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VARIATIONAL ANALYSIS OF THE TWO-BODY PROBLEM WITH HARMONIC-OSCILLATOR STATES

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RESUMEN

En este trabajo llevamos a cabo un análisis variacional para el problema del deuterón, usando una función de onda de ensayo que es una combinación lineal de estados del oscilador armónico hasta con 10 cuantos. Se calcula la energía como función de la frecuencia del oscilador armónico para varios potenciales gaussianos propuestos por Feenberg para el deuterón, y para el potencial de Eikemeier y Hackenbroich que tiene un carozo repulsivo blando. Los resultados del análisis variacional para ambos casos se comparan con el resultado exacto, es decir, con la energía de enlace del deuterón. Se indica también la variación del radio medio

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cuadrado del deuterón en función del número máxima de cuantos empleados en el desarrollo. El objeto principal del trabajo es establecer la rapidez de convergencia para el proceso de aproximación, las cuales podrían emplearse posteriormente en problemas involucrando más de dos partículas.

ABSTRACT

In the present paper we carry out a variational analysis of the deuteron problem with a trial wave function that is a linear combination of harmonic oscillator states of up to 10 quanta. The energy is computed as a function of the frequency of the harmonic oscillator for several Gaussian potentials for the deuteron proposed by Feenberg, and for the Eikemeier-Hackenbroich potential which has a soft repulsive core. The variational analysis is compared with the exact result, i.e., the binding energy of the deuteron, for both potentials. We also indicate the variation of the root-mean square radius of the deuteron as function of the maximum number of quanta considered in the expansion. The main purpose of this paper is to establish bound and convergence rates for the approximation process, which we could use later in problems involving more than two particles.

I. INTRODUCTION

Variational analysis of the ground states of light nuclei has been carried out almost since the achievement of a clear understanding of the nuclei as proton neutron systems. Usually the trial wave functions used were of a very simple type, e.g., for the α particle one took a Gaussian in the relative coordinates of the four nucleons[1]. The parameters in these trial wave functions were adjusted so as to minimize the expectation value of the Hamiltonian proposed, hoping to obtain binding energy in reasonable agreement with experiment for an interaction potential between the nucleons deduced from two body binding and scattering data.

Up to relatively recently no direct experimental information on the trial wave function itself could be obtained. This situation has changed radically with

the measurements of form factors by electron scattering experiments[2]. From the Fourier transform of the form factor we get the charge distribution which we could also determine directly from the trial wave function, thus testing these wave functions in much more detail than was done before through the evaluation of the binding energy.

It is interesting to note that, for example, in the case of the α particle, the simple Gaussian wave function mentioned above is no longer sufficient as it leads to a Gaussian charge-density distribution and a Gaussian form factor. The actual form factor as measured in recent experiments[3] contains besides a Gaussian factor a polynomial of sixth order in the square of the momentum transfer. This implies that if the wave function of the four-nucleon system is developed in terms of harmonic oscillator states in the relative coordinates, it would have to include states of up to 6 quanta in addition to the states of 0 quanta involved in the standard symmetric Guassian wave function. Construction of these four-nucleon states, and more generally of the n-nucleon states, characterized by definite irreducible representation of the rotation group R_3 and the symmetric group S_n in a translationally invariant harmonic oscillator potential has been achieved by Kramer and Moshinsky[4] These states could then be used for an expansion of the trial wave functions of the ground states of light nuclei, which would have a much better chance of giving at the same time the binding energy and the correct charge distribution.

Before engaging in a program of this type it is of great interest to see whether a superposition of harmonic oscillator states could constitute a good trial wave function for the two-body problem. It is important to see how quick is the convergence rate to the binding energy as a function of the maximum number of quanta considered. It is also of interest to see how this convergence rate depends on whether the two body interaction has a repulsive core or not.

In the present paper we shall analyze three types of potentials between two particles:

(a) A family of Gaussian potentials without repulsive core introduced be Feenberg for the proton-neutron system whose parameters were adjusted numerically to give the appropriate binding -2.2 MeV for the deuteron.

(b) A superposition of three Gaussian potentials, one of which is positive and represents a repulsive core, which has introduced by Eikemeier and Hackenbroich[5] to adjust numerically both the binding energy of the deuteron and the nucleon scattering in the triplet state.

(c) The Coulomb potential between electron and proton. In all three cases we shall calculate the matrix elements of the Hamiltonian with respect to harmonic oscillator states of up to ten quanta and then proceed to study the variation of the binding as a function of the frequency ω of the oscillator. We shall, in particular, be interested in the values of the root mean square radius of the two particle state for this frequency.

We first discuss the details of the variational analysis and then proceed to apply them to the potentials considered.

II. THE VARIATIONAL ANALYSIS

The Hamiltonian for the two-body problem would be written as

$$\mathcal{H} = (2\mu)^{-1} p'^2 + V(r') , \tag{1}$$

where μ is the reduced mass of the two particles and r', p' are the relative coordinates and momenta of the two particles. It proves very convenient to use instead of r', p' the following dimensionless coordinates and momenta

$$r = (\mu\omega/\hbar)^{\frac{1}{2}} r' , \tag{2a}$$

$$p = (\mu\omega\hbar)^{-\frac{1}{2}} p' , \tag{2b}$$

where ω is some frequency which we shall later identify with the frequency of the harmonic oscillator states we use. The Hamiltonian $\mathcal{H}$ takes then the form

$$\mathcal{H} = \frac{1}{2}\, h\omega p^2 + V\left((\hbar/\mu\omega)^{1/2} r\right). \tag{3}$$

Now the harmonic oscillator states of frequency ω could be designated by the kets

$$|n\,l\,m>\,, \tag{4a}$$

$$N = 2n + l\,, \tag{4b}$$

where n is the radial quantum number, l the orbital angular momentum, m its projection and N the total number of quanta.

The trial wave function we shall use will be a linear combination of (4a), but as we shall be concerned only with states with orbital angular momentum $l = 0$, both in the deuteron and hydrogen atom case, we could restrict ourselves to the wave function $|n, 0, 0, >$, which in what follows we shall designate in the short hand notation

$$|n> \equiv R_{n0}(r)\,. \tag{5}$$

We have then that

$$\psi = \sum_n a_n\, |n>\,, \tag{6a}$$

$$\sum_n a_n^2 = 1\,, \tag{6b}$$

and a variational analysis of the expectation value of $\mathcal{H}$ subject to the restriction that ψ is normalized leads to a system of linear equations in the a_n. The secular determinant is then

$$\det \left| \langle n' | \mathcal{H} | n \rangle - E \delta_{n',n} \right| = 0, \tag{7}$$

with n', n being integers restricted to

$$0 \leqq n', n \leqq \frac{1}{2} N, \tag{8}$$

with N being the maximum even number of quanta considered in the problem.

The matrix elements in (7) can be easily shown to be[6]

$$\langle n' | \mathcal{H} | n \rangle = \frac{1}{2} \hbar\omega \left\{ \left[n(n + \tfrac{1}{2}) \right]^{\frac{1}{2}} \delta_{n',n-1} + (2n + \tfrac{3}{2}) \delta_{nn'} + \left[n(n + \tfrac{3}{2}) \right]^{\frac{1}{2}} \delta_{n',n+1} \right\}$$

$$+ \sum_{p=0}^{n+n'} B(n'0, n0p) I_p , \tag{9}$$

where I_p are the Talmi integrals[7]

$$I_p = \left[2/\Gamma(p + \tfrac{3}{2}) \right] \int_0^\infty r^{2p+2} e^{-r^2} V((\hbar/\mu\omega)^{\frac{1}{2}} r)\, dr , \tag{10}$$

and $B(n'0, n0, p)$ are coefficients algebraically determined and explicitly tabulated by Brody and Moshinsky[8].

We shall now consider the particular cases of the deuteron and the hydrogen atom separately, to introduce dimensionless Hamiltonians convenient in each case. For the deuteron case we define

$$H \equiv \frac{\mathcal{H}}{B} = \frac{\mathcal{H}}{4.3\, mc^2},$$

B = Binding energy of deuteron,

$$\mu = \frac{1}{2} M \tag{11}$$

with m, M being the masses of the electron and proton respectively. Instead of the frequency ω, it is more convenient to introduce in the matrix elements of H corresponding to (9) the dimensionless parameter

$$\epsilon = \frac{\hbar\omega}{mc^2} . \tag{12}$$

The potential will be given by a Gaussian or a superposition of Gaussian's of the form

$$- V_0 \exp\left[- \left(\frac{r'}{r_0}\right)^2 \right] . \tag{13}$$

It is more convenient to give the parameters of this potential in terms of the dimensionless variables

$$A = \frac{V_0}{4.3mc^2} , \tag{14a}$$

$$\alpha = \frac{\hbar^2}{Mmc^2 r_0^2} ,$$

$$\left(\frac{\hbar^2}{Mmc^2}\right)^{\frac{1}{2}} = 9.0196 \text{ fm} , \tag{14b}$$

thus being able to write the ratio of this potential to $4.3mc^2$ as

$$-A \exp\left[-\frac{2\alpha r^2}{\epsilon}\right]. \tag{15}$$

From (10) we conclude that the Talmi integral for potential (15) is

$$I_p = -A\left(1+\frac{2\alpha}{\epsilon}\right)^{-p-3/2}. \tag{16}$$

In the units $4.3mc^2$ of energy used here the binding energy of the deuteron is

$$E_b \equiv -B = -1.00, \tag{17}$$

and this is the number we must approach in our variational calculations.

For the Hydrogen atom case[6] a convenient dimensionless Hamiltonian is defined by

$$H = \left(\frac{me^4}{2\hbar^2}\right)^{-1} \mathcal{H},$$

$$\mu = m, \tag{18}$$

with m being the mass of the electron, so we are dividing by the energy of the first Bohr orbit. Instead of the frequency ω it is more convenient in this case to use the parameter

$$\beta^2 = \hbar\omega \left(\frac{me^4}{2\hbar^2}\right)^{-1}. \tag{19}$$

The Talmi integral for the Coulomb potential divided by the energy of the first Bohr orbit, is then given[5] by

$$-\sqrt{2}\ \beta\left[p!/\Gamma(p+\frac{3}{2})\right], \tag{20}$$

and the binding energy of the hydrogen atom in these units is

$$E_b = -1.00\ . \tag{21}$$

As a final point in this general discussion we shall analyze the expectation value of the square of the radius r'^2 with respect to the state (6), i.e.,

$$\langle r'^2\rangle \equiv \int \psi^* r'^2 \psi d\tau = \frac{\hbar}{\mu\omega} \sum_{n\,n'=0}^{\frac{1}{2}N} a_{n'}^* a_n \langle n'|r^2|n\rangle$$

$$= \frac{\hbar}{\mu\omega}\left[\frac{3}{2} + 2\sum_{n=1}^{\frac{1}{2}N}\left\{n a_n^2 - \left[n(n+\frac{1}{2})\right]^{\frac{1}{2}} a_{n-1}\, a_n\right\}\right], \tag{22}$$

where we made use both of the matrix element[6] of r^2 and the normalization condition (6b), the latter to eliminate a_0. Using the dimensionless parameters discussed above we see that we can write in the case of the deuteron

$$(\hbar/\mu\omega) = 2(\hbar^2/Mmc^2)\,\epsilon^{-1} = 2\times 81.353\, fm^2 \times \epsilon^{-1}, \tag{23a}$$

and in the case of the hydrogen atom

$$(\hbar/\mu\omega) = 2(\hbar^2/me^2)\,\beta^{-2} = 2\,a_B^2\,\beta^{-2}, \tag{23b}$$

with a_B being the Bohr radius. The root-mean-square radius is of course $\sqrt{<r'^2>}$.

III. THE FEENBERG POTENTIAL FOR THE NEUTRON-PROTON SYSTEM

Feenberg determined numerically a series of Gaussian potentials that gave the same binding energy -4.3 in units of mc^2, for the lowest bound state of the neutron proton system. In table 1 we reproduce the values of V_0, r_0 in MeV and fermis respectively and the corresponding values of A and α for these nine potentials enumerated in order of increasing range.

To analyze how well the harmonic oscillator states approach the lowest bound state of these potentials we first discuss the binding energy as a function of ϵ defined in (12) in the zero quantum approximation for the potentials 1, 3, 6, 9 of table 1. The other cases were also analyzed, but as the corresponding curves lie between those presented in figs. 1, 2, 3, 4 we do not reproduce them here.

An interesting point is that the energy, which in this case is just

$$E_0(\epsilon) = <0|H|0>, \tag{24}$$

has very different behavior as a function of ϵ in these four cases, despite the fact that the binding energy of the ground state in all of them is the same. For the very short range potential 1 of fig. 1, $E_0(\epsilon)$ is not only always positive, it does not even show a minimum. If the range increases somewhat as for the potential 3, the $E_0(\epsilon)$ shows a minimum at $\epsilon = 30.86$, though this minimum is still slightly above zero thus not leading to a bound state in this approximation. For the longer-range potentials 6 and 9 the minimum is negative and in the case 9, we already in this very rough approximation get 69.72% of the binding energy at the minimum $\epsilon = 13.88$.

We see then that for a Gaussian potential the validity of approximating the ground state of fixed energy by a Gaussian wave function, i.e., an harmonic-oscil-

lator state of 0 quanta, is strongly dependent on the range the Gaussian potential, becoming better as the range increases.

We now extend the analysis of potentials 3 and 9 from zero to up to ten quanta. We selected the potentials 3 and 9 for this more thorough analysis for the following reasons:

(a) Case 3 not only gives the correct binding energy of the deuteron as do all the others, but also can be shown to give an effective range[9] close to the one required experimentally for the scattering of nucleons in the triplet state at low energy. Furthermore, the minimum of $E(\epsilon)$ at 0 quanta comes quite close to the one obtained at 0 quanta for the Hackenbroich potential, thus allowing us to compare the speed of convergence to the binding energy of the deuteron of two potentials, one without and one with repulsive core.

(b) Case 9 has the longest range of all the Feenberg potentials and so it is interesting to see the speed of convergence in this case, the most favourable one in the case of the zero quantum approximation.

The quantum numbers of the states are $n = 0,1,2,3,4,5$; as the number of quanta is $2n$, for the last, $n = 5$, we have 10 quanta. Altogether we have then a 6×6 matrix in eq. (7). we evaluated the matrix elements as a function of ϵ and proceeded to diagonalize and find the lowest eigenvalue, not only for the 6×6 matrix but also for its submatrices of $5\times5, 4\times4, 3\times3, 2\times2$, and 1×1. Thus we were able to draw fig. 5 for potential 3 and fig. 6 for potential 9, indicating by E_n for each of the curves the $(n+1)\times(n+1)$ matrix to which it belongs.

We immediately notice in both figures that we approach the binding energy -1 when we increase the number of quanta, but that this approach does not take place in a uniform way. First when we include only states with $n = 0,1$, i.e., a 2×2 matrix, the minimum does not even diminish though the new curve comes below the 1×1 curve as it should. But if we include $n = 0,1,2$, i.e., a 3×3 matrix, we get a considerable jump toward the correct binding energy. Again when we pass to $n = 0,1,2,3$, i.e., a 4×4 matrix we do not decrease the minimum of the 3×3 matrix, but for the 5×5 matrix, i.e., $n = 0,1,2,3,4$ we get another jump followed by no improvement for the 6×6 matrix, i.e., $n = 0,1,2,3,4,5$. Clearly

then, our binding energy for the value of ϵ for which it is a minimum, decreases only when we increase our states by jumps of four and not by only two quanta, i.e., when we go from number of quanta = 0, ... N to number of quanta = 0, ... N, $N+2$, $N+4$ with N a multiple of 4. This is a general result as will be seen in the following sections.

For potential 3 the 0-quanta approximation does not even give binding as seen in fig. 2, but already we get binding for $N = 4$, with lowest energy $E_2 = -0.68$ for $\epsilon = 30.8$, and for $N = 8$ with lowest energy $E_4 = -0.87$ for $\epsilon = 27.8$. In the latter case we get already 88% of the binding energy.

For potential 9 the zero quantum approximation already gives a considerable part of the binding. Nevertheless the binding increases with increasing number of quanta though by smaller steps and again significant changes occur only by jumps of four quanta. For $N = 10$ we get the lowest energy $E_5 = -0.93$ for an $\epsilon = 10.8$. It is clear that if we carry our calculations up to 10 quanta, we get almost as good binding energy for case 3 as for case 9, despite the fact that in the latter our starting point fo $N = 0$ was much better.

In the process of diagonalizing the $1\times1, 2\times2, \ldots 6\times6$ matrices we get the corresponding a_n, $n = 0,1,\ldots\frac{1}{2}N$, $N = 0,2,4,\ldots 10$. For the ϵ that gives the minimum for the respective submatrices we calculate the root mean square radius $\sqrt{<r'^2>}$ using (22). In table 2a we give the results for the potential 3 and in table 2b for the potential 9. Though not strictly equivalent it is interesting to compare these values with the so called radius $R = \sqrt{\hbar^2/MB} = 4.31$ fm of the deuteron where B is the binding energy. For potential 3, $\sqrt{<r'^2>}$ increases from 2.81 to 3.76 fm when N goes from 0 to 10, thus approaching, but keeping below R, while for potential 9 $\sqrt{<r'^2>}$ increases from 4.080 to 5.056 fm when N goes from 0 to 10, thus exceeding almost immediately the value of R. This seems to indicate that potential 3 provides a more realistic description of the deuteron though both potential 3 and 9 give the same binding energy.

In conclusion we see that the ground state in a Gaussian potential could be approached quite effectively by a superposition of harmonic oscillator states of up to ten quanta, though an approximation by a state of zero quanta is only reasonable for a long range Gaussian potential.

IV. THE EIKEMEIER-HACKENBROICH POTENTIAL FOR THE NEUTRON-PROTON SYSTEM

Recently Eikemeier and Hackenbroich proposed a central potential for the two-nucleon system that includes a soft repulsive core and describes correctly both the binding energy of the deuteron and the scattering of nucleons by nucleons for up to 300 MeV in the laboratory system. This potential is a superposition of three Gaussians and for the triplet case, which is the only one appearing in the discussion of the binding energy of the deuteron, the parameters V_0, r_0 and the corresponding A, α of (14) are given in table 3.

The calculations were carried out in the same way as for the case of the Feenberg potential. In fig. 7 we give the energy as a function of ϵ in the 0-quanta approximation (24). We note the presence of a minimum, but as in case 3 of the Feenberg potentials, the minimum though close to zero, is positive, thus not giving a bound state in this approximation. When we increase the number of quanta up to $N = 10$, we get the six curves of fig. 8 in which the size $(n+1) \times (n+1)$ of the matrix diagonalized is indicated by E_n for each curve. Again we note the property that increases in the binding energy appears for jumps of four quanta and for $N = 4$ we get binding with a lowest value of the energy $E_2 = -0.54$ for $\epsilon = 24$, while for $N = 8$, $E_2 = -.635$ for $\epsilon = 22$. The percentage of the binding energy of the deuteron that we get even for $N = 10$ is 64% considerably lower than the 88% which we got for the comparable case 3 of the Feenberg potential. This is to be expected in view of the presence of the repulsive core.

In the same way as indicated in Section 3, the root mean square radius was calculated for up to $N = 0, 2, 4, 6, 8, 10$ quanta and the results are given in table 4. As Eikemeier and Hackenbroich give also a wave function

$$U(r') = r'\psi(r') \,, \tag{25}$$

for the deuteron by means of a graph, we tried to estimate the expectation value of r'^2 with respect to this wave function. For this purpose we approximated the

wave function $U(r')$ by an analytical function of the following type

$$U(r') = C \sin(\pi r'/2r_0) \quad \text{for } 0 \leqslant r' \leqslant r_1 , \tag{26a}$$

$$U(r') = D \exp(-r'/R) \quad \text{for } r_1 \leqslant r' \leqslant \infty , \tag{26b}$$

where the parameters take the following values

$$C = 0.55\,\text{fm}^{-\frac{1}{2}}, \quad D = 0.8476\,\text{fm}^{-\frac{1}{2}}, \quad R = 4.31\,\text{fm}, \quad r_1 = 1.99\,\text{fm}, \quad r_0 = 1.72\,\text{fm}. \tag{26c}$$

In fig. 9 we reproduce, marking it with a full line, the function $U(r')$ of Eikemeier and with a broken line draw curve (26). The exponential tail is the correct one obtained from the binding energy of the deuteron at distances in which the attractive potential almost vanishes.

With (26) the root mean square radius is

$$\sqrt{\langle r'^2 \rangle} = 3.8439\,\text{fm} , \tag{27}$$

which compares reasonably with some of the values of table 4.

We shall discuss the implications of the analysis of the Hackenbroich potential in section 6, but before that we also analyze for comparison value, the results for the Coulomb potential.

V. THE COULOMB POTENTIAL IN THE HYDROGEN ATOM

This problem was discussed fully by Moshinsky and Novaro[6]. Here we would only like to present in fig. 10 the graph of their results for the energy as function of the parameter β for up to $N = 10$ quanta. In this case the 0-quanta

approximation (24) gives already 84% of the binding energy which is improved up to 97.5% when we go up to $N = 10$ quanta. As before, the gains in the binding energy occur at jumps of four quanta, rather than two.

The exact wave function for the hydrogen atom ground state is

$$\psi = (\pi a_B^3)^{-\frac{1}{2}} \exp\left(-\frac{r'}{a_B}\right) , \tag{28}$$

where a_B is the Bohr radius. The root-mean-square radius for this wave function is

$$\sqrt{\langle r'^2 \rangle} = \sqrt{3}\ a_B = 1.732\, a_B . \tag{29}$$

For the approximate wave function for up to ten quanta the root mean square radius is given in table 5. In this case we see that all the approximate values are quite close to the exact value (29).

VI. DISCUSSION

The potential of Eikemeier and Hackenbroich would be a reasonably realistic potential for the description of two nucleon interactions. In section 5 we saw the approximation we could get with this interaction for the binding of the two nucleon systems, i.e., the deuteron when we used a trial wave function built from harmonic oscillator states. The question arises as to what sort of approximation we could get for heavier nuclei such as the α particle and beyond, with the same potential but with an n-particle trial wave function built from harmonic oscillator states of the type discussed by Kramer and Moshinsky[4].

While no rigorous answer can be given before carrying out the calculation, it is interesting to note that for the Feenberg potentials the harmonic oscillator trial wave function is much better for long range potentials than for short range ones. This means that it works much better if most of the wave function is inside

the potential, and not, as is the case of the Eikemeier-Hackenbroich potential and other two nucleon potentials, where most of the deuteron wave function is outside the potential. In the case of the α-particle though, the strong binding will keep most of the wave function inside the potential, thus possibly favouring the harmonic oscillator trial wave function *vis a vis* the corresponding wave function for the deuteron problem. In this case, the approximation for the α-particle when we increase the number of quanta should possibly converge more rapidly than in the two body problem, and thus the latter problem provides us with a kind of lower bound for convergence. It is interesting to note also that for a reasonable Feenberg potential, such as case 3, we get a closer approximation to the correct binding energy than for the Eikemeier-Hackenbroich potential which has a repulsive core. This clearly indicates that for a realistic potential, the contribution of states with a higher number of quanta is fundamental which seems to be in agreement with the fact that these states appear in the α-particle as discussed above.

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Table 1

The nine potentials of Feenberg

Type	1	2	3	4	5	6	7	8	9
V_0	122.33	83.04	61.00	47.62	38.6	32.24	27.53	23.93	21.13
r_0	1.082	1.353	1.624	1.894	2.164	2.435	2.706	2.976	3.247
A	55.1	37.4	27.48	21.45	17.38	14.52	12.4	10.78	9.56
α	69.4	44.4	30.86	22.68	17.36	13.72	11.11	91.82	7.72
A/α	0.79	0.84	0.89	0.95	1.001	1.06	1.116	1.186	1.25

V_0 in MeV, r_0 in fm, A and α are dimensionless

Table 2

(a) Root-mean-square radius of the Feenberg potential 3 (fm)

n	0	1	2	3	4	5
$\sqrt{<r^2>_n}$	2.8122	2.8175	3.2985	3.2681	3.8624	3.7649

(b) Root-mean-square radius of the Feenberg potential 9 (fm)

n	0	1	2	3	4	5
$\sqrt{<r^2>_n}$	4.080	4.1045	4.6103	4.624	5.048	5.056

Table 3

The parameters of the Eikemeier-Hackenbroich potential $A_i e^{-2\alpha_i r^2/\epsilon}$

i	1	2	3
V_{0i}	600	-70.0	-27.6
r_{0i}	0.4264	1.414	1.622
α_i	447.44	40.6765	30.914
A_i	270.3	-31.53	-12.43

V_0 in MeV, r_0 in fm, A, α are dimensionless

Table 4

Root-mean-square radius for Eikemeier-Hackenbroich potential (fm)

n	0	1	2	3	4	5
$\sqrt{<r^2>_n}$	3.0638	3.055	3.69	3.669	4.1592	4.2777

Table 5

Root-mean-square radius for Coulomb potential in units of the Bohr radius

n	0	1	2	3	4	5
$\sqrt{<r^2>_n}$	1.634	1.5956	1.622	1.63	1.669	1.67

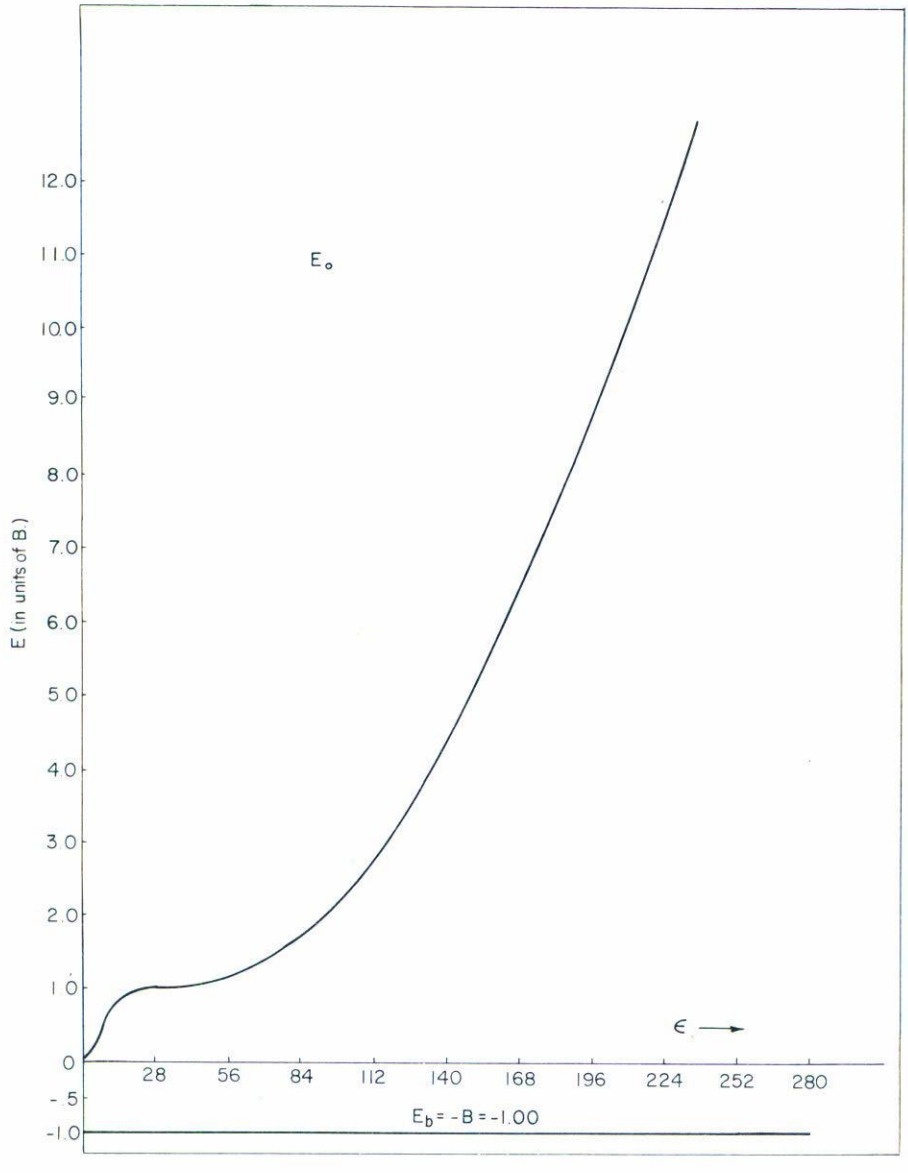

Fig. 1. Energy $E_0(\epsilon)$ for the Feenberg potential of case 1, in the zero quantum approximation as function of $\epsilon = (\hbar\omega/B)$. The energy is given in units of the binding energy B of the deuteron in this and the following figures except for figs. 9 and 10.

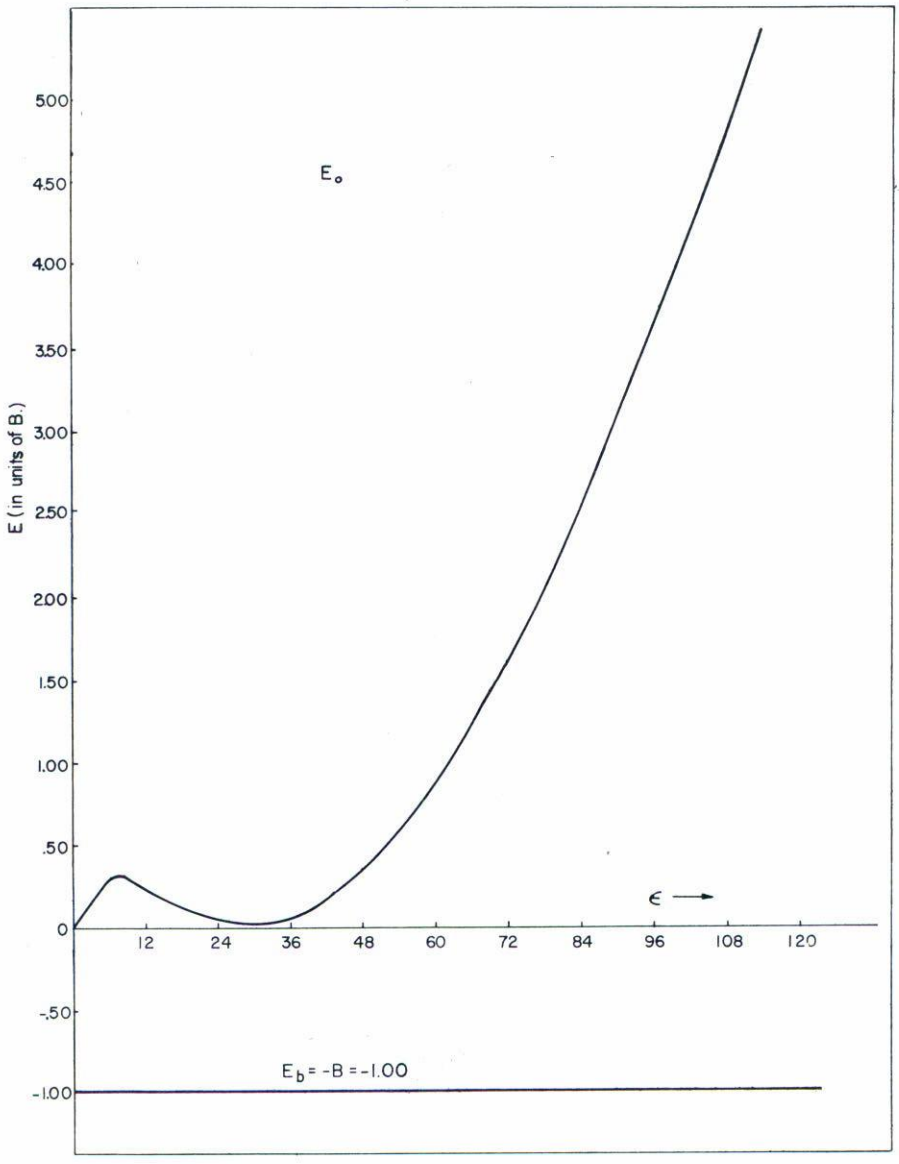

Fig. 2. Energy $E_0(\epsilon)$ for the Feenberg potential of case 3 in the zero quantum approximation.

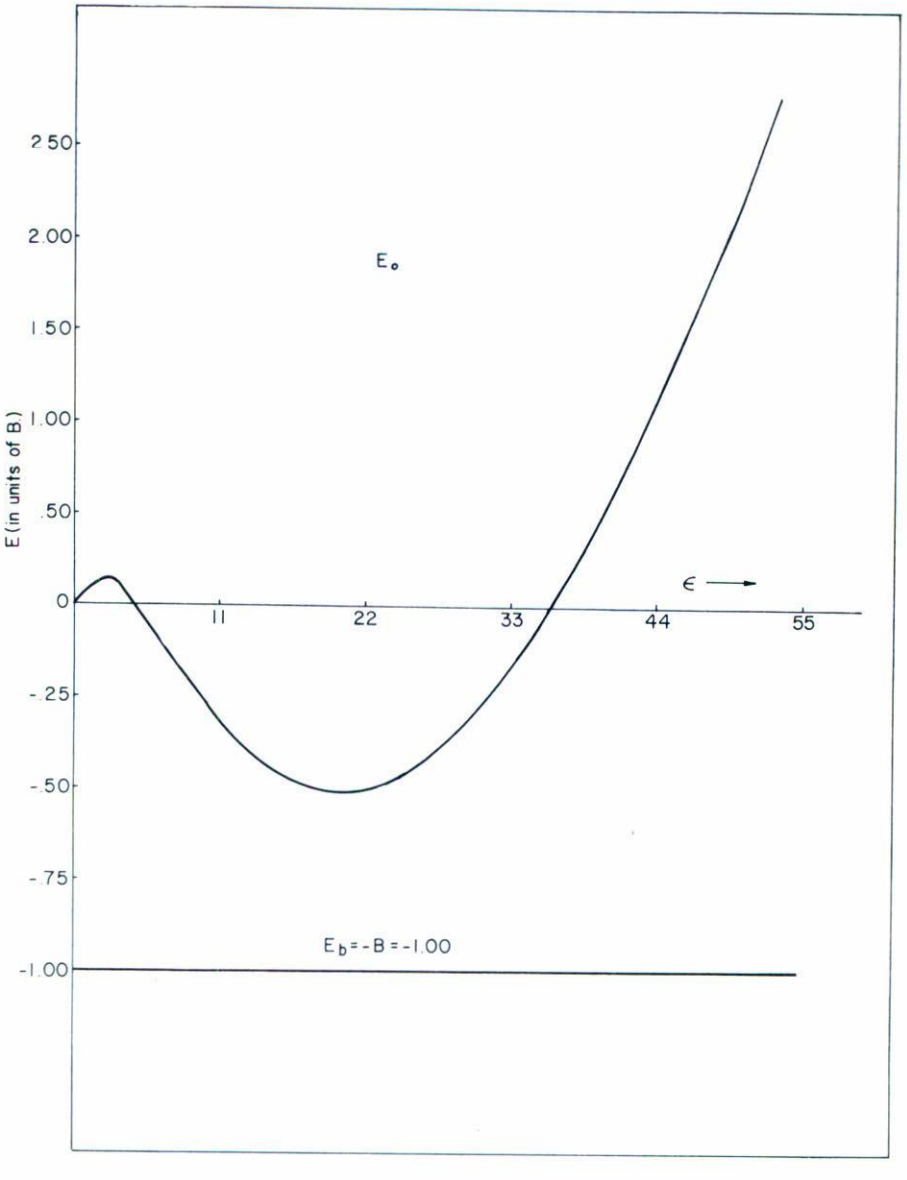

Fig. 3. Energy $E_0(\epsilon)$ for the Feenberg potential of case 6 with zero quantum approximation.

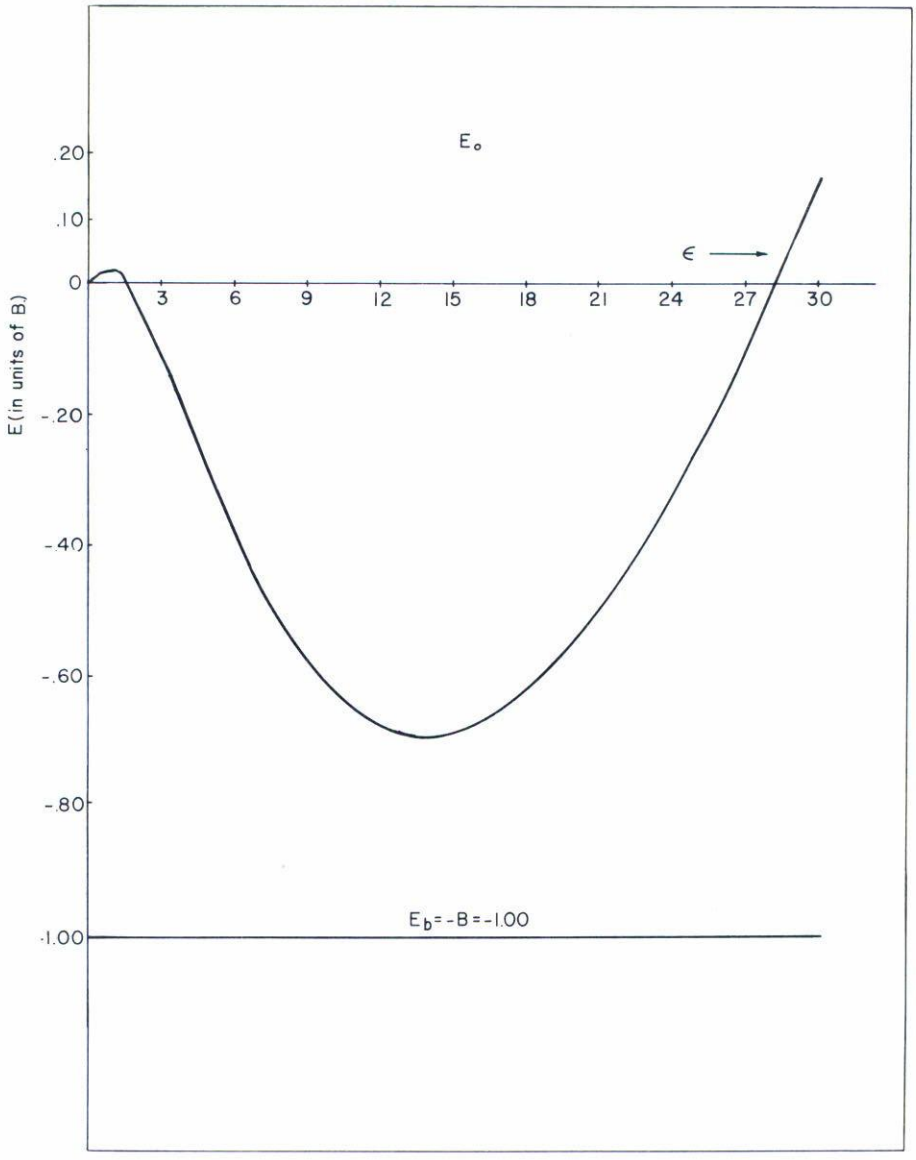

Fig. 4. Energy $E_0(\epsilon)$ for the Feenberg potential of case 9 in the zero quantum approximation.

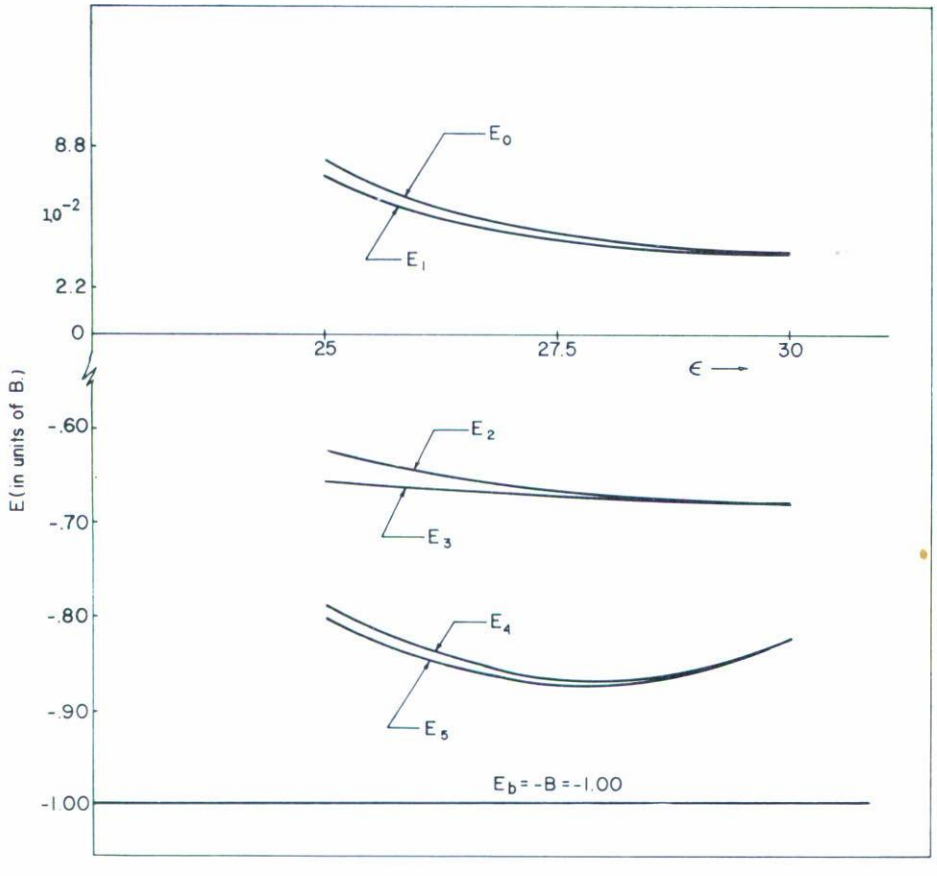

Fig. 5. Energy $E_0(\epsilon) \ldots E_5(\epsilon)$ for approximations of up to ten quanta for the Feenberg potential of case 3. Note the difference in scale for the E_0, E_1, curves and the $E_2, \ldots E_5$ curves.

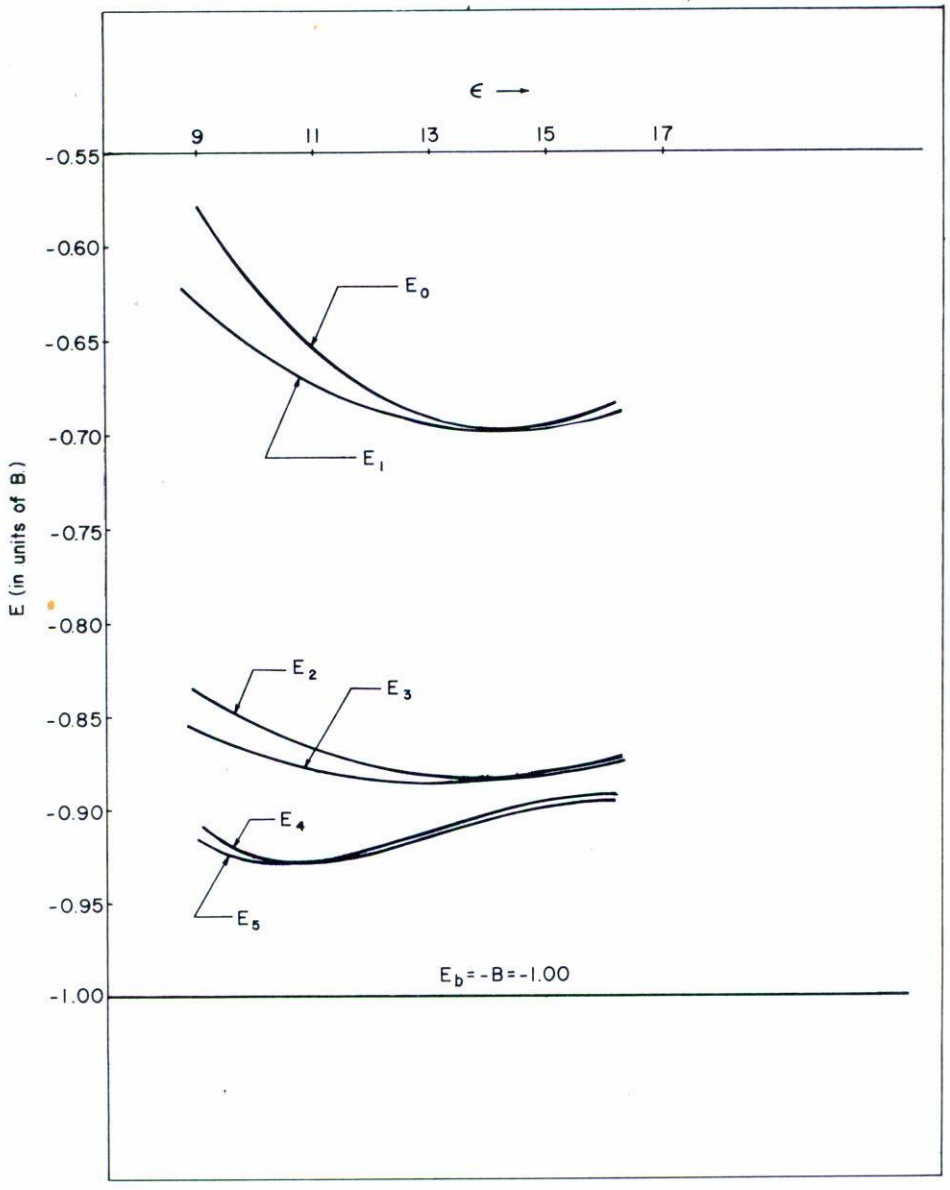

Fig. 6. Energy $E_0(\epsilon) \ldots E_5(\epsilon)$ for approximation of up to ten quanta for the Feenberg potential of case 9.

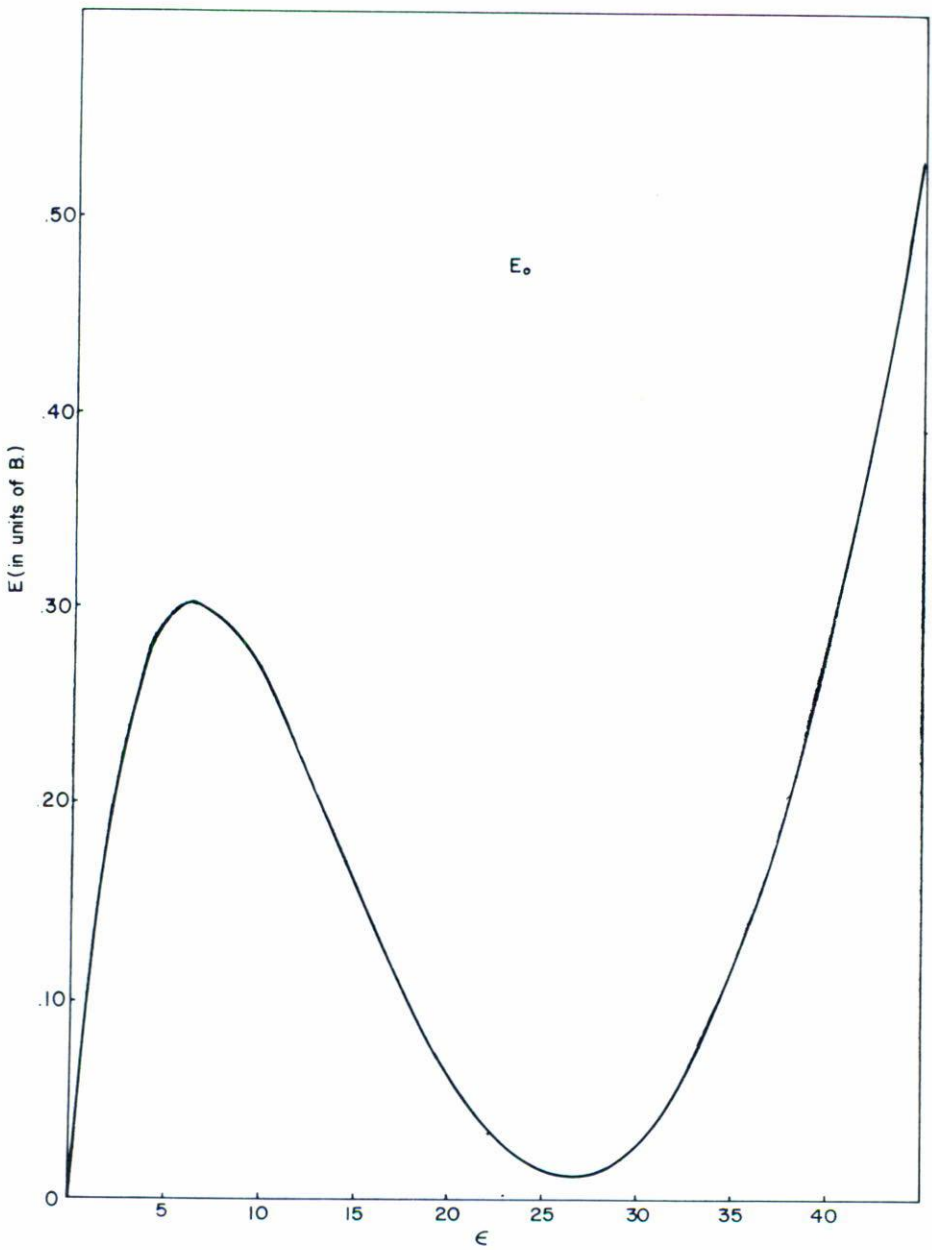

Fig. 7. Energy $E_0(\epsilon)$ for the Eikemeier-Hackenbroich potentials in the zero quantum approximation.

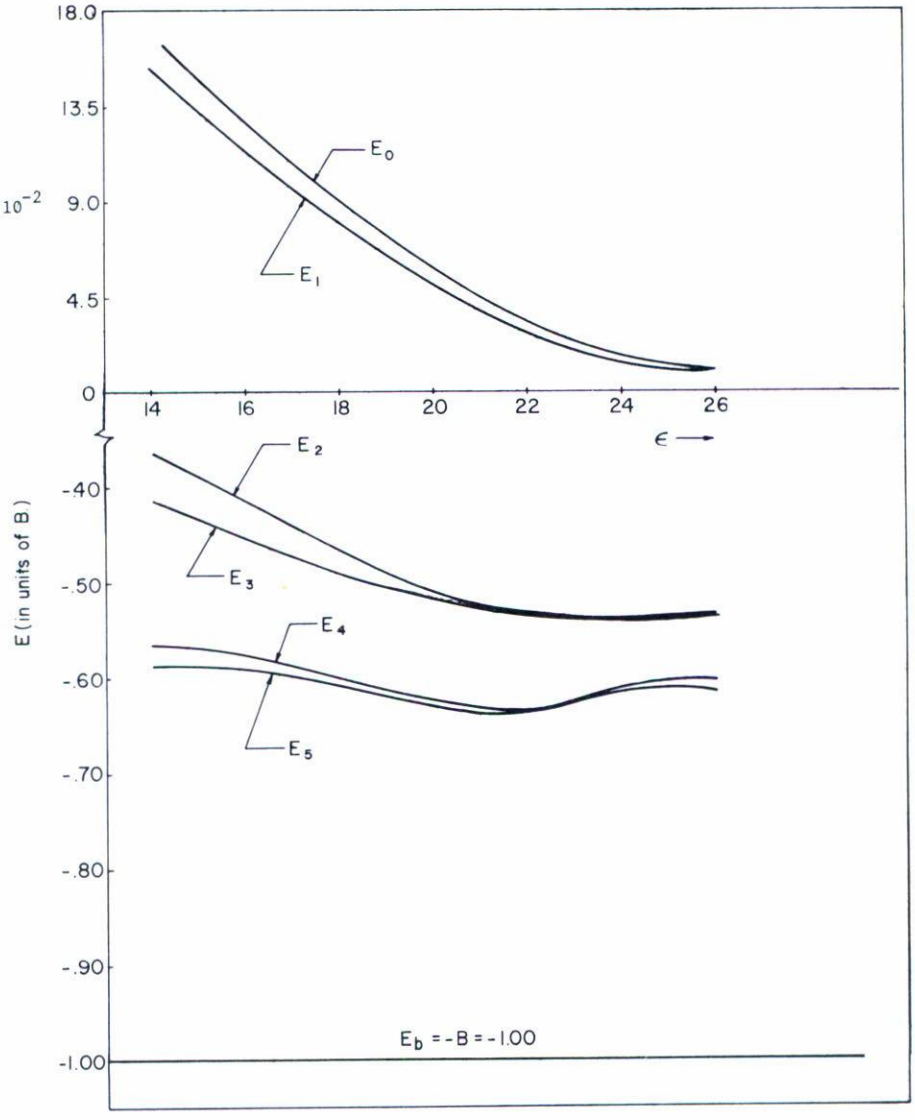

Fig. 8. Energies $E_0(\epsilon) \ldots E_5(\epsilon)$ for approximations of up to ten quanta for the Eikemeier-Hackenbroich potentials. Note the differences in scale for the E_0, E_1 curves and the E_2, $\ldots E_5$ curves.

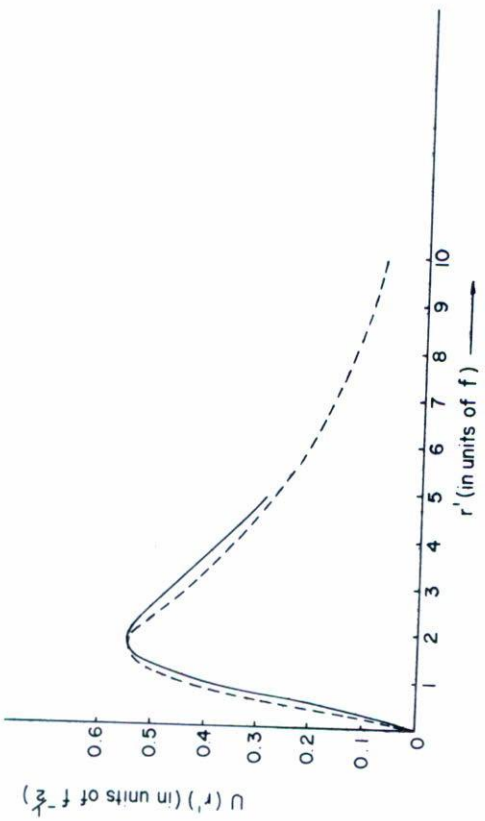

Fig. 9. Wave function $U(r')$ of Eikemeier and Hackenbroich (full line) and its approximation by the analytic wave function (dotted line) given in text.

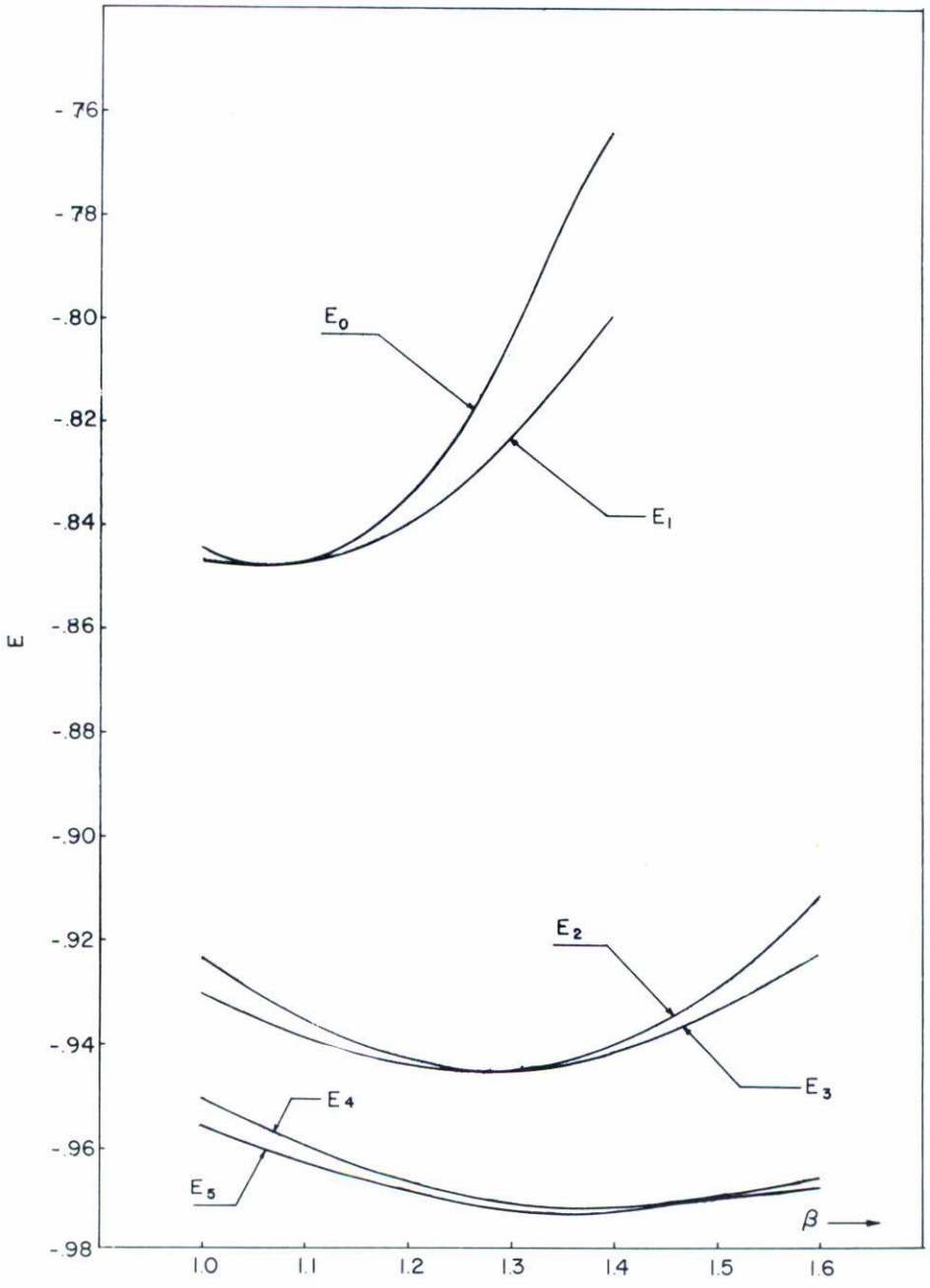

Fig. 10. Energies $E_0(\epsilon)$ for approximation of up to ten quantum for the hydrogen atom problem in terms of the parameter $\beta = [\hbar\omega/(me^4/2\hbar^2)]^{1/2}$. The unit of energy is that of the first Bohr orbit.