NON-INVARIANCE GROUPS OF DYNAMICAL PROBLEMS

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RESUMEN

Recientemente ha habido gran interés en los grupos de no-invariancia para sistemas dinámicos, fundamentalmente con la intención de generalizar estas ideas al campo de las partículas elementales. En éste trabajo vamos a analizar la construcción de grupos de no-invariancia en forma directa haciendo uso de las variables dinámicas básicas de varios sistemas físicos como el rotor rígido y el oscilador armónico.

La técnica general consiste en construir operadores que dependan solo de las variables dinámicas del problema y que satisfagan el álgebra de Lie de algún grupo. Este grupo así construido debe tener las siguientes características: contener como subgrupo del grupo de simetrías del sistema y sus operadores de Casimir ser tales que todas las eigenfunciones del hamiltoniano físico pertenezcan a una misma representación del grupo mayor. Esto nos permitirá construir todos los posibles estados del sistema con solo conocer los estados de máximo peso mediante una técnica general.

ABSTRACT

There has been recently a great deal of interest in non-invariance groups of dynamical systems mainly with the purpose of generalizing the ideas to the field of elementary particles. We want to discuss here the direct construction of non-invariance groups, using the dynamical variables of several physical systems like the rigid rotator and the harmonic oscillator.

The general technique consists in building up operators that depend only on the dynamical variables of the problem and that satisfy some group's Lie algebra. Such a group must have the following characteristics: it must contain as a sub-group, the symmetry group of the system and its Casimir operators be such that all the Hamiltonian's eigenfunctions belong to a single representation of the larger group. This will allow us to construct all possible states of the system by the mere knowledge of the maximum weight states by the use of a general technique.

INTRODUCTION

In the present work we will try to show the most relevant aspects of non-invariance dynamical groups[1] that may provide us with a new technique in the treatment of dynamical systems. To illustrate this let us start with some very simple examples.

First let us take the one-dimensional harmonic oscillator, whose Hamiltonian is:

$$H = \frac{1}{2} (p^2 + x^2) = \frac{1}{2} (a^+ a + a a^+)$$

with solutions

$$\frac{(a^+)^n}{\sqrt{n!}} | 0 > \qquad (1)$$

As is well known[2] the invariance group of this Hamiltonian is the unitary group in one-dimension SU(1). But let us take the following viewpoint; we shall use the dynamical variables of the problem either coordinate and momentum (x, p) or alternatively the creation and annihilation operators $\left(a^+ = \frac{1}{\sqrt{2}}(x - ip)\right.$; $\left.a = \frac{1}{\sqrt{2}}(x + ip)\right)$ to build new operators that generate the Lie algebra of a group that contains SU(1) as a subgroup. Such operators are[3]

$$I_- = -\frac{a^+ a^+}{2}, \quad I_0 = -\frac{1}{4}(a^+ a + a a^+); \quad I_+ = \frac{a a}{2} \qquad (2)$$

These form a Lie algebra, in fact they are the generators of a group 0(2,1) as Lipkin[4] has shown. We shall call this a dynamical group on account of the fact that we build it from the dynamical variables and a non-invariance group because not all of its generators commute with H. In spite of this we can derive useful information of the system from this group. We construct the Casimir operator of 0(2,1) and we see from (2) that it is equal to a constant:

$$I^2 = I_- I_+ + I_0(I_0 + 1) = -3/16 \qquad (3)$$

This implies that all the eigenfunctions of the problem are eigenfunctions of I^2 with same eigenvalue, that is, all the Hamiltonian's solutions belong to the same irreducible representation of 0(2,1). We can then build all the possible states the problem from the maximum weight state by applying to it the weight lowering operator I_-. In our example the two states $|0\rangle$ and $a^+|0\rangle$ are highest weight states because:

$$I_+|0\rangle = 0; \quad I_+ a^+|0\rangle = 0 \qquad (4a)$$

Their respective weights are

$$I_0 \,|\, 0 > \; = - 1/4 \,|\, 0 > \qquad I_0 \, a^+ |\, 0 > \; = - 3/4 \, a^+ |\, 0 > \tag{4b}$$

Yet they belong to the same eigenvalue of the Casimir operator of $0(2,1)$.

This is similar to what happens when we go from $O(2) \supset O^+(2)$, the Casimir operator of $O(2)$ is L_z^2 but L_z itself is invariant before $O^+(2)$ and not so in the larger group so by using L_z^2 as the Casimir operator of $O^+(2)$ we get two highest weight states with the same eigenvalue m^2 of L_z^2.

But the relevant point is that all the even eigenfunctions can be obtained by applying powers of I_- to the state $|\, 0 >$ and all the odd states are obtained by applying I_-^k to $a^+ |\, 0 >$.

We can also construct the matrix elements of the dynamical group generators once we know the representation eigenfunctions, in fact we have

$$< n' \,|\, I_+ \,|\, n > \; = - \frac{1}{2} \cdot \sqrt{n(n+1)} \; \delta_{n',\, n-2}$$

$$< n' \,|\, I_- \,|\, n > \; = - \frac{1}{2} \sqrt{(n+1)(n+2)} \; \delta_{n',\, n+2} \tag{5}$$

$$< n' \,|\, I_0 \,|\, n > \; = - \frac{1}{4} \, (2n+1) \, \delta_{n',\, n}$$

This serves to illustrate our main interest in non-invariance dynamical groups as an extension of the symmetry group of a physical system. The eigen-solutions belong to many irreducible representations of the symmetry group (usually an infinite number of them). We construct the larger (non-invariance) group in such a way as to have all the solutions as eigenfunctions of a single I.R. of this group.

We note that this implies that the dynamical group is not compact and that all its Casimir operators must be constants.

Let us analyse another simple example, the point rotator in a plane, that is a particle constrained to move in a circle. Now the symmetry group is $O^+(2)$ and we want to extend it to a larger non-invariance group. As we have only two dynamical variables, the rotation angle φ and its conjugate momentum we want our generators to depend only on them.

We proceed as follows: one natural way of extending $O^+(2)$ would be to take the x and y components of the angular momentum vector in three dimensions: (L_x and L_y); eliminate their dependence on the second angle θ and see if they, together with the generator of $O^+(2)$ L_z will form a Lie algebra. These operators:

$$L_x = i \cos \varphi \frac{\partial}{\partial\varphi}; \quad L_y = i \sin \varphi \frac{\partial}{\partial\varphi}; \quad L_z = \frac{1}{i} \frac{\partial}{\partial\varphi} \tag{6}$$

do form a Lie algebra. However the first two do not have a defined hermiticity. That is, while their commutation rules correspond to the complex extension of an $O^+(3)$ Lie algebra, they are not the generators of an $O^{+(3)}$ group. In fact if we want to get an extension of the symmetry group we must build with these another three operators that are hermitian, by symmetrization. These turn out to be

$$p_x \equiv \frac{1}{2i}(L_y^+ + L_y) = \frac{1}{2} \cos \varphi; \quad p_y \equiv \frac{1}{2i}(L_x^+ + L_x) = \frac{1}{2} \sin \varphi; \quad L_z = \frac{1}{i} \cdot \frac{\partial}{\partial\varphi} \tag{7}$$

And they are the generators of an Euclidean group in two dimensions $E(2)$:

$$[p_x, p_y] = 0; \quad [L, p_x] = i\, p_y; \quad [L, p_y] = -i\, p_x \tag{8}$$

The Casimir operator of $E(2)$ is:

$$p_x^2 + p_y^2 = 8/4 , \tag{9}$$

and again all the problem's solutions belong to the same I.R. of the larger group so that starting from any state we can obtain all the rest by just applying the $E(2)$ generators. In this case we also have the matrix elements of the weight raising ($L_+ = p_x + i\, p_y$) weight lowering ($L_- = p_x - i\, p_y$) and weight operators ($L_0 = L_z$)

$$\langle m' | L_0 | m \rangle = m\, \delta_{m',\, m}$$

$$\langle m' | L_+ | m \rangle = -\, m\, \delta_{m',\, m+1} \qquad (10)$$

$$\langle m' | L_- | m \rangle = m\, \delta_{m',\, m-1}$$

The process we followed above is not unique. We could have tried to extend the invariance group in the following manner, let us try to build operators

$$C_{ij} = x_i \frac{\partial}{\partial x_j} - i,\ j = 1,2 \qquad (11)$$

that is the usual generators of a unimodular unitary group SU(2) but eliminating their dependence on $r = \sqrt{x^2 + y^2}$ which is not a dynamical variable. We get the operators:

$$C_{11} = -\sin\varphi \cos\varphi \frac{\partial}{\partial\varphi}\ ; \quad C_{12} = \cos^2\varphi \frac{\partial}{\partial\varphi}$$

$$C_{22} = \sin\varphi \cos\varphi \frac{\partial}{\partial\varphi}\ ; \quad C_{21} = -\sin^2\varphi \frac{\partial}{\partial\varphi} \qquad (12)$$

They form a Lie algebra but again they are not hermitian, so we construct three hermitian operators from them (note that only 3 C_{ij} are independent) and they form a Lie algebra of a non-invariance group

$$\mathfrak{p}_x = -\cos 2\varphi \; ; \qquad \mathfrak{p}_y = -\sin 2\varphi \; ; \qquad \mathfrak{L} = \frac{\partial}{\partial\varphi}$$

In the simple two-dimensional case this group is again $E(2)$ but in general this second procedure gives a different dynamical group that is known as a double-jump in contrast with the single-jump group obtained in the direct fashion mentioned earlier.

These examples illustrate what we expect from this technique of building from the dynamical variables the generators of a non-invariance dynamical group, that while not leaving the Hamiltonian invariant nevertheless permits us to solve the problem just from the knowledge of the particular representation of the larger group we are working with, and from the determination of one of the eigenstates we can construct all the other states directly as we have explicitly shown in the above examples.

We will summarize here our results of the corresponding three-dimensional cases of these problems.

The three-dimensional point rotator has as its symmetry group $O^+(3)$, the motion is described using two rotation angles θ and φ their conjugate momenta giving four dynamical variables. We use them to build the generators of a non-invariance group. These are:

$$L_x = \frac{1}{i}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) ; \quad L_y = \frac{1}{i}\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) ; \quad L_z = \frac{1}{i}\frac{\partial}{\partial\varphi}$$

$$p_x = \frac{x}{r} \; ; \quad p_y = \frac{y}{r} \; ; \quad p_z = \frac{z}{r} \tag{1}$$

The first three are the generators of $O^+(3)$ and together with p_x, p_y and p_z they generate Lie algebra of an $E(3)$ group. We can also check that the Casimir generators of dynamical groups are constants, for example we have in this case the Casimir operator

$$p_x^2 + p_y^2 + p_z^2 = 1 \qquad (2)$$

This dynamical group will be referred as the single-jump group.

Let us now obtain the double-jump group. We build the operators

$$C_{ij} = x_i \frac{\partial}{\partial x_j} \qquad (3)$$

That depend only on θ, φ; p_θ and p_φ and form a Lie algebra, we use them to build hermitian operators and get the generators of an eight parameter non-compact group. This group is not isomorphic to the single-jump group, but is can be identified. Let us suppose a five dimensional space and take the $E(5)$ group that has 15 generators, ten of them associated to rotations $O^+(5)$ and five to translations T(5). But let us take an $O^+(3)$ subgroup, the one that acts on the space of the representation $\mathfrak{D}^{(2)}(O^+(3))$ this is a five-dimensional space but characterized by only three-parameters. So we can identify our double-jump group as a subgroup of $E(5)$.

Now let us analize the three-dimensional harmonic oscillator whose symmetry group is the unitary group U(3) and has the following dynamical variables:

$$\partial_i^+ = \frac{1}{\sqrt{2}}(x_i - i p_i) \; ; \qquad \partial_i = \frac{1}{\sqrt{2}}(x_i + i p_i) \qquad i = 1,2,3 \qquad (4)$$

Using them we can construct the 21 operators

$$\frac{1}{2}(\partial_i^+ \partial_j - \partial_j \partial_i^+) \; ; \quad \partial_i^+ \partial_j^+ \; ; \quad \partial_i \partial_j \; ; \qquad (5)$$

which form a Lie algebra of a dynamical group of the problem. The first set of nine operators are actually the generators of U(3) and the other two sets of six operators are irreducible tensors with respect to U(3). Of these 21 operators of the larger group nine are weight raising operators , three give the weight and the other nine are lowering operators. The irreducible representations of the non-invariance group are characterized by three numbers (λ_1, λ_2, λ_3) and we see from the form of the three weight operators:

$$\frac{1}{2}(2a_i^+ a_i + 1)$$

That these representations must be the (1/2, 1/2, 1/2) I.R. and the (3/2, 1/2, 1/2) I.R. and that all states with an even number of quanta belong to the first one and the odd states to the other one.

The dynamical group is probably a non-compact version of O(7) or Sp(6), but again we are able to construct all the problem's eigenstates from the knowledge of the representation of the dynamical group and the determination of just one of the eigenstates.

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