THREE-NUCLEON MATRIX ELEMENTS IN TRANSLATION-INVARIANT OSCILLATOR STATES*

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RESUMEN

Se deduce una fórmula cerrada apta para programación de computadora electrónica para los elementos de matriz de un hamiltoniano arbitrario de uno mas dos cuerpos que actúa entre estados traslacionalmente invariantes del oscilador armónico. La fórmula resulta ser una suma de productos de: dos paréntesis de transformación, dos coeficientes de precedencia fraccional de spin-isospín, un coeficiente de 12-j y los elementos de matriz de dos cuerpos. Se hace hincapié sobre el hecho que será útil esta forma de calcular el problema de tres nucleones en la medida que los elementos de matriz de dos cuerpos representen una interacción efectiva verdadera.

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ABSTRACT

A closed formula, suitable for computer programming, is obtained for three-nucleon matrix elements of an arbitrary one-plus-two body hamiltonian operator acting between translation-invariant three-body oscillator states. The formula is a sum of products of: two transformation brackets, two spin-isopin fractional-parentage coefficients, one 12-j symbol and the two-body matrix elements in relative coordinate. It is emphasized that the usefulness of the scheme will depend on the extent to which the two-body *matrix elements represent a true effective interaction.*

I. INTRODUCTION

We consider the translational-invariant A-particle hamiltonian

$$H = \sum_{i=1}^{A} \frac{p_i^2}{2m} - \frac{1}{2Am}\left(\sum_{i=1}^{A} p_i\right)^2 + \sum_{i<j}^{A} v_{ij} \equiv \sum_{i<j}^{A} H_{ij} \qquad (1a)$$

$$H_{ij} \equiv \frac{1}{2Am}(p_i - p_j)^2 + v_{ij} \qquad (1b)$$

where v_{ij} is an arbitrary nucleon-nucleon interaction and the second term on the r.h.s. of (1a) is the center of mass kinetic energy substracted. The A-particle Schrödinger equation

$$H\Psi_\alpha = E_\alpha \Psi_\alpha \qquad (3.a)$$

will be satisfied for state energies E_α and associated translational-invariant eigenfunctions Ψ_α. These may be expanded in some complete set of states as

$$\Psi_\alpha = \sum_{\nu=1}^{\infty} a_\nu^\alpha \,|\, \nu > \qquad (3.b)$$

where $|\nu>$ are functions of appropriately chosen $A-1$ *intrinsic* spatial variables, as well as of A spin and isospin variables. To satisfy the exclusion principle these states $|\nu>$ must in addition be totally antisymmetric under exchange of spatial, spin and isospin variables of any pair of particles. Now, the ***spatial part*** of a possible complete set will of course be the eigenfunctions of A particles interacting by pairs with an oscillator potential of frequency $\omega/\sqrt{A}$, viz.,

$$H^0 = \frac{1}{2Am} \sum_{i<j}^{A} (\mathbf{p}_i - \mathbf{p}_j)^2 + \frac{m\omega^2}{2A} \sum_{i<j}^{A} (\mathbf{r}_i - \mathbf{r}_j)^2 \tag{4a}$$

$$H^0 \,|\nu> = E^0_\nu \,|\nu> \tag{4b}$$

which is a manifestly translational-invariant problem. Defining the dimensionless *Jacobi coordinates*[1]

$$\left.\begin{aligned} \dot{\mathbf{x}}_i &\equiv \frac{1}{b} \frac{1}{\sqrt{i(i+1)}} \left(\sum_{j=1}^{i} \mathbf{r}_j - i\mathbf{r}_{i+1} \right) \\ \dot{\mathbf{p}}_i &\equiv \frac{b}{\hbar} \frac{1}{\sqrt{i(i+1)}} \left(\sum_{j=1}^{i} \mathbf{p}_j - i\mathbf{p}_{i+1} \right) \end{aligned}\right\} \qquad (1 \leqslant i < A-1)$$

$$\dot{\mathbf{x}}_A \equiv \frac{1}{b} \frac{1}{\sqrt{A}} \sum_{j=1}^{A} \mathbf{r}_j \;; \qquad \dot{\mathbf{p}}_A \equiv \frac{b}{\hbar} \frac{1}{\sqrt{A}} \sum_{j=1}^{A} \mathbf{p}_j$$

$$b \equiv \sqrt{\hbar/m\omega} \tag{5}$$

it is easily verified that the A ***cartesian*** coordinate (momentum) vectors are connected with the A ***Jacobi*** coordinate (momentum) vectors by an ***orthogonal*** $A \times A$ matrix. It is then seen that the hamiltonian (4a) can be expressed as

$$H^0 = \frac{1}{2}\hbar\omega \sum_{i=1}^{A-1} \{(\dot{\mathbf{p}}_i)^2 + (\dot{\mathbf{x}}_i)^2\} \tag{6}$$

that is, in terms only of the first $A-1$ (intrinsic) Jacobi coordinate and momentum vectors.

It is convenient to express H^0 in terms of the dimensionless boson creation $\dot{\eta}_i$ and annihilation $\dot{\xi}_i$ operators

$$\dot{\eta}_i \equiv \frac{1}{\sqrt{2}}(\dot{\mathbf{x}}_i - i\dot{\mathbf{p}}_i)\ ; \qquad \dot{\xi}_i \equiv \dot{\eta}_i^+ = \frac{1}{\sqrt{2}}(\dot{\mathbf{x}}_i + i\dot{\mathbf{p}}_i) \tag{7}$$

which obey the usual commutation relations (superindices refer to *components*, indices to different *vectors*)

$$\left[\dot{\eta}_i^k, \dot{\eta}_j^l\right] = \left[\dot{\xi}_i^k, \dot{\xi}_j^l\right] = 0\ ; \quad \left[\dot{\xi}_i^k, \dot{\eta}_j^l\right] = \delta^{kl}\delta_{ij} \tag{8}$$

$$(i, j = 1, 2, \ldots, A-1) \qquad (k, l = x, y, z)$$

so that one has the oscillator hamiltonian (6) becoming

$$H^0 = \left[\sum_{i=1}^{A-1} \dot{\eta}_i \cdot \dot{\xi}_i + \frac{3}{2}(A-1)\right]\hbar\omega \tag{9}$$

As shown explicitly by Moshinsky[2], a normalized eigenstate of this hamiltonian is just

$$\langle \dot{\mathbf{x}}_1, \dot{\mathbf{x}}_2, \ldots, \dot{\mathbf{x}}_{A-1} | \dot{n}_1 \dot{l}_1 \dot{m}_1, \dot{n}_2 \dot{l}_2 \dot{m}_2, \ldots, \dot{n}_{A-1} \dot{l}_{A-1} \dot{m}_{A-1} \rangle$$

$$\equiv \prod_{i=1}^{A-1} A_{\dot{n}_i \dot{l}_i} (\dot{\eta}_i \cdot \dot{\eta}_i)^{\dot{n}_i} \mathcal{Y}_{\dot{l}_i \dot{m}_i}(\dot{\eta}_i) \, |0> \; ; \tag{10}$$

$$A_{nl} \equiv (-)^n \sqrt{4\pi/(2n+2l+1)!!(2n)!!}$$

$$\mathcal{Y}_{lm}(\dot{\eta}) \equiv (\dot{\eta})^l Y_{lm}(\hat{\dot{\eta}}) \; ; \qquad |0> \equiv (b^2\pi)^{-3A/4} \exp\left(-\frac{1}{2b^2} \sum_{i=1}^{A} \dot{x}_i^2 \right)$$

and has associated the eigenvalue

$$\left[\sum_{i=1}^{A-1} (2\dot{n}_i + \dot{l}_i) + \frac{3}{2}(A-1) \right] \hbar\omega . \tag{11}$$

One way of constructing the states $|\nu>$ in expansion (3b), which obey the exclusion principle in *all* fermion variables[2] and are eigenstates of total angular momentum J, is by the sum over products of spatial with spin-isospin functions

$$|\nu> \equiv |n_1 l_1, n_2 l_2, \ldots n_{A-1} l_{A-1}, \Lambda, f; STM_T, \tilde{f}; JM_J>$$

$$= \frac{1}{\sqrt{d_f}} \sum_r (-)^r \left[\Phi^{(x_1, x_2, \ldots, x_{A-1})}_{n_1 l_1, n_2 l_2, \ldots, n_{A-1} l_{A-1}, \Lambda, fr} \quad \Gamma^{(\sigma_1 \tau_1, \sigma_2 \tau_2, \ldots, \sigma_A \tau_A)}_{STM_T, \tilde{f}\tilde{r}} \right]_{JM_J} \tag{12}$$

Here, Φ is the *spatial* function of appropriate (*not* necessarily Jacobi) *intrinsic* variables $\{x_i \mid i = 1,2,\ldots,A-1\}$ and which must have a definite symmetry under permutation of the ***spatial cartesian particle*** variables $\{r_i \mid i = 1,2,\ldots,A\}$, and also definite total intrinsic orbital angular momentum Λ. The spatial symmetry can be designated by the Young partition f of A particles and the Yamanouchi symbol r, according to the usual rules associated with the permutation group[13]. Also, Γ is the spin-isospin function (of the $2A$ variables indicated) and must correspond to spin-isospin permutation symmetry $(\tilde{f}\tilde{r})$ ***conjugate*** to the spatial symmetry (fr) in order to yield total antisymmetry. The sum in Eq. (12) is over possible Yamanouchi symbols associated with a given (f), d_f is the dimensionality of the irreducible representation (f) of the permutation group $S(A)$ and $(-)^r$ the so-called "signature" of a given r-symbol, to be described below.

For the case of $A = 3$ and 4 particles, Moshinsky has shown[2] that the spatial function Φ with the aforementioned properties can be constructed as linear combinations of the Jacobi variable eigenstates (10) if one defines intrinsic boson creation operators η_i related to the Jacobi $\dot{\eta}_i$ by a specific unitary transformation

$$\eta_i = \sum_{j=1}^{A-1} M_{ij}\,\dot{\eta}_j\,; \qquad MM^+ = I = M^+M \tag{13}$$

The elements M_{ij} of the $(A-1)\mathrm{x}(A-1)$ matrix M for $A = 3$ and 4 are given in ref. 2, where explicit rules relevant to the construction of 3- and 4-nucleon states Φ are also found. Under the condition (13) the oscillator hamiltonian (9) if form-invariant, i.e.,

$$\frac{H^0}{\hbar\omega} - \frac{3}{2}(A-1) = \sum_{i=1}^{A-1} \dot{\eta}_i \cdot \dot{\xi}_i = \sum_{i=1}^{A-1} \eta_i \cdot \xi_i \tag{14}$$

so that H^0 will also be an eigenoperator in states (10) but with all "dots" removed, and with eigenvalue given by (11), also with all "dots" removed.

Using expansion (3b), the energy E_α associated with a given normalized eigenfunction Ψ_α of the A-particle Schrödinger Eq. (3a) will be given by the *infinite* sum

$$\sum_{\nu, \bar{\nu} = 1}^{\infty} a_{\nu}^{\alpha^*} a_{\bar{\nu}}^{\alpha} \langle \nu | H | \bar{\nu} \rangle = E_\alpha \, . \tag{15a}$$

Since A-particle states (12) are totally antisymmetric, matrix elements of the symmetric two-body hamiltonian (1) will be[3]

$$\langle \nu | H | \bar{\nu} \rangle = \binom{A}{2} \langle \nu | H_{12} | \bar{\nu} \rangle$$
$$\binom{A}{2} \equiv \frac{1}{2} A(A-1) \tag{15b}$$

so that (15a) becomes

$$\binom{A}{2} \sum_{\nu, \bar{\nu} = 1}^{\infty} a_{\nu}^{\alpha^*} a_{\bar{\nu}}^{\alpha} \langle \nu | H_{12} | \bar{\nu} \rangle = E_\alpha \tag{15c}$$

In practice, of course, the sum must be ***truncated***. However, the convergence of the left-hand member of (15c) to the desired eigenvalues E_α may be very slow but, if the "bare" hamiltonian H_{12} were adequately replaced by an appropriately defined "effective" hamiltonian $\mathcal{H}_{12}$ which included short and long-range two-body correlations, the series with the operator $\mathcal{H}_{12}$ would certainly have a better chance of convergence than that with the "bare" H_{12}. An example of $\mathcal{H}_{12}$ is that which involves the Brueckner two-body reaction matrix. In what follows, however, we shall continue to speak in terms of H_{12}, leaving questions of convergence aside.

The spatial part of H_{12}, from (1b) and (5), depends *only* on the first Jacobi vectors $\vec{x}_1$ and $\vec{p}_1$. If then, the states $|\nu\rangle$ are expressible as linear combinations of states whose spatial part is the eigenstate (10), one can visualize that matrix elements in (15) will be given in terms of the relatively simple two-body matrix elements

$$<nlS'jm_j, T'M_T'|H_{12}|\bar{n}\bar{l}\bar{S}'\bar{j}\bar{m}_j, \bar{T}'M_T'> =$$

$$= \delta_{\bar{S}\bar{S}'}\delta_{\bar{j}j}\delta_{\bar{T}'T'} <nlS'j|H_{12}|\bar{n}\bar{l}S'j> \quad (16)$$

where all *primed* quantities $S'T'M_T'$ refer to total *two* body spin and isospin quantum numbers, and the states used are defined by

$$|nlS'j, T'M_T'> \equiv R_{nl}(r)\, \mathcal{Y}_{lS'j}(\hat{r})\,|T'M_T'>$$

$$\mathcal{Y}^{m_j}_{lS'j}(\hat{r}) \equiv [Y_l(\hat{r})X_{S'}]_{jm_j} \quad (17)$$

with $R_{nl}(r)$ being normalized oscillator radial functions in the relative coordinate $r = \frac{1}{\sqrt{2}}(r_1 - r_2)$. The last step in (16) follows on assuming a charge-independent, parity-conserving and scalar two-body hamiltonian acting in two-body states which, to obey the exclusion principle, must satisfy $l + S' + T' =$ odd. In Appendix I a closed formula is derived for the intrinsic kinetic energy contribution to Eq. (16).

We proceed now to deduce explicit formulae for the coefficients required to calculate the matrix elements in (15) for the three-nucleon problem from matrix elements (16); these "geometrical" coefficients can then be evaluated via standard computor codes.

II. THREE-NUCLEON MATRIX ELEMENTS

To conform more closely with standard notation, we shall relabel the oscillator quantum numbers in eigenstates (10) for $A = 3$ particles by

$$(\dot{n}_1 \dot{l}_1 \dot{n}_2 \dot{l}_2) \rightarrow (nlNL) \tag{18}$$

so that (nl) refer to oscillator states in the first Jacobi vector $\dot{x}_1$ and (NL) to those in the second vector $\dot{x}_2$. We first focus on the spatial states Φ in Eq. (12) for $A = 3$ which have a definite spatial symmetry (fr) and instrinsic orbital angular momentum ΛM. The *normalized* explicit form of Φ which can be deduced from that given by Moshinsky and coworkers[7] is

$$\Phi_{n_1 l_1 n_2 l_2, \Lambda M, fr}(x_1, x_2) =$$

$$= [2(1 + \delta_{n_1 n_2} \delta_{l_1 l_2})]^{-\frac{1}{2}} \sum_{nlNL} (-)^{n + q\lambda + \frac{1}{2}(l+q)} \left[1 + (-)^{l+q}\right] \cdot$$

$$\cdot \langle nlNL\Lambda | n_1 l_1 n_2 l_2 \Lambda \rangle \langle \dot{x}_1 \dot{x}_2 | nlNL, \Lambda M \rangle \;, \tag{19}$$

where λ is such that $2n_1 + l_1 - 2n_2 - l_2 \equiv \lambda \pmod 3$, q is an index defined in the table below:

λ	q	fr	$(-)^r$
0	0	[3] (111)	+1
	1	[111] (321)	+1
1,2	0	[21] (211)	+1
	1	[21] (121)	-1

and where $<nlNL\Lambda | n_1 l_1 n_2 l_2 \Lambda>$ is a standard Moshinsky bracket[8] and the states $<\vec{x}_1 \vec{x}_2 | nlNL, \Lambda M>$ are just

$$<\vec{x}_1 \vec{x}_2 | nlNL, \Lambda M> = [<\vec{x}_1 | nl><\vec{x}_2 | NL>]^{\Lambda}_{M} \quad , \tag{20}$$

$$<\vec{x} | nlm> \equiv R_{nl}(\vec{x})\, Y_{lm}(\hat{\vec{x}}) \quad , \tag{21}$$

with normalized radial functions $R_{nl}(\vec{x})$ as defined in Ref. 2.

Next we consider the three-nucleon normalized spin-isospin functions Γ of definite spin-isospin symmetry $(\tilde{f}\,\tilde{r})$ appearing in Eq. (12). These can be decomposed in terms of a function $|\gamma^2 \tilde{f}' \tilde{r}' S' M_S' T' M_T'>$ in nucleons 1 and 2, and a function $|\gamma \sigma \tau>$ in the third nucleon, via spin-isospin fractional parentage coefficients as defined and evaluated, e.g., by Jahn[9]:

$$\Gamma_{\gamma^3 \tilde{f} \tilde{r} S T M_S M_T} =$$

$$= \sum_{\tilde{f}' S' T'} <\gamma^2 \tilde{f}' S' T'; \gamma | \} \gamma^3 \tilde{f} S T> \, [\,|\gamma^2 \tilde{f}' \tilde{r}' S' T'>|\gamma>]_{S M_S, T M_T} \tag{22}$$

where the symbol γ stands for (1/2, 1/2).

Combining (19) and (22) into Eq. (12), and writing

$$|nlm>|\gamma^2 \tilde{f}' \tilde{r}' S' M_S' T' M_T'> =$$

$$= \sum_{j m_j} <l S' m M_S' | j m_j> | n l S' j m_j, \tilde{f}' \tilde{r}', T' M_T'> \tag{23}$$

one obtains the explicit orthonormalized state

$$|\nu\rangle = |n_1 l_1 n_2 l_2 \Lambda, f; STM_T, \tilde{f}; JM_J\rangle =$$

$$= [2(1+\delta_{n_1 n_2}\delta_{l_1 l_2})]^{-\frac{1}{2}} \sum_{\substack{nlS'jT'M'_T \\ NL\tau}} (d_f)^{-\frac{1}{2}} (-)^r (-)^{n+q\lambda+\frac{1}{2}(l+q)} \cdot$$

$$\cdot [1+(-)^{l+q}] \langle nlNL\Lambda | n_1 l_1 n_2 l_2 \Lambda\rangle \langle \gamma^2 \tilde{f}' S' T'; \gamma |\} \gamma^3 \tilde{f} ST\rangle \cdot \tag{24}$$

$$\cdot \langle T' 1/2\, M'_T \tau | TM_T\rangle \sum_{\substack{MM_S\, mM'_S \\ M_L \sigma m_j}} \langle \Lambda SMM_S | JM_J\rangle \langle lLmM_L | \Lambda M\rangle \cdot$$

$$\cdot \langle lS' mM'_S | jm_j\rangle \langle S' 1/2\, M'_S \sigma | SM_S\rangle \cdot$$

$$\cdot |nlS'jm_j, T'M'_T\rangle |NLM_L\rangle |\gamma\sigma\tau\rangle ,$$

where the "signature" $(-)^r$ is *plus* for Yamanouchi symbols $r = (111), (321), (211)$ and *minus* for $r = (121)$, (cf. Ref. 1). The sums over r and $\tilde{f}'$ are redundant as there is already a sum over S' and T'.

Writing

$$|nlS'jm_j, T'M'_T\rangle |NLM_L\rangle |\gamma\sigma\tau\rangle =$$

$$= \sum_{\substack{\mathfrak{j}\mathfrak{m} \\ J^\star M_J^\star}} \langle L\, 1/2\, M_L\, \sigma | \mathfrak{j}\mathfrak{m} \rangle \langle j \mathfrak{j} m_j \mathfrak{m} | J^\star M_J^\star \rangle \; | nlS' j, T' M_T', \gamma\tau, \mathfrak{j}; J^\star M_J^\star \rangle \; , \qquad (24a)$$

it is easy to see, from the fact that $J^\star M_J^\star = JM_J$, and from Ref. 3, page 122, that the state (24) can also be expressed as (putting $[\alpha] \equiv 2\alpha + 1$)

$$|\nu\rangle = [2(1 + \delta_{n_1 n_2} \delta_{l_1 l_2})]^{-\frac{1}{2}} \sum_{\substack{nlS'T'M_T' \\ NL\tau}} (d_f)^{-\frac{1}{2}} (-)^r (-)^{n + q\lambda + \frac{1}{2}(l+q)} \; .$$

$$\cdot [1 + (-)^{l+q}] \langle nlNL\Lambda | n_1 l_1 n_2 l_2 \Lambda \rangle \langle \gamma^2 \tilde{f}' S' T'; \gamma | \} \gamma^3 \tilde{f} S T \rangle \; .$$

$$\cdot \langle T' 1/2\, M_T' \tau | T M_T \rangle \sqrt{[\Lambda][S]} \; \sum_{j\mathfrak{j}} \sqrt{[j][\mathfrak{j}]} \begin{Bmatrix} l & S' & j \\ L & 1/2 & \mathfrak{j} \\ \Lambda & S & J \end{Bmatrix} \; \cdot \qquad (24b)$$

$$\cdot | nlS' j, T' M_T'; NL, \gamma\tau, \mathfrak{j}; JM_J \rangle \; .$$

The convenience of this expansion is apparent on realizing that the nuclear charge form factor operator for the three-nucleon system can be expressed[2] solely in terms of the 2nd Jacobi vector $\dot{x}_2$ and the isospin projection operator $\hat{t}_{z3}$ associated with the 3rd particle: the operator thus acts only on that portion of (24b) which is characterized by $(NL, \gamma\tau)$, the evaluation of the corresponding matrix elements being trivial, as shown in Ref. 2.

A list of positive parity states corresponding to $2n_1 + l_1 + 2n_2 + l_2 \equiv \mathfrak{n} = 0$ and 2 quanta is given in Table I. The integer λ in the second column is such that[2,7] $2n_1 + l_1 - 2n_2 - l_2 \equiv \lambda$ (mod. 3). For details regarding the construction of such tables, for any number $\mathfrak{n}$ of quanta, the reader is referred to Refs. 2 and 7. The asterisk on the last column merely indicated states with $J^\pi T = 1/2^+ 1/2$

(ground state of He^3). As for a general interaction only $J^{\pi}T$ would be "good" quantum numbers one observes, e.g., that a 0-quanta calculation of the ground state of He^3 would thus involve a 1x1 matrix; a 2-quanta calculation a 9x9 matrix, etc. A calculation in this "$\mathfrak{N}$-quanta scheme" is equivalent to doing, in shell-model language, a non-spurious, particle-hole calculation of arbitrary complexity. In Fig. 1 are illustrated the possible particle-hole configurations included in a 4-quanta calculation.

To calculate the matrix elements in Eq. (15) we take note of (16), as well as independence of (15) with respect to projection M_J, so that using Eqs. (24) and (24a) one has

$$[J]^{-1} \sum_{M_J} \langle \nu | H | \nu \rangle \equiv$$

$$\equiv \langle n_1 l_1 n_2 l_2 \Lambda, f; STM_T, \tilde{f}; J | H | \bar{n}_1 \bar{l}_1 \bar{n}_2 \bar{l}_2 \bar{\Lambda}, \bar{f}; \bar{S}\bar{T} M_T, \tilde{\bar{f}}; J \rangle$$

$$= \frac{A(A-1)}{4} [(1 + \delta_{n_1 n_2} \delta_{l_1 l_2})(1 + \delta_{\bar{n}_1 \bar{n}_2} \delta_{\bar{l}_1 \bar{l}_2}) d_f d_{\bar{f}}]^{-\frac{1}{2}} \cdot$$

$$\cdot \sum_{\substack{n l \bar{n} \bar{l} \\ jNL}} (-)^{n + \bar{n} + q\lambda + \bar{q}\bar{\lambda} + \frac{1}{2}(l + \bar{l} + q + \bar{q})} [1 + (-)^{l+q}][1 + (-)^{\bar{l}+\bar{q}}] \cdot$$

$$\cdot \langle n l N L \Lambda | n_1 l_1 n_2 l_2 \Lambda \rangle \langle \bar{n} \bar{l} N L \bar{\Lambda} | \bar{n}_1 \bar{l}_1 \bar{n}_2 \bar{l}_2 \bar{\Lambda} \rangle \cdot$$

$$\cdot \sum_{S' T'} \langle \gamma^2 \tilde{f}' S' T'; \gamma | \} \gamma^3 \tilde{f} S T \rangle \langle \gamma^2 \tilde{f}' S' T'; \gamma | \} \gamma^3 \tilde{\bar{f}} \bar{S} \bar{T} \rangle \cdot$$

$$\begin{Bmatrix} l & & \Lambda & & S & & S' & \\ & L & & J & & 1/2 & & j \\ \bar{l} & & \bar{\Lambda} & & \bar{S} & & \bar{S}' & \end{Bmatrix} \quad [j]\sqrt{[\Lambda][\bar{\Lambda}][S][\bar{S}]}\,(-)^{S+\bar{S}+J+L+j+\frac{1}{2}}\,\delta_{\bar{S}',S'}\,\cdot$$

$$\cdot <nlS'j\,|\,H_{12}\,|\,\bar{n}\bar{l}S'j> \,, \tag{25}$$

where $[\alpha] \equiv (2\alpha + 1)$ and, as discussed in Appendix 2, the 12-j symbol used here is

$$\begin{Bmatrix} l & & \Lambda & & S & & S' & \\ & L & & J & & 1/2 & & j \\ \bar{l} & & \bar{\Lambda} & & \bar{S} & & \bar{S}' & \end{Bmatrix} \equiv$$

$$\equiv \sum_x [x] \begin{Bmatrix} \Lambda & J & S \\ \bar{S} & x & \bar{\Lambda} \end{Bmatrix} \begin{Bmatrix} l & L & \Lambda \\ \bar{\Lambda} & x & \bar{l} \end{Bmatrix} \begin{Bmatrix} S' & 1/2 & S \\ \bar{S} & x & \bar{S}' \end{Bmatrix} \begin{Bmatrix} l & j & S' \\ \bar{S}' & x & \bar{l} \end{Bmatrix} = \tag{26}$$

$$= \frac{(-)^{S+\bar{S}+J+L+j+\frac{1}{2}}}{[J][j]\sqrt{[\Lambda][\bar{\Lambda}][S][\bar{S}]}} \sum_{(\text{all } m'\text{s})} <\Lambda S M M_S\,|\,JM_J><\bar{\Lambda}\bar{S}\bar{M}\bar{M}_S\,|\,JM_J> \cdot$$

$$\cdot <lLmM_L\,|\,\Lambda M><\bar{l}L\bar{m}M_L\,|\,\bar{\Lambda}\bar{M}><S'\,1/2\,M'_S\,\sigma\,|\,SM_S><\bar{S}'\,1/2\,\bar{M}'_S\,\sigma\,|\,\bar{S}\bar{M}_S> \cdot$$

$$\cdot <lS'mM'_S\,|\,jm_j><\bar{l}\bar{S}'\,\bar{m}\bar{M}'_S\,|\,jm_j>$$

in which $\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}$ are the standard 6-j symbols as defined, e.g., in Ref. 10. The 12-j symbol defined here *differs* from the apparently more common one which involves a phase factor multiplying each summand in (26), the coupling sequence of the 12 parameters also being slightly different. A computor code to evaluate

12-j symbols using expression (26) turns out to be relatively easy to write[12].

In deducing Eq. (25) one notes that since H_{12} conserves parity then $(-)^{l} = (-)^{\bar{l}}$ which implies that $\tilde{f}' = \tilde{\bar{f}}'$ and ultimately that f and $\bar{f}$ must have the same symmetry with respect to particles 1 and 2. Eq. (25) was also deduced starting directly from Eq. (24b).

For the case $\mathfrak{n} = 0$ we may use: a) the identity (Ref. 3, page 132)

$$\begin{Bmatrix} j_1 & j_1' & 0 \\ j_2 & j_2' & j_3 \end{Bmatrix} = \frac{\delta_{j_1 j_1'} \delta_{j_2 j_2'} (-)^{j_1 + j_2 + j_3}}{\sqrt{[j_1][j_2]}}$$

in (26) to evaluate the 12-j symbol, b) the spin-isospin cfp tables of Ref. 9, reproduced in Table II here, and c) the results of Appendix I, to reduce Eq. (25) to the simple result

$$\langle H \rangle_{\mathfrak{n} = 0} = \frac{3}{2} \; [\hbar\omega + \langle n = 0, {}^3s_1 | v_{12} | n = 0, {}^3s_1 \rangle + \langle n = 0, {}^1s_0 | v_{12} | n = 0, {}^1s_0 \rangle] \; . \tag{28}$$

This coincides with the result obtained by calculating the expectation of H between 3-nucleon Slater determinants corresponding to the configuration $(os)^3$, and substracting the center-of-mass kinetic energy evaluated by the virial theorem for the harmonic oscillator.

III. CONCLUSION

A closed formula for the matrix elements of an arbitrary one-plus-two-body hamiltonian between three-nucleon harmonic oscillator states, both hamiltonian and states being translational invariant, has been given as a sum of products of the following recoupling coefficients: two transformation brackets, two spin-isospin fractional parentage coefficients and one 12-j symbol. Once the two-body matrix elements are given, the formula can be computor-coded with relative ease for evaluation. The hope, of course, is that one may be able to represent the three-nucleon problem by a *finite* number of terms in the expansion (3b), i.e., by those corresponding to the *lowest* eigenvalues of Eq. (11). This will be feasible if short-ranged correlations as well as the effect of other particles (e.g., "healing" at large separations) can be adequately built into the *two-body* matrix elements as seems to be the case, e.g., with those deduced by Elliott and co-workers[6] based on a method requiring only knowledge of the nucleon-nucleon phase shifts as functions of scattering energy.

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REFERENCES

1. Kramer, P. and M. Moshinsky, in *"Group Theory and Applications"*, ed. by E.M. Loebl, Academic Press, 1968.
2. Moshinsky, M. *Lectures*, Latin American School of Physics held in Mexico City, 1968. (to be published by Gordon and Breach).
3. C.f., for example, A. de Shalit and I. Talmi, *"Nuclear Shell Theory"* (Academic Press, 1963).
4. Moshinsky, M. and T.A. Brody, Rev.Mex.Fís., 9 (1960) 203.
5. Grillot, D.V. (private communication).
6. Elliott, J.P., A.D. Jackson, H.A. Mavromatis, E.A. Sanderson and B. Singh, Nucl. Phys. (to be published).
7. Aguilera, V.C., (private communication).
8. Brody, T.A. and M. Moshinsky, "Tables of Transformation Brackets" (Gordon and Breach, 1967).
9. Jahn, H.A., Proc.Roy.Soc. (Lond.) A201 (1950) 516.

10. Rose, M.E., *"Elementary Theory of Angular Momentum"* (Wiley, 1957).

11. Ord-Smith, R.J., Phys.Rev. 94 (1954) 1227; Sharp, W.T., Bull.Am.Phys. Soc. (Ser. II) Q (1956) Nº 4; Brody, T.A., J. Flores and P.A. Mello, Rev.Mex.Fís. 17 (1968) 301.

12. Mello, P.A. (private communication).

13. Hammermesh, M. *"Group Theory"* (Addison-Wesley, 1962).

APPENDIX I.

To evaluate the kinetic energy contribution to the two-body matrix element (16), we have from Eq. (1b) that

$$H_{12} = \frac{1}{2Am}(p_1 - p_2)^2 + v_{12} = \frac{\hbar^2}{Amb^2}(\dot{p}_1)^2 + v_{12}$$

with $\dot{p}_1$ the first Jacobi momentum vector as defined in (5). Now, it is evident that

$$< nlm \,|\, \frac{1}{2}(\dot{p}_1)^2 \,|\, n\,l\,m > \; = \; (2n + l + \frac{3}{2})\, \delta_{n\bar{n}}\, \delta_{l\bar{l}}\, \delta_{m\bar{m}}$$

$$- < nlm \,|\, \frac{1}{2}(\dot{x}_1)^2 \,|\, \bar{n}\,\bar{l}\,\bar{m} > .$$

But the last matrix element can be evaluated by standard radial-integral techniques[4]. The final result is just

$$< nlS'j \,|\, \frac{\hbar^2}{Amb^2}(\dot{p}_1)^2 \,|\, \bar{n}\,\bar{l}\,S'j > =$$

$$= \delta_{l\bar{l}}\, \frac{\hbar\omega}{A} \left[\delta_{\bar{n}n}(2n + l + \frac{3}{2}) + \delta_{\bar{n},\, n-1} \sqrt{n\,(n + l + 1/2)} \right.$$

$$\left. + \, \delta_{\bar{n},\, n+1} \sqrt{(n+1)\,(n + l + 3/2)} \right].$$

APPENDIX II.

Our purpose here is to sketch how the identity of expressions (26) and (27) can be established. We begin with the sum

$$\sum_{(\text{all } m'\text{s})} <\Lambda S M M_S | J M_J><\bar{\Lambda}\bar{S}\bar{M}\bar{M}_S | J M_J><l L m M_L | \Lambda M><\bar{l} L \bar{m} M_L | \bar{\Lambda}\bar{M}> \cdot$$

$$\cdot <s' \, 1/2 \, M_S' \, \sigma | S M_S><\bar{s}' \, 1/2 \, \bar{M}_S' \, \sigma | \bar{S}\bar{M}_S><l s' m M_S' | j m_j><\bar{l}\bar{s}' \bar{m}\bar{M}_S' | j m_j>$$

which appears in writing down the left-hand member of (25). Next, use the identity (Ref. 3, page 109)

$$<j_1 j_2 m_1 m_2 | j_3 m_3> = (-)^{j_1 - j_2 + m_3} \sqrt{[j_3]} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & -m_3 \end{pmatrix}$$

and also use the 3-j symbol symmetry properties to express the above sum as a sum (with respect to all m's) over products

$$\begin{pmatrix} \Lambda & S & J \\ -M & -M_S & M_J \end{pmatrix} \begin{pmatrix} \bar{S} & \bar{\Lambda} & J \\ \bar{M}_S & \bar{M} & -M_J \end{pmatrix} \begin{pmatrix} l & \Lambda & L \\ m & -M & M_L \end{pmatrix} \begin{pmatrix} \bar{\Lambda} & \bar{l} & L \\ \bar{M} & -\bar{m} & -M_L \end{pmatrix} \cdot$$

$$\cdot \begin{pmatrix} s' & s & 1/2 \\ M_S' & -M_S & \sigma \end{pmatrix} \begin{pmatrix} \bar{s} & \bar{s}' & 1/2 \\ \bar{M}_S & -\bar{M}_S' & -\sigma \end{pmatrix} \begin{pmatrix} l & s' & j \\ -m & -M_S' & m_j \end{pmatrix} \begin{pmatrix} \bar{s}' & \bar{l} & j \\ \bar{M}_S' & \bar{m} & -m_j \end{pmatrix}$$

Now carry out the sum over the *third* projection quantum number of each *pair* of 3-j's (starting with the first two, etc.) and in each of these four cases apply the relation.

$$\sum_{m_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & j_3 \\ m_1' & m_2' & -m_3 \end{pmatrix} =$$

$$= \sum_{l_3 m_3'} (-)^{l_3 + j_3 + m_1 + m_1'} \begin{pmatrix} j_1 & l_2 & l_3 \\ m_1 & m_2' & m_3' \end{pmatrix} \begin{pmatrix} l_1 & j_2 & l_3 \\ m_1' & m_2 & -m_3' \end{pmatrix} \begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix} (2l_3 + 1)$$

given in Ref. 3, page 131, where the factor $(2l_3 + 1)$ has been omitted! One may then employ the orthogonality relation (Ref. 3, page 110)

$$\sum_{m_1 m_2} \begin{pmatrix} j_1 & j_2 & x \\ m_1 & m_2 & m \end{pmatrix} \begin{pmatrix} j_1 & j_2 & x' \\ m_1 & m_2 & m' \end{pmatrix} = [x]^{-1} \delta_{xx'} \delta_{mm'}$$

for appropriate pairs of 3-j's and, after removing all remaining Kronecker deltas by the corresponding sums, there results a sum over **one** index only of products of four 6-j symbols. Q.E.D.

TABLE I

THREE-NUCLEON STATES FOR $\omega = 0$ AND 2 QUANTA, WHERE $\omega = 2n_1 + l_1 + 2n_2 + l_2$

$n_1 l_1 n_2 l_2$	λ	$[f]$	Λ	S	T	J^+	$nlNL$			$J^\pi T = 1/2^+ 1/2$
0000	0	[3]	0	1/2	1/2	1/2	0000			*
1000	2	[21]	0	1/2	1/2, 3/2	1/2	1000	0101	0010	*
"	"	"	"	3/2	1/2	3/2	"	"	"	
0010	"	"	"	1/2	1/2, 3/2	1/2	"	"	"	*
"	"	"	"	3/2	1/2	3/2	"	"	"	
0101	0	[3]	0	1/2	1/2	1/2	1000	0010	"	*
0101	0	[111]	0	1/2	1/2	1/2	0101			*
"	"	"	"	3/2	3/2	3/2	0101			
"	"	[111]	1	1/2	1/2	1/2, 3/2	0101			*
"	"	"	"	3/2	3/2	1/2, 3/2, 5/2	"			
"	"	"	2	1/2	1/2	3/2, 5/2	0101			
"	"	"	"	3/2	3/2	1/2, 3/2, 5/2, 7/2	0101			*
"	"	[3]	2	1/2	1/2	3/2, 5/2	0200	0002		
0200	2	[21]	2	1/2	1/2, 3/2	3/2, 5/2	0200	0101	0002	
"	"	"	"	3/2	1/2	1/2, 3/2, 5/2, 7/2	"	"	"	*
0002	"	"	"	1/2	1/2, 3/2	3/2, 5/2	"	"	"	
"	"	"	"	3/2	1/2	1/2, 3/2, 5/2, 7/2	"	"	"	*

TABLE II.

ONE-PARTICLE SPIN-ISOSPIN *fpc* FOR THE THREE-NUCLEON SYSTEM

$$<\gamma^2 \tilde{f}' S' T', \gamma | \} \gamma^3 \tilde{f} S T>$$

$\tilde{f}$		{111}	{21}	{21}	{21}	{3}	{3}
$\tilde{f}'$	$S'T'$ \ ST	1/2 1/2	1/2 1/2	3/2 1/2	1/2 3/2	1/2 1/2	3/2 3/2
{11}	10	$-1/\sqrt{2}$	$1/\sqrt{2}$	1			
{11}	01	$1/\sqrt{2}$	$1/\sqrt{2}$		1		
{2}	00		$1/\sqrt{2}$			$1/\sqrt{2}$	
{2}	11		$-1/\sqrt{2}$	1	1	$1/\sqrt{2}$	1

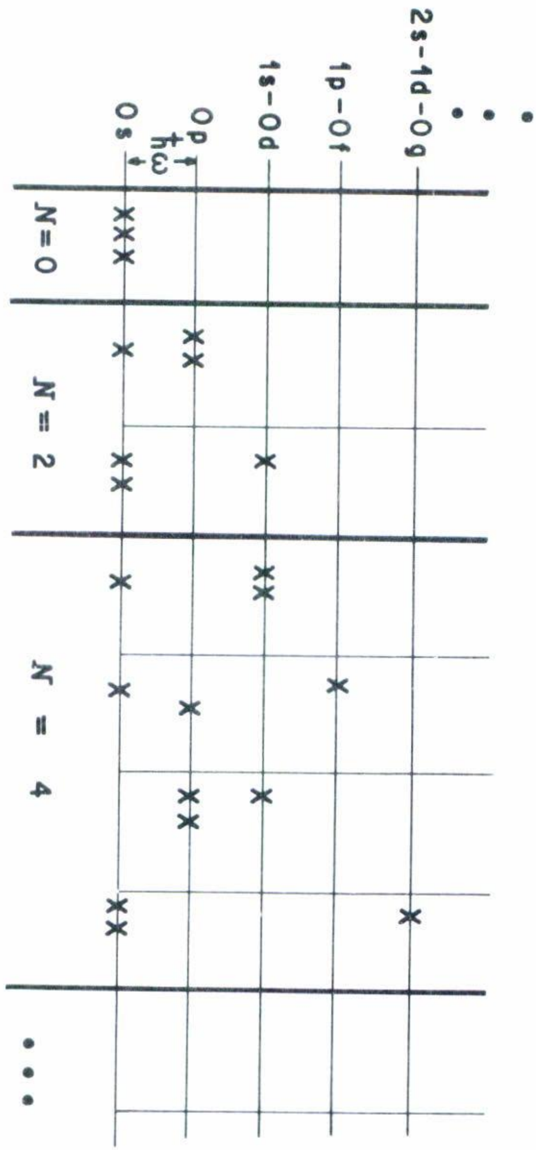

Fig. 1. Correspondence of "η-quanta-scheme" with shell model "particle-hole" configurations (without spurious effects) for η = 0, 2 and 4 quanta (i.e., positive parity states only).