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REPRESENTATION COEFFICIENTS FOR THE SU(3) GROUP*

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ABSTRACT

The general SU(3) matrix is factorized into the product of five matrices, of which three belong to the same SU(2) subgroup and the other two are the matrix of the transposition (2, 3). Using the previously determined matrix elements of the transposition (2, 3), the representation coefficients of SU(3) are obtained as linear combination of products of three representation coefficients of SU(2). From this expression it is verified that the SU(3) Gelfand basis states are a particular case of the representation coefficients.

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RESUMEN

La matríz general de SU(3) se factoriza en un producto de cinco matrices, de las cuales tres pertenecen a un mismo subgrupo SU(2) y las otras dos son la matríz de la transposición (2,3). Usando los elementos de matríz de esta transposición, obtenidos en un trabajo anterior, se deducen los coeficientes de representación de SU(3) como combinación lineal de productos de tres coeficientes de representación de SU(2). A partir de esta expresión se comprueba que los estados de Gelfand de SU(3) son un caso particular de los coeficientes de representación.

I. INTRODUCTION

In a former paper[1] by M. Moshinsky and the present author, we gave a derivation of the representation coefficients of SU(3). In reference 1 we adopted a method of factorization of SU(3) matrices proposed by Murnaghan[2]. In this paper we adopt a different factorization method, essentially analogous to the familiar factorization of SO(3) rotations into three successive rotations[3] by the Euler angles α, β, γ. By this method we obtain for the representation coefficients of SU(3) an expression simpler than the one quoted in reference 1. In Section II we describe the new method of factorization and obtain the explicit expression for the representation coefficients.

In Section III we verify that, as should be expected, the basis states of an Irreducible Representation (I R) of SU(3), i.e. in this case the Gelfand states[4], are obtained when we restric: the row indices and parameters of the general representation coefficient to some specific values. An analogous property to this is possessed by the SU(2) representation coefficients. In fact, the general SU(2) matrix is usually written as

$$U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}, \quad |a|^2 + |b|^2 = 1,$$

and the representation coefficient corresponding to this transformation on the basis states of the I R $[j]$ of SU(2), which is given below in equation (14), reduces when $m' = j$ to

$$\mathcal{D}^{(j)}_{m,j} = \sqrt{(2j)!}\ \frac{a^{*j+m}(-b)^{j-m}}{\sqrt{(j+m)!\,(j-m)!}} = \sqrt{(2j)!}\ \frac{u^{*j+m}_{11}\ u^{*j-m}_{21}}{\sqrt{(j+m)!\,(j-m)!}}$$

If we now interpret u_{11} and u_{21} as boson creation operators and make them operate on the vacuum state $|0\rangle$, we have

$$\mathcal{D}^{(j)*}_{m,j}\ (U)\ |0\rangle = \sqrt{(2j)!}\ \ |jm\rangle$$

the ket on the right hand side being the familiar SU(2) basis states. The corresponding SU(3) result will be derived in Section III.

II. SU(3) REPRESENTATION COEFFICIENTS

The general element of the SU(3) group can be written as

$$U = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{pmatrix} \tag{1}$$

with the matrix elements u_{is} subject to the additional restrictions

$$(C_1, C_1) = 1,\ (C_1, C_2) = (C_1, C_3) = 0,\ \text{Det } U = 1 \tag{2a}$$

$$(C_2, C_2) = (C_3, C_3) = 1,\quad (C_2, C_3) = 0 \tag{2b}$$

In formulas (2) C_s $(s = 1,2,3)$ is a vector with components $\{u_{1s}, u_{2s}, u_{3s}\}$, and the scalar product has the usual definition $(C_r, C_s) = u^*_{1r} u_{1s} + u^*_{2r} u_{2s} + u^*_{3r} u_{3s}$. In particular, we have

$$u^*_{11} u_{12} + u^*_{21} u_{22} + u^*_{31} u_{32} = 0$$

$$u^*_{11} u_{13} + u^*_{21} u_{23} + u^*_{31} u_{33} = 0 .$$

Viewed this as a system of homogeneous equations in u^*_{i1} $(i = 1,2,3)$, it has the non-trivial solution[5]

$$u^*_{11} = \Delta^{23}_{23} , \; u^*_{21} = \Delta^{23}_{31} , \; u^*_{31} = \Delta^{23}_{12} \tag{3}$$

where

$$\Delta^{rs}_{ij} \equiv u_{ir} u_{js} - u_{jr} u_{is} . \tag{4}$$

In this way we can conclude that a SU(3) matrix can be written as

$$U = \begin{pmatrix} \Delta^{23\,*}_{23} & u_{12} & u_{13} \\ \Delta^{23\,*}_{31} & u_{22} & u_{23} \\ \Delta^{23\,*}_{12} & u_{32} & u_{33} \end{pmatrix} \tag{5}$$

and the conditions (2a) are automatically satisfied. Let us now, instead of the two vectors C_2, C_3 obeying the restrictions (2b), introduce six new paramaters $a_1, b_1, a_2, b_2, a_3, b_3$ obeying other restrictions, by means of the definitions

$$a_1 = \frac{u_{13}}{b_2}, \quad a_2 = u_{33}^*, \quad a_3 = -\frac{\Delta_{12}^{23\,*}}{b_2}$$

$$b_1 = -\frac{u_{23}^*}{b_2}, \quad b_2 = \sqrt{|u_{13}|^2 + |u_{23}|^2}, \quad b_3 = -\frac{u_{32}}{b_2} \tag{6a}$$

Using the conditions (2b) it is readily verified that the new parameters obey the restrictions

$$b_2 = b_2^*, \quad |a_1|^2 + |b_1|^2 = |a_2|^2 + b_2^2 = |a_3|^2 + |b_3|^2 = 1 \tag{6b}$$

From the explicit form of the matrix (5) and the orthogonality $(C_2, C_3) = 0$, we obtain the inverse relations

$$\begin{aligned} u_{12} &= b_1 a_3^* + a_1 a_2 b_3 \quad , \quad u_{13} = a_1 b_2 \\ u_{22} &= a_1^* a_3^* - b_1^* a_2 b_3 \quad , \quad u_{23} = -b_1^* b_2 \\ u_{32} &= -b_2 b_3 \quad , \quad u_{33} = a_2^* \end{aligned} \tag{7}$$

It is clear that definitions (6a) are valid only when $b_2 \neq 0$. When this is not the case, i.e. when $u_{13} = u_{23} = u_{31} = u_{32} = 0$, the formulas (7) are still valid if we take $b_2 = b_3 = 0$ and $a_3 = 1$.

In terms of the parameters $a_1, \ldots, b_3$ the general matrix of SU(3) now reads

$$U = \begin{pmatrix} a_1 a_2 a_3 - b_1 b_3^* & a_1 a_2 b_3 + b_1 a_3^* & a_1 b_2 \\ - b_1^* a_2 a_3 - a_1^* b_3^* & - b_1^* a_2 b_3 + a_1^* a_3^* & - b_1^* b_2 \\ - b_2 a_3 & - b_2 b_3 & a_2^* \end{pmatrix} \tag{8}$$

By simple matrix multiplication we can verify that $U = U_1 \, U_2' \, U_3$, where

$$U_1 = \begin{pmatrix} a_1 & b_1 & 0 \\ - b_1^* & a_1^* & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad U_2' = \begin{pmatrix} a_2 & 0 & b_2 \\ 0 & 1 & 0 \\ - b_2 & 0 & a_2^* \end{pmatrix}, \quad U_3 = \begin{pmatrix} a_3 & b_3 & 0 \\ - b_3^* & a_3^* & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{9}$$

Furthermore, the matrix U_2' can be decomposed as $U_2' = (2,3) \; U_2 \; (2,3)$, with

$$(2,3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad U_2 = \begin{pmatrix} a_2 & b_2 & 0 \\ - b_2 & a_2^* & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{10}$$

In this way we arrive at our final expression

$$U = U_1 \cdot (2,3) \cdot U_2 \cdot (23) \cdot U_3 \tag{11}$$

and we see that the three matrices U_1, U_2, U_3 belong to the same SU(2) subgroup.

We shall next consider a set of basis states belonging to an IR of SU(3). We choose this set to be the Gelfand basis states[4], and denote them by

$$\left| \begin{matrix} h_1 & & h_2 & & 0 \\ & p & & q & \\ & & r & & \end{matrix} \right\rangle \qquad h_1 \geq p \geq h_2 \geq q \geq 0 , \quad p \geq r \geq q \tag{12}$$

In (12) $[h_1\ h_2\ 0]$ is the label of the I R of U(3), and $[p\ q]$, $[r]$ are the labels of the I R of the canonical subgroups U(2) and U(1), respectively. We would like to remark that normally the Gelfand states are defined as bases of I R of the unitary group U(n) classified by the canonical chain of subgroups $U(n) \supset U(n-1) \supset U(n-2) \ldots$ However, as the I R $[h_1\ h_2\ \ldots\ h_k]$ of U(k) is isomorphic to an I R of SU(k) ⊗ U(1), we can speak of the states (12) as classified by the chain SU(3) ⊃ SU(2) ⊃ U(1). Also, as the transformations we shall consider does not take us off an I R of SU(3), in the following we shall suppress the indexes $[h_1\ h_2]$ in the notation for the states.

Now, in reference 1 we have evaluated the matrix elements of the transposition (2,3) with respect to the states (12). The result is

$$\left\langle \begin{matrix} p & q \\ & r \end{matrix} \middle| (2,3) \middle| \begin{matrix} p' & q' \\ & r' \end{matrix} \right\rangle = \delta_{rr'}\, \delta_{h_1+h_2-p-q,\ p'+q'-r'} \sqrt{(p-q+1)(p'-q'+1)}$$

$$\times W\left[\frac{h_1+h_2-p-q}{2}, \frac{r+h_2-q-q'}{2}, \frac{p+p'-r-h_2}{2}, \frac{p'+q'-r}{2}; \frac{h_1+r-p-q'}{2}, \frac{h_1+r-p'-q}{2}\right] \tag{13}$$

where W(abcd; ef) is a Racah coefficient. Moreover, the transformations U_1, U_2, U_3 are SU(2) elements and the states (12) belong to a basis for an I R of SU(2); hence the matrix elements of these transformations with respect to the states (12) are the familiar SU(2) representation coefficients[3]

$$\left\langle \begin{matrix} p & q \\ & r \end{matrix} \middle| U_s(a_s, b_s) \middle| \begin{matrix} p' & q' \\ & r' \end{matrix} \right\rangle = \delta_{pp'}\, \delta_{qq'}\, \mathfrak{D}^{\frac{1}{2}(p-q)}_{r-\frac{1}{2}(p+q),\ r'-\frac{1}{2}(p+q)}(a_s, b_s)$$

$$= \delta_{pp'}\, \delta_{qq'} \frac{(p-q)!}{\sqrt{\binom{p-q}{r-q}\binom{p-q}{r'-q}}} \sum_k (-)^k \frac{a_s^{p-r-k}\, a_s^{*\,r'-q-k}\, b_s^k\, b_s^{*\,k+r-r'}}{(p-r-k)!\,(r'-q-k)!\,k!\,(k+r-r')!} \tag{14}$$

Using the results in (13) and (14), we obtain for the representation coefficient corresponding to the general transformation U of SU(3) given by (11), the expression

$$\mathcal{D}^{[h_1 h_2]}_{pqr,\, p'q'r'}(a_1, \ldots, b_3) = \sum_{\sigma=0}^{h_2} \sum_{\tau=h_2}^{h_1} \sqrt{(p-q+1)(p'-q'+1)}\; (\tau-\sigma+1)$$

$$\times\; W\left[\frac{h_1+h_2-p-q}{2}, \frac{p-h_1+\tau}{2}, \frac{h_1-q-\sigma}{2}, \frac{h_1+h_2-\sigma-\tau}{2}; \frac{q-h_2+\tau}{2}, \frac{p-h_2+\sigma}{2}\right]$$

$$\times\; W\left[\frac{h_1+h_2-p'-q'}{2}, \frac{p'-h_1+\tau}{2}, \frac{h_1-q'-\sigma}{2}, \frac{h_1+h_2-\sigma-\tau}{2}; \frac{q'-h_2+\tau}{2}, \frac{p'-h_2+\sigma}{2}\right]$$

$$\times\; \mathcal{D}^{\frac{1}{2}(p-q)}_{r-\frac{1}{2}(p+q),\, \sigma+\tau-h_1-h_2+\frac{1}{2}(p+q)}(a_1, b_1)$$

$$\mathcal{D}^{\frac{1}{2}(p'-q')}_{\sigma+\tau-h_1-h_2+\frac{1}{2}(p'+q'),\, r'-\frac{1}{2}(p'+q')}(a_3, b_3)$$

$$\times\; \mathcal{D}^{\frac{1}{2}(\tau-\sigma)}_{p+q-h_1-h_2+\frac{1}{2}(\sigma+\tau),\, p'+q'-h_1-h_2+\frac{1}{2}(\sigma+\tau)}(a_2, b_2) \qquad (15)$$

III. DERIVATION OF GELFAND STATES FROM THE REPRESENTATION COEFFICIENTS

Let us consider the particular case of the representation coefficient (15) when $r' = p' = h_1$, $q' = h_2$. In this case both Racah coefficients in (15) are of a simple type that does not involve summations, and furthermore the second one is different from zero only when $\sigma = 0$. The last two SU(2) representation coefficients in (15) also reduce to a single term, and we obtain, using (14)

$$\mathcal{D}^{[b_1 b_2]}_{pqr,\, b_1 b_2 b_1}(a_1, \ldots, b_3) = (-)^p N b_2^{b_1 + b_2 - p - q} \sum_{k\tau} \frac{(-)^{k+\tau} a_1^{p-r-k} a_1^{*\,p-b_1-b_2+\tau-k}}{k!\,(p-b_1-b_2+\tau-k)!}$$

$$\times \frac{(b_1+b_2-q-\tau)!\, b_1^k\, b_1^{*\,b_1+b_2-p-q+r+k-\tau}\, a_2^{*\,p+q-b_1-b_2+\tau}\, a_3^{*\,\tau-b_2}\, b_3^{b_1-\tau}}{(b_1-\tau)!\,(b_1+b_2-p-q+r+k-\tau)!\,(p-r-k)!\,(b_2-q)!} \tag{16}$$

with

$$N = \sqrt{\frac{(b_1+1)!\, b_2!\,(b_1-b_2)!\,(p-b_2)!\,(b_2-q)!\,(r-q)!\,(p-r)!}{(b_1-q+1)!\,(b_1-p)!\,(p+1)!\,q!}\,(p-q+1)} \tag{17}$$

Now, in the product of factorials that appears in (16) we shall replace some factors through the use of the identity

$$\frac{(b_1+b_2-q-\tau)!}{(b_1-\tau)!\,(b_1+b_2-p-q+r+k-\tau)!\,(p-r-k)!\,(b_2-q)!}$$

$$= \sum_{s=0} \frac{1}{s!\,(b_1-\tau-s)!\,(b_1+b_2-p-q+r+k-\tau-s)!\,(p-b_1-r+\tau-k+s)!}$$

and moreover, instead of τ we shall introduce as new dummy index $n = p - b_1 - r + s - k + \tau$. With these changes (16) becomes

$$\mathfrak{D}^{[h_1 h_2]}_{pqr,\, h_1 h_2 h_1}(a_1, \ldots, b_3) = (-)^{h_1 + r} N b_2^{h_1 + h_2 - p - q} \sum_{nks} (-)^{s+n} a_1^{p-r-k} a_1^{*r-h_2+n-s}$$

$$\times \frac{b_1^k \, b_1^{*h_2-q-n+s} \, a_2^{*q-h_2+r+k-s+n} \, a_3^{*h_1-h_2-p+r+k-s+n} \, b_3^{p-r-k+s-n}}{k!\,(r-h_2+n-s)!\,s!\,(p-r-k-n)!\,(h_2-q-n)!\,n!}$$

(18)

A careful examination of this formula reveals that the summations over s and k come from the expansion of two binomials, namely

$$\mathfrak{D}^{[h_1 h_2]}_{pqr,\, h_1 h_2 h_1}(a_1, \ldots, b_3) = (-)^{h_1 + r} N \sum_n (-)^n \frac{(a_1^* a_2^* a_3^* - b_1^* b_3)^{r-h_2+n}}{n!\,(r-h_2+n)!\,(p-r-n)!\,(h_2-q-n)!}$$

$$\times \quad (a_1 b_3 + b_1 a_2^* a_3^*)^{p-r-n} (b_2 a_3^*)^{h_1-p} a^{*q} (b_1^* b_2)^{h_2-q-n} (a_1 b_2)^n \qquad (19)$$

Then, if we return to the elements u_{is} of the original matrix U by means of the equivalences in (8), we have

$$\mathfrak{D}^{[h_1 h_2]^*}_{pqr,\, h_1 h_2 h_1}(U) =$$

$$= (-)^{h_2 - q} N \sum_n (-)^n \frac{u_{11}^{r-h_2+n} \, u_{21}^{p-r-n} \, u_{31}^{h_1-p} (\Delta^{12}_{12})^q (\Delta^{12}_{31})^{h_2-q-n} (\Delta^{12}_{23})^n}{n!\,(r-h_2+n)!\,(p-r-n)!\,(h_2-q-n)!}$$

(20)

In writing formula (20) we have used the fact that the property expressed by equations (3) is valid for the elements of any column (or row) of U, i.e. in a unitary unimodular matrix the elements of any column (or row) are equal to the conjugate of the corresponding cofactors in the matrix. Owing to this property we could write in (20) the cofactors Δ_{ij}^{12} defined according to (4), instead of the elements u_{k3}^{*}.

If we interpret now the u_{is} as boson creation operators, the right hand side of (20) when acting on the vacuum state $|0>$ is proportional[4] to the Gelfand state $\left|\begin{matrix} h_1 & & h_2 & & 0 \\ & p & & q & \\ & & r & & \end{matrix}\right\rangle$, and taking care of the appropriate normalization we arrive at the final result

$$\mathcal{D}^{[h_1 h_2]^*}_{pqr,\, h_1 h_2 h_1}(U)\,|0> = \sqrt{\frac{(h_1+1)!\,h_2!}{(h_1-h_2+1)}}\left|\begin{matrix} h_1 & & h_2 & & 0 \\ & p & & q & \\ & & r & & \end{matrix}\right\rangle \tag{21}$$

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