prove that $\rho(0) = 0$ for an arbitrary combination of states (47) by noticing that the charge density $\rho(\mathbf{x})$ becomes a linear combination of matrix elements[10]

$$< \dot{n}_1' 2\dot{l}' + 1, \dot{n}_2' 2\dot{l}' + 1, 00 \,|\, \delta\left(\mathbf{x} + \sqrt{\frac{2}{3}}\, \dot{\mathbf{x}}^2\right) |\, \dot{n}_1 2\dot{l} + 1, \dot{n}_2 2\dot{l} + 1, 00 >$$

$$= \delta_{\dot{n}_1' \dot{n}_1}\, \delta_{\dot{l}' \dot{l}}\, \frac{1}{\sqrt{4\pi}}\, R_{\dot{n}_2' 2\dot{l}+1}\left(\sqrt{\frac{3}{2}}\, r\right) R_{\dot{n}_2 2\dot{l}+1}\left(\sqrt{\frac{3}{2}}\, r\right) \tag{51}$$

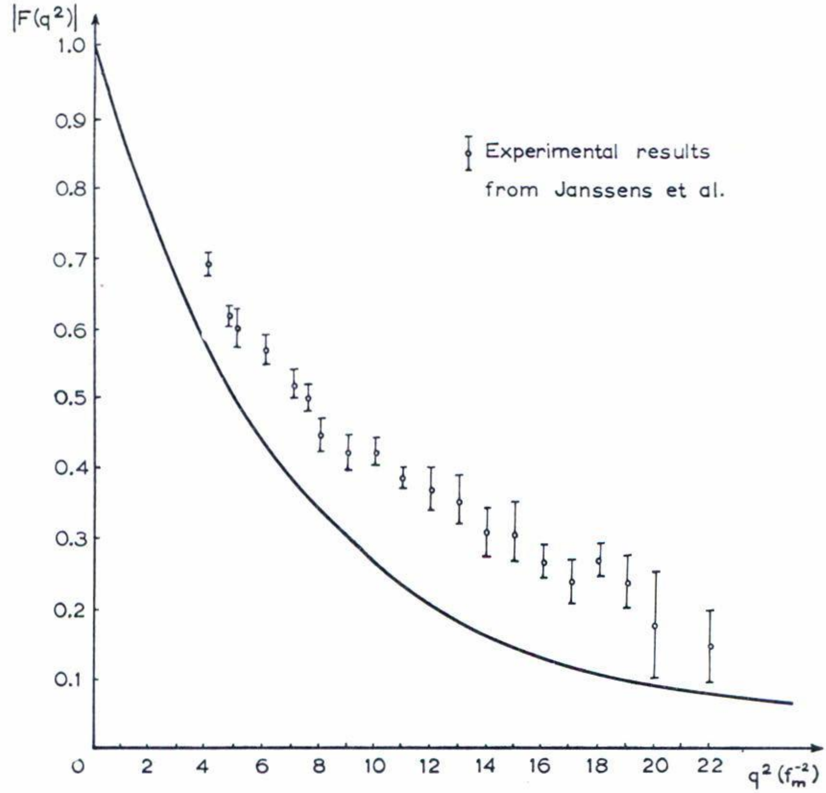


Fig. 1. Form factor of the proton as a function of the square of the momentum transfer when this particle is represented as a system of three quarks. The wave function of this system is taken as an harmonic-oscillator state of six quanta whose frequency was adjusted to give the experimental slope of the curve at $q^2 = 0$. As discussed in the article this is the state of lowest number of quanta antisymmetric under exchange of the coordinates of the three quarks and of orbital angular momentum zero.

where $r = |\mathbf{x}|$. From the form of R_{nl} given in reference 10, we see that (51) is a gaussian in r multiplied by a polynomial whose lowest order term is $r^{4\dot{l}+2}$. Clearly then even for $\dot{l} = 0$ we start with a term r^2 and so the matrix element (51) vanishes for $r = 0$, *q. e. d.*.

The experimental data on the form factor of the proton up to the q^2 measured so far[11] do not suggest that $F(q^2)$ can become negative and so are not in agreement with the possibility that the charge density of the proton vanishes at the center of mass of this particle. It is clear therefore that the quark model of the proton, with the assumptions made at the beginning of this section, does not describe correctly the form factor of the proton. Some of the assumptions could be relaxed but as this paper deals with the harmonic oscillator and not the quark model, we shall not discuss the problem further here.

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