A CLOSED FORMULA FOR THE $3nj$ COEFFICIENTS OF R_3^*

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RESUMEN

Se da, en forma cerrada, una fórmula general para los coeficientes de $3nj$ para R_3, considerándolos como generalizaciones del coeficiente de Racah. La fórmula es aplicable a una definición del coeficiente de $3nj$ que incluye como casos particulares las formas usuales de coeficientes de $9j$ y de $12j$. Se discuten algunas propiedades del coeficiente de $3nj$.

ABSTRACT

A general formula is given in closed form for the $3nj$ coefficients of R_3, considered as generalisations of the Racah coefficient. The formula applies to a type of $3nj$ coefficient which includes the standard forms of $9j$ and $12j$ coefficients. Some properties of the $3nj$ coefficient are discussed.

I. INTRODUCTION

Recently, considerable effort has been devoted to the investigation of nuclear structure through the technique of obtaining successive moments of the interaction Hamiltonian (French 1967); in developing the matrix elements of powers of the Coulomb interaction in order to study isospin mixing (Flores and Mello, to be published), $12j$ coefficients appeared for the square, and higher moments will require coefficients with 15 and more j's. This led the authors to search for a general expression. Such a formula for the $3nj$ coefficient seems to have been obtained (Sharp 1956), but we have not been able to trace it in the literature. Since these coefficients clearly have a number of other applications, it appeared useful to publish the general expression for them, together with an exact description of the recoupling scheme it corresponds to. This is done in section II of this paper. Section III discusses some properties of the $3nj$ coefficient. In section IV some particular cases are presented that either give rise to useful sum rules or because they are required for work at present being carried out.

II. THE GENERAL $3nj$ COEFFICIENT

The simplest case of interest of a $3nj$ coefficient occurs for $n = 2$, as the overlap between two wave functions determined by three angular momenta coupled in two different ways. It is the well-known Racah coefficient (Racah 1942, 1943), defined as follows:

$$ W(j_1, j_{21}, j_{23}, j_3; j_2 J) = \frac{< (j_2 j_3) j_{23}, j_1; J | (j_2 j_1) j_{21}, j_3; J >}{([j_{21}][j_{23}])^{\frac{1}{2}}} , \quad (1) $$

where $[a] = 2a + 1$. The two couplings are made explicit in the diagramme of Fig. 1. In this and the diagrammes that follow, the right-hand side shows the coupling scheme for the ket, and the left-hand side that for the bra; the vertical line in the centre represents the total angular momentum J; the outer sides of the

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polygon stand for the fundamental momenta and are therefore repeated on each side, though in a different order; the interior lines are the momenta which occur as intermediate steps in the coupling. The order of the coupling is given by considering all lines as vectors which are always taken to be pointing upwards rather than down[*].

When the number of fundamental angular momenta is four, we have the case $n = 3$, i.e. the $9j$ coefficient. The coupling scheme employed by Wigner (1951) and John and Hope (1954) is shown in Fig. 2 and corresponds to the definition

$$
\left\{
\begin{array}{ccc}
j_2 & j_1 & j_{21} \\
j_4 & j_3 & j_{43} \\
j_{24} & j_{13} & J
\end{array}
\right\}
= \frac{<(j_2 j_4) j_{24}, (j_1 j_3) j_{13}; J \mid (j_2 j_1) j_{21}; (j_4 j_3) j_{43}; J>}{([j_{21}][j_{43}][j_{24}][j_{13}])^{1/2}} .
$$

(2)

In Figs. 1 and 2, the coupling of the bra is simply a reflection of that for the ket in the vertical line associated with J. For three fundamental angular momenta all other possible diagrammes reduce to that of Fig. 1 by a trivial renaming of the angular momenta; however, this is not so for Fig. 2, nor for $n > 3$. In this paper we shall only consider the "symmetric" case, where the difference between bra and ket consists only in the ordering of the fundamental angular momenta.

The restriction to the "symmetric" case does not eliminate completely the multiplicity of possible diagrammes; thus for $n = 3$, another symmetric diagramme is shown in Fig. 3, which will not reduce to that of Fig. 2. It is trivial to show that, whatever the type of diagramme, there are exactly $n - 1$ intermediate angular momenta on each side; they may not intersect, of course, since this would imply that one and the same angular momentum is coupled twice, in two different triangles. Since the number of fundamental angular momenta, excluding J, is $n + 1$, the total

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[*] In the diagrammes, this could be indicated by arrowheads on all the lines; but since they would all point the same way, they have been omitted for simplicity.

number of angular momenta involved in the $[3nj]$ coefficient is

$$2(n-1) + (n+1) + 1 = 3n .$$

To eliminate the remaining multiplicity among $[3nj]$ diagrammes, we shall adhere to the following rule: All interior lines of the ket shall go through the same vertex of the outer polygon; the fundamental angular momenta are numbered downwards from this vertex, up to $j_k (k = 1, \ldots, n)$, continuing clockwise with j_{k+1} from the upper end of J. Thus there are $k - 1$ interior lines from the chosen vertex downwards, labelled successively $l_2, l_3, \ldots, l_k$; and $n - k$ interior lines upwards, labelled $l_{k+1}, \ldots, l_n$. The bra has the reflected configuration and the value of k is thus the same; the numbering of the fundamental angular momenta is the same except that j_1 and j_{n+1} (which adjoin the chosen vertex) are interchanged and the interior momenta are labelled L_i. It should be noted that $k = 0$ or $k = n+1$ give no new cases, since they only differ by a phase factor from $k = 1$ and $k = n$, respectively. The k-dependence is further discussed in section III.

This rule reproduces the coupling schemes normally used for $n = 2, 3$ and 4 (Jahn 1954, Ord-Smith 1954), if the resulting diagramme is interpreted as follows: in the ket, j_2 and j_1 are coupled (in that order) to l_2; j_3 and l_2 are coupled to l_3; and so on, up to l_k. Then j_{n+1} and j_n are coupled to l_n; l_n and j_{n-1} are coupled to l_{n-1}; and so on, up to l_{k+1}. Finally, l_k and l_{k+1} are coupled to J. The rule for the bra is analogous.

Denoting the bra and ket so obtained by $< j_i L_i; J |$ and $| j_i l_i; J >$, the general $[3nj]$ coefficient is then defined by the corresponding diagramme as

$$[3nj] = \frac{< j_i L_i; J | j_i l_i; J >}{\prod\limits_{i=2}^{n} ([l_i][L_i])^{1/2}} \tag{3}$$

It should be noted that for $n = 2$, Eq. (3) defines the Racah coefficient and not the

usual $6j$ symbol; for $n = 3$ or 4, the $9j$ and $12j$ coefficients are obtained in their standard forms.

The general expression for the $3nj$ coefficient may now be obtained straightforwardly by writing out explicitly the coupling of bra and ket in (3) in terms of Clebsch-Gordan coefficients. Applying the well-known result[*]

$$\sum_{m_3} < j_1\, j_2\, m_1\, m_2 \,|\, j_3 - m_3 > < l_1\, l_2\, m_1'\, m_2' \,|\, j_3\, m_3 > =$$

$$= \sum_{l_3\, m_3'} (-)^{l_3 + j_3 + m_1 + m_1'} (2 j_3 + 1) < j_1\, l_2\, m_1\, m_2' \,|\, l_3 - m_3' >$$

$$< l_1\, j_2\, m_1'\, m_2 \,|\, l_3\, m_3' > \begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix} \tag{4}$$

and the orthonormality of the Clebsch-Gordan coefficients, one obtains

$$[3nj] = \sum_x (-)^{\phi} [x] \begin{Bmatrix} j_1 & j_2 & l_2 \\ L_2 & x & j_{n+1} \end{Bmatrix} \begin{Bmatrix} l_2 & j_3 & l_3 \\ L_3 & , & L_2 \end{Bmatrix} \cdots \begin{Bmatrix} l_k & J & l_{k+1} \\ L_{k+1} & x & L_k \end{Bmatrix}$$

$$\begin{Bmatrix} l_{k+1} & j_{k+1} & l_{k+2} \\ L_{k+2} & x & L_{k+1} \end{Bmatrix} \cdots \begin{Bmatrix} l_{n-1} & j_{n-1} & l_n \\ L_n & x & L_{n-1} \end{Bmatrix} \begin{Bmatrix} l_n & j_n & j_{n+1} \\ j_1 & x & L_n \end{Bmatrix} , \tag{5}$$

where

$$\phi = R + (n-1)x = \sum_{i=1}^{n+1} j_i + \sum_{i=2}^{n} (l_i + L_i) + J + (n-1)x , \tag{6}$$

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[*] It may be useful to note that in giving this result as their eq. (15.14), de-Shalit and Talmi (1963) omit the factor $(2 j_3 + 1)$.

writing R for the sum of all angular momenta involved in the $[3nj]$ coefficient. The sum over x in (5) is over all values compatible with the triangle conditions of the $6j$ symbols (not Racah coefficients).

III. SOME PROPERTIES OF THE $[3nj]$ COEFFICIENT

Since there is no generally accepted notation for the $[3nj]$ coefficient, we will use the obvious extension to the general case of the Möbius-strip notation introduced by Ord-Smith (1954). The $3n$ angular momenta involved in the coefficient are written on three lines with alternating spacing, and the ends are considered to be joined after a turn through $180°$ to form a Möbius-strip. Corresponding to the diagramme of Fig. 4, the coefficient (5) then becomes

$$
\begin{bmatrix}
j_1 & & l_2 & & l_3 \ldots l_k & & l_{k+1} \cdots l_n \\
& j_2 & & j_3 \cdots j_k & & J & & j_{k+1} \cdots j_n \\
j_{n+1} & & L_2 & & L_3 \ldots L_k & & L_{k+1} \cdots L_n
\end{bmatrix}
\tag{7}
$$

One of the advantages of this notation is that it displays clearly which groups of angular momenta must obey the triangle condition

$$
|a - b| \leqslant c \leqslant a + b ,
\tag{8}
$$

if the coefficient is to differ from 0: they form triangles with their vertices on the central line. There are two triangles not immediately obvious in (7), $(l_n j_n j_{n+1})$ and $(L_n j_n j_1)$, which are obtained at once by considering the closed Möbius strip.

Another advantage of the notation (7) is that it allows to write the general expression (5) in a very compact way: Calling the elements in the three lines of the Möbius strip α_i, β_i, γ_i, respectively, with $i = 1, \ldots, n$, and defining $\alpha_{n+1} = \gamma_1$, $\gamma_{n+1} = \alpha_1$ (which corresponds to closing up the Möbius strip), the

expression (5) may be rewritten

$$
\begin{bmatrix}
\alpha_1 \cdots \alpha_n \\
\quad \beta_1 \cdots \beta_n \\
\gamma_1 \cdots \gamma_n
\end{bmatrix}
= \sum_x (-)^{\phi}[x] \prod_{i=1}^{n}
\begin{Bmatrix}
\alpha_i & \beta_i & \alpha_{i+1} \\
\gamma_{i+1} & x & \bar{\gamma}_i
\end{Bmatrix}
$$

$$(9)$$

In this new notation, it is trivial to show that the Möbius strip representing the $[3nj]$ coefficient may be opened up anywhere, without any change in value. Furthermore, the Möbius strip may be read either from left to right or from right to left. These two operations generate $4n$ symmetry operations with respect to which the coefficient is invariant.

It should be noted that (9) is invariant under a change in the value of k in the range 1 to n, in the sense that the formal expression does not change; the interpretation of the angular momenta in (9), however, does change, as will be seen by comparison with (5).

Two interesting relations are obtained by putting some angular momentum equal to zero. If the zero is chosen on the middle row of the Möbius strip, say β_m, then only a single $6j$ symbol in the expansion (9) is affected:

$$
\begin{Bmatrix}
\alpha_m & 0 & \alpha_{m+1} \\
\gamma_{m+1} & x & \gamma_m
\end{Bmatrix}
= \frac{(-)^{\alpha_m + \gamma_m + x}}{([\alpha_m][\gamma_m])^{\frac{1}{2}}} \, \delta_{\alpha_m \alpha_{m+1}} \, \delta_{\gamma_m \gamma_{m+1}}
$$

$$(10)$$

This yields at once a reduction to the $[3(n-1)j]$ coefficient:

$$
\begin{bmatrix}
\alpha_1 \cdots \alpha_m & \alpha_m & \alpha_{m+2} \cdots \alpha_n \\
\beta_1 \cdots 0 & \beta_{m+1} & \cdots & \beta_n \\
\gamma_1 \cdots \gamma_m & \gamma_m & \gamma_{m+2} \cdots \gamma_n
\end{bmatrix}
= \frac{1}{([\alpha_m][\gamma_m])^{\frac{1}{2}}}
\begin{bmatrix}
\alpha_1 & \cdots & \alpha_m & \alpha_{m+2} \cdots \alpha_n \\
\beta_1 \cdots \beta_{m-1} & \beta_{m+1} & \cdots & \beta_n \\
\gamma_1 & \cdots & \gamma_m & \gamma_{m+2} \cdots \gamma_n
\end{bmatrix}
$$

$$(11)$$

If on the other hand, the zero is on the first or third row, two $6j$ symbols now contain a zero and the sum over x disappears, due to (10). We then have, taking α_m to be zero,

$$
\begin{bmatrix}
\alpha_1 \cdots \beta_{m-1} & 0 & \beta_m \cdots & \alpha_n \\
\beta_1 \cdots \beta_{m-1} & \beta_m & \beta_{m+1} \cdots \beta_n \\
\gamma_1 \cdots \gamma_{m-1} & \gamma_m & \gamma_{m+1} \cdots \gamma_n
\end{bmatrix} =
$$

$$
= \frac{(-)^{R+(n+1)\gamma_m + \beta_m + \beta_{m-1} + \gamma_{m-1} + \gamma_{m+1}}}{([\beta_m][\beta_{m-1}])^{\frac{1}{2}}}
\begin{Bmatrix} \alpha_1 & \beta_1 & \alpha_2 \\ \gamma_2 & \gamma_m & \gamma_1 \end{Bmatrix} \cdots
$$

$$
\begin{Bmatrix} \alpha_{m-2} & \beta_{m-2} & \beta_{m-1} \\ \gamma_{m-1} & \gamma_m & \gamma_{m-2} \end{Bmatrix}
\begin{Bmatrix} \beta_m & \beta_{m+1} & \alpha_{m+2} \\ \gamma_{m+2} & \gamma_m & \gamma_{m+1} \end{Bmatrix} \cdots
\begin{Bmatrix} \alpha_n & \beta_n & \gamma_1 \\ \alpha_1 & \gamma_m & \gamma_n \end{Bmatrix}
$$

$$(12)$$

Special cases of (11) and (12) are well known (e.g. Jahn 1954). When $n = 2$ or 3, the results (11) and (12) coincide after some rearrangement. For $n = 4$, the Biedenharn-Elliott identity (see e.g. De Shalit eq. 15.32) is obtained from a similar argument.

IV. SOME PARTICULAR CASES

For the case $n = 1$, we have from (9)

$$
\begin{bmatrix} j_1 \\ & J \\ j_2 \end{bmatrix} = \sum_x (-)^{j_1 + j_2 + J} [x] \begin{Bmatrix} j_1 & J & j_2 \\ j_1 & x & j_2 \end{Bmatrix}.
$$

$$(13)$$

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The corresponding coupling diagramme, Fig. 5, yields the matrix element

$$< (j_1 \, j_2) \, J \, | \, (j_2 \, j_1) \, J > = (-)^{j_1 + j_2 - J} \Delta (j_1 \, j_2 \, I) \, .$$

where $\Delta = 1$ or 0 according as the triangle condition between its arguments is satisfied or not. Hence we obtain the sum rule (de-Shalit 1963, eq. (15.17))

$$\sum_x [x] \left\{ \begin{array}{ccc} j_1 & J & j_2 \\ & & \\ j_1 & x & j_2 \end{array} \right\} = (-)^{2(j_1 + j_2)} \Delta (j_1 \, j_2 \, J) \, . \tag{14}$$

For $n = 2$, Eqs. (1) and (5) give (see Fig. 1)

$$W (j_1 \, j_{21} \, j_{23} \, j_3 \, ; \, j_2 \, J) = \sum_x (-)^{j_1 + j_2 + j_3 + j_{21} + j_{23} + J + x}$$

$$[x] \left\{ \begin{array}{ccc} j_1 & j_2 & j_{21} \\ & & \\ j_{23} & x & j_3 \end{array} \right\} \left\{ \begin{array}{ccc} j_{21} & J & j_3 \\ & & \\ j_1 & x & j_{23} \end{array} \right\} \tag{15}$$

which corresponds to Eq. (15.11) of de-Shalit (1963).

The $9j$ coefficient, with the coupling of Fig. 2, is seen to be

$$\left\{ \begin{array}{ccc} j_2 & j_1 & j_{21} \\ j_4 & j_3 & j_{43} \\ j_{24} & j_{13} & J \end{array} \right\} \equiv \left[\begin{array}{ccc} j_1 & j_{21} & j_{43} \\ j_2 & J & j_3 \\ j_4 & j_{24} & j_{13} \end{array} \right] = \sum_x (-)^{R + 2x}$$

$$[x] \left\{ \begin{array}{ccc} j_1 & j_2 & j_{21} \\ & & \\ j_{24} & x & j_4 \end{array} \right\} \left\{ \begin{array}{ccc} j_{21} & J & j_{43} \\ & & \\ j_{13} & x & j_{24} \end{array} \right\} \left\{ \begin{array}{ccc} j_{43} & j_3 & j_4 \\ & & \\ j_1 & x & j_3 \end{array} \right\} \tag{16}$$

which, with an interchange of columns in the 9j symbol, coincides with a result given by Jahn (1954).

For the sake of completeness, we give the formulae for $n = 4$ and $n = 5$, which are occasionally used in work at present being carried on.

$$
\begin{bmatrix} j_1 & l_2 & l_3 & l_4 \\ & j_2 & j_3 & J & j_4 \\ j_5 & L_2 & L_3 & L_4 \end{bmatrix} = \sum_x (-)^{R-x} [x] \begin{Bmatrix} j_1 & j_2 & l_2 \\ L_2 & x & j_5 \end{Bmatrix} \begin{Bmatrix} l_2 & j_3 & l_3 \\ L_3 & x & L_2 \end{Bmatrix}
$$

$$
\begin{Bmatrix} l_3 & J & l_4 \\ L_4 & x & L_3 \end{Bmatrix} \begin{Bmatrix} l_4 & j_4 & j_5 \\ j_1 & x & L_4 \end{Bmatrix}
\tag{17}
$$

$$
\begin{bmatrix} j_1 & l_2 & l_3 & l_4 & l_5 \\ & j_2 & j_3 & J & j_4 & j_5 \\ j_6 & L_2 & L_3 & L_4 & L_5 \end{bmatrix} =
$$

$$
= \sum_x (-)^R [x] \begin{Bmatrix} j_1 & j_2 & l_2 \\ L_2 & x & j_6 \end{Bmatrix} \begin{Bmatrix} l_2 & j_3 & l_3 \\ L_3 & x & L_2 \end{Bmatrix} \begin{Bmatrix} l_3 & J & l_4 \\ L_4 & x & L_3 \end{Bmatrix} \begin{Bmatrix} l_4 & j_4 & l_5 \\ L_5 & x & L_4 \end{Bmatrix}
$$

$$
\begin{Bmatrix} l_5 & j_5 & j_6 \\ j_1 & x & L_5 \end{Bmatrix}
\tag{18}
$$

Eq. (17) is a result given by Ord-Smith (1954). Eq. (18) has not, to our knowledge, been given previously.

To obtain numerical values needed in applications, the general expression in the form (9) is easily programmable, if a suitable routine for the evaluation of 6j symbols is available. This has been done in FORTRAN by one of the authors (TAB) and is available from him.

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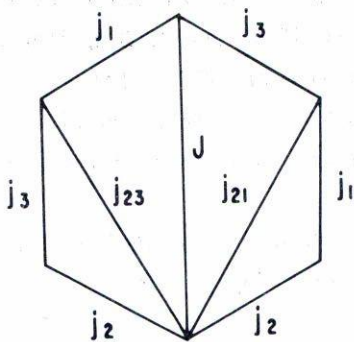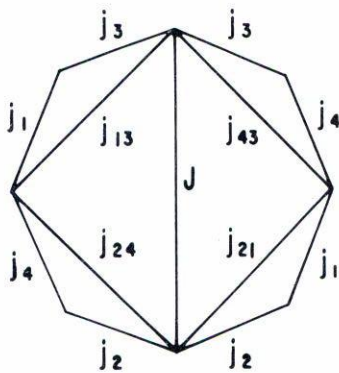
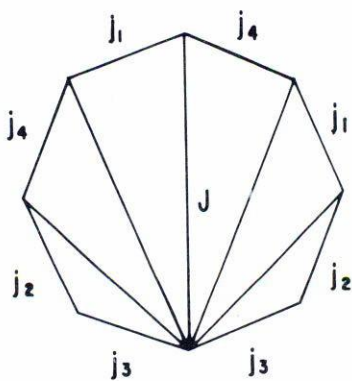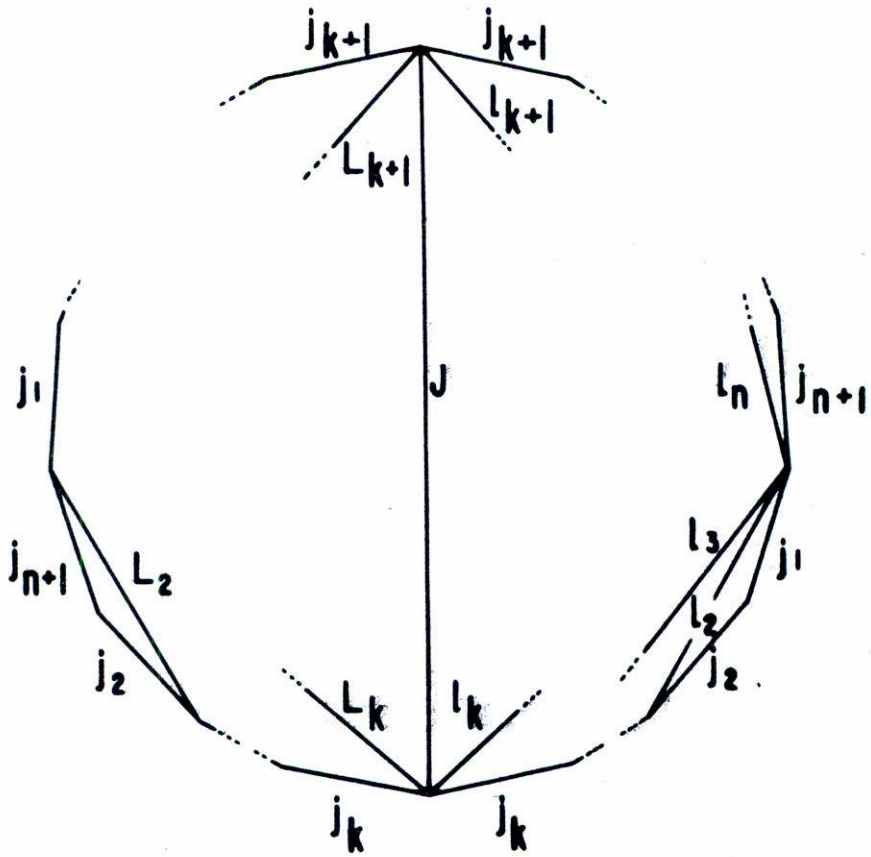7. Wigner, E.P. Amer.J.Math. **63**, 57, (1941).

Fig. 1.

Fig. 2.

Fig. 3.

Fig. 4.

Fig. 5.