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A NEW FORMULATION OF STOCHASTIC THEORY AND QUANTUM MECHANICS
GENERAL INTEGRATION OF THE FUNDAMENTAL EQUATIONS

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RESUMEN

En esta nota se presenta un método directo para integrar las ecuaciones maestras de una teoría estocástica propuesta recientemente. Se estudian dos casos importantes: el movimiento de la partícula bajo la acción de una fuerza conservativa y de un campo electromagnético externo. Las ecuaciones de Schrödinger y de Fokker-Planck siguen inmediatamente como casos particulares de las ecuaciones generales.

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ABSTRACT

In this note, we present a direct method of integration of the master equations of a stochastic theory recently proposed. Two important cases are studied, namely, the motion of the particle under a conservative force and under an external electromagnetic field. The Schrödinger and Fokker-Planck equations follow immediately as particular cases of the more general equations.

I. INTRODUCTION

In a recent paper[1], one of us presented a new formulation of stochastic theory developed from first principles. Two basic equations of motion were thus obtained, as a generalization of Newtonian mechanics for the case of a stochastic force. It was further shown that the first integral of these equations leads to Schrödinger's equation when non-Markoffian terms are neglected and some parameters take on specified values, and that quantum mechanics can therefore be interpreted as a stochastic process.

The scope of this paper is to present a more general method of integration of the equations of motion. This integrated set may be considered the fundamental system of equations for further development of the theory.

Two different cases are considered: firstly, the motion under a conservative force and secondly, the general electromagnetic problem. In both cases the corresponding Fokker-Planck and Schrödinger equations are obtained.

II. MOTION UNDER THE ACTION OF A POTENTIAL

The two fundamental equations of stochastic motion are[1]:

$$\mathcal{D}_C \mathbf{v} - \lambda \mathcal{D}_S \mathbf{u} = \mathbf{f}_0 ,$$
$$\mathcal{D}_C \mathbf{u} + \mathcal{D}_S \mathbf{v} = 0 , \qquad (1)$$

where the systematic and stochastic derivatives, $\mathbb{D}_C$ and $\mathbb{D}_S$ respectively, are expressed as follows:

$$\mathbb{D}_C = \frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla - D_-\nabla\nabla\cdot + \hat{\mathbf{L}}_C\nabla \, .$$

$$\mathbb{D}_S = \mathbf{u}\cdot\nabla + D_+\nabla\nabla\cdot + \hat{\mathbf{L}}_S\nabla \, . \tag{2}$$

$\mathbf{v}$ and $\mathbf{u}$ are the systematic and stochastic velocities respectively, given by $\mathbf{v} = \mathbb{D}_C\mathbf{x}$ and $\mathbf{u} = \mathbb{D}_S\mathbf{x}$; $\mathbf{f}_0$ is the external force (per unit mass) and D_+, D_- are related to the diffusion coefficient[1]. $\hat{\mathbf{L}}_C$ and $\hat{\mathbf{L}}_S$ are linear operators containing derivatives of order $\geqslant 2$; the value of their coefficients depends upon the nature of the particle's stochastic interaction with its surroundings.

Let us assume that D_+, D_- and the coefficients appearing in $\hat{\mathbf{L}}_C$ and $\hat{\mathbf{L}}_S$ do not depend on the coordinates; in this case, $\mathbb{D}_C$ and $\mathbb{D}_S$ satisfy the commutation rules

$$[\partial_i, \mathbb{D}_S] = \sum_j (\partial_i u_j)\,\partial_j \, ,$$

$$[\partial_i, \mathbb{D}_C] = \sum_j (\partial_i v_j)\,\partial_j \, , \tag{3}$$

These rules can be readily generalized for variable coefficients, in which case the corresponding derivatives must be added to the right side of eqs. (3).

In studying the case of conservative forces, we may write $\mathbf{f}_0 = -\nabla V$. Let us introduce two real dimensionless functions R and S, satisfying

$$\mathbf{v} = 2D_0\nabla S \, , \tag{4a}$$

$$\mathbf{u} = 2D_0\nabla R \tag{4b}$$

with D_0 constant. For simplicity, we restrict ourselves to the case $\lambda = 1$. Eqs. (1) can be readily integrated with the aid of (3) and (4), to obtain:

$$\mathbb{D}_C R + \mathbb{D}_S S - 2D_0 \nabla R \cdot \nabla S = 0 , \tag{5a}$$

$$2D_0 (\mathbb{D}_C S - \mathbb{D}_S R) - 2D_0^2 [(\nabla S)^2 - (\nabla R)^2] = - V , \tag{5b}$$

which are the integrated fundamental equations. A combination of them allows us to write

$$2D_0 \mathbb{D}_Q w - 2D_0^2 (\nabla w)^2 = V , \tag{6}$$

where we have introduced the following definitions:

$$\mathbb{D}_Q = \mathbb{D}_S + i \mathbb{D}_C , \tag{7a}$$

$$w = R + iS . \tag{7b}$$

Eq. (6) is our new fundamental equation of stochastic theory. It describes the motion of a particle subject to the action of an external conservative force f_0 and a stochastic force due to the interaction of the particle with its surroundings. In particular, we can derive from it the equation of motion for a quantum-mechanical particle, i.e., Schrödinger's equation. For this purpose, we introduce the function ψ given by

$$\psi = e^{R + iS} \tag{8}$$

and rewrite (6) with the aid of (8), to obtain

$$2iD_0 \frac{\partial\psi}{\partial t} = -2D_0^2 \nabla^2\psi + [V + 2D_0 (D_0 - D_Q) \nabla^2 ln\psi - 2D_0 \hat{\mathbf{L}}_Q \cdot \nabla ln\psi] \psi , \tag{9}$$

where $\hat{\mathbf{L}}_Q = \hat{\mathbf{L}}_S + i\hat{\mathbf{L}}_C$ and $D_Q = D_+ - iD_-$.

This general, non-linear equation reduces to Schrödinger's equation by making $D_Q = D_+ = D_0$, i.e. $D_- = 0$; $\hat{\mathbf{L}}_Q = 0$ and taking $D_0 = \hbar/2m$, (also recall that $\lambda = 1$):

$$i\hbar \frac{\partial\psi}{\partial t} = -\hbar^2/2m \nabla^2\psi + U\psi . \tag{10}$$

$U = mV$ is the total potential.

III . MOTION UNDER THE ACTION OF AN APPLIED ELECTROMAGNETIC FIELD.

Let us now study the more general problem of a particle under the action of an external electromagnetic field characterized by the potentials ϕ and $\mathbf{A}$.

As a first step, we must generalize the original system of basic equations, to take into account all the terms of the external force. Clearly, in this case, $\mathbf{f}_0$ is given by the Lorentz force, which contains a term proportional to the total velocity $\mathbf{c} = \mathbf{v} + \mathbf{u}$:[*,1]

$$\mathbf{f}_0 = \frac{e}{m} \left[\mathbf{E} + \frac{1}{c} (\mathbf{v} + \mathbf{u}) \times \mathbf{H} \right] \tag{11}$$

[*] The term $\mathbf{u} \times \mathbf{H}$ must be taken into account as becomes obvious during the working out of the algebra: without it, it is impossible to recover the usual formulation of quantum mechanics for the electromagnetic case.

Since by hypothesis $\mathbf{v}$ changes its sign under time inversion, while $\boldsymbol{u}$ does not, we see that $\boldsymbol{f}_0$ has two components, which we shall call $\boldsymbol{f}_0^{(+)}$ and $\boldsymbol{f}_0^{(-)}$, behaving differently under time inversion, namely,

$$\boldsymbol{f}_0^{(+)} = \frac{e}{m}\left[\boldsymbol{E} + \frac{1}{c}\,\mathbf{v}\times\boldsymbol{H}\right] ,$$

$$\boldsymbol{f}_0^{(-)} = \frac{e}{mc}\,\boldsymbol{u}\times\boldsymbol{H} \quad . \tag{12}$$

The set of eqs. (1), which was written under the assumption $\boldsymbol{f}_0^{(-)} = 0$, must now be rewritten as follows:

$$\mathcal{D}_C\,\mathbf{v} - \mathcal{D}_S\,\boldsymbol{u} = \boldsymbol{f}_0^{(+)} \quad ,$$

$$\mathcal{D}_C\,\boldsymbol{u} + \mathcal{D}_S\,\mathbf{v} = \boldsymbol{f}_0^{(-)} \quad . \tag{13}$$

Furthermore, we must modify our definition of the function S. We know that in the Newtonian limit, $\hbar S$ may be identified with the action[2]. In electrodynamics, there exists a well-known relation between the action, the vector potential and the velocity of the particle, namely[3]

$$\mathbf{v} = \frac{1}{m}\,\nabla(\hbar S) - \frac{e}{mc}\,\mathbf{A} \quad .$$

This relation is a direct generalization of eq. (4a) for the electromagnetic case; therefore, we define the new function S by

$$\mathbf{v} = 2D_0\nabla S - \frac{e}{mc}\,\mathbf{A} \quad . \tag{14}$$

Eq. (14) implies that the mean value of $m\mathbf{v}$ is given by the expectation value[2] of $-i\hbar\nabla - \frac{e}{c}\mathbf{A}$, i.e., that the electromagnetic field contributes to the momentum with $-\frac{e}{c}\mathbf{A}$, as is well known. However, for the stochastic velocity $\boldsymbol{u}$ we use the same relation (4b), which guarantees that its mean value remains equal to zero. With the aid of eqs. (4b) and (14) the integration is readily performed.

Following now a procedure similar to that used in the preceding section, we obtain for the fundamental integrated equations:

$$2D_0(\mathcal{D}_C R + \mathcal{D}_S S) - 4D_0^2 \nabla R \cdot \nabla S = \frac{e}{mc}(D_+ \nabla \cdot \mathbf{A} + \hat{\mathbf{L}}_S \cdot \mathbf{A}) , \qquad (15a)$$

$$2D_0(\mathcal{D}_C S - \mathcal{D}_S R) + 2D_0^2 [(\nabla R)^2 - (\nabla S)^2] =$$

$$-V - \frac{e^2}{2m^2c^2}\mathbf{A}^2 + \frac{e}{mc}(-D_- \nabla \cdot \mathbf{A} + \hat{\mathbf{L}}_C \cdot \mathbf{A}) . \qquad (15b)$$

Combining these results with the aid of (7), we obtain:

$$2D_0 \mathcal{D}_Q w - 2D_0^2 (\nabla w)^2 = V + \frac{ie}{mc}(D_Q \nabla \cdot \mathbf{A} - \frac{ie}{2mc}\mathbf{A}^2 + \hat{\mathbf{L}}_Q \cdot \mathbf{A}) , \qquad (16)$$

which can be considered the basic equation of motion for a stochastic particle under the action of an electromagnetic field. Following the previous procedure, we may rewrite it in terms of ψ:

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{1}{2m}(-i\hbar\nabla - \frac{e}{c}\mathbf{A})^2\psi + [\phi + \hbar(D_0 - D_Q)\nabla^2 ln\psi - \hbar\hat{\mathbf{L}}_Q \cdot \nabla ln\psi]\psi ,$$

(17)

where $\phi = mV$ is the total scalar potential. As is expected, (17) reduces to Schrödinger's equation in the limit $\hat{L}_Q = 0$, with $D_0 = D_+ = \hbar/2m$ and $D = 0$:

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{1}{2m}(-i\hbar\nabla - \frac{e}{c}\mathbf{A})^2 \psi + \phi \psi .$$

It is easy to show that ψ plays the role of a probability amplitude. In fact, by introducing

$$\rho = e^{2R} = |\psi|^2 \tag{18}$$

and rewriting (15a) in terms of ρ, we obtain

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\mathbf{v}\rho) = \frac{1}{D_0}\rho\Big[D_- \nabla \cdot \mathbf{u} + (D_0 - D_+)\nabla \cdot \mathbf{v} - \hat{\mathbf{L}}_S \cdot \mathbf{v} - \hat{\mathbf{L}}_C \cdot \mathbf{u}\Big], \tag{19}$$

i.e., a continuity equation with sources for the probability density ρ. Thus, one of the basic equations of stochastic theory is essentially the continuity equation, while the second one, eq. (15b), expresses the conservation of energy, as has been shown in earlier papers[1,2].

It is evident from (19) that the non-Markoffian terms account for self-interactions. The usual quantum mechanics is obtained by postulating that the process is Markoffian, i.e., that such self-interactions may be neglected.

Eq. (19) was written having in mind the continuity equation of quantum mechanics. Clearly, we can give it the form of a Fokker-Planck equation; in fact, it is simple to show that eq. (19) is equivalent to

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\mathbf{c}\rho) - D\nabla^2 \rho =$$

$$= - \frac{1}{D_0} \rho \left[(D_+ - D_0) \left(\nabla \cdot \mathbf{c} + \frac{u^2}{D_0} \right) - 2D_- \left(\nabla \cdot \boldsymbol{u} + \frac{u^2}{2D_0} \right) + \hat{L}_S \cdot \boldsymbol{v} + \hat{L}_C \cdot \boldsymbol{u} \right] ; \tag{20}$$

where $D = D_+ - D_-$ is the diffusion coefficient. For the quantum-mechanical case, eq. (20) reduces to the usual Fokker-Planck equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{c}\rho - D\nabla^2 \rho = 0 \ . \tag{21}$$

Since eq. (20) is written in terms of the velocities $\mathbf{v}$ and $\boldsymbol{u}$, it is also valid for the previous case of a conservative force.

REFERENCES

1. L. de la Peña-Auerbach, Phys. Lett., **27A**, 594 (1968) ; University of Mexico preprint (extended version, to be published).
2. L. de la Peña-Auerbach and L.S. García-Colín, J. Math. Phys., **9**, 916 (1968).
3. See e.g. L. Landau and E. Lifshitz, *The Classical Theory of Fields*, Addison-Wesley, 1951.