# ON THE LADDER REPRESENTATIONS OF THE O(4,2) GROUP.

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(Recibido: 13 de mayo 1969)

*RESUMEN*

*Se hace un estudio de las representaciones escalera del grupo O(4,2) mediante métodos algebráicos, utilizando la cadena $O(4,2) \supset O(4,1) \supset O(4)$; se calculan explícitamente los elementos de matriz de los generadores infinitesimales.*

*ABSTRACT*

*A study of the ladder representations of the O(4,2) group is done, using algebraic methods, for the chain $O(4,2) \supset O(4,1) \supset O(4)$, and the matrix elements of the infinitesimal generators between ladder-states are explicitly calculated.*

## INTRODUCTION

Recently considerable attention has been devoted to the O(4,2) group in connection with the Majorana-type equations, as a theoretical scheme for describing, in an unified way, particles and resonances of fixed unitary-spin structure[1]. Further, this group has been shown to be the dynamical group for a charged particle in a Coulomb field[2]. In most of the cases of physical interest, one has to do with a special class of representations, the so-called ladder representations, which are multiplicity-free, unitary and irreducible representations (UIR). Although these representations have been treated before in the literature[3], it seems to be interesting for the applications to have them derived and exhibited explicitly, as far as the matrix elements of the group generators between ladder states are concerned. This has been done in this work, using algebraic methods, in the case of the decomposition chain O(4,2) $\supset$ O(4,1) $\supset$ O(4). The ladder representations decompose, with respect to the O(4) subgroup, into an infinite sequence of O(4) multiplets, characterized by $(j_0, n)$, these two labels describing its spin content (the $j_0$ label gives the minimum spin of the O(4) multiplet). The case $j_0 = 0$ is the one realized in the H atom[2] and the ladders with $j_0 \neq 0$ are of interest in hadron physics[4].

Section 1 is devoted to definitions and notations. In Section 2, the matrix elements of the group generators are deduced. Finally, in Section 3, a discussion is devoted to representation relations[5] which characterize the ladder representations.

## 1. ALGEBRAIC PRELIMINARIES

The non-compact O(4,2) Lie algebra is defined by

$$[S_{AB}, S_{CD}] = -i(g_{AC}S_{BD} + g_{BD}S_{AC} - g_{AD}S_{BC} - g_{BC}S_{AD}) \tag{1.1}$$

where the capital Latin indices assume the values 0, 1, 2,3,4, 5 and the signature of the metric tensor $g_{AB}$ is $(+ - - - - +)$.

Since $S_{AB} = - S_{BA}$, the number of independent generators is fifteen. They may be written as:

$$
\begin{aligned}
M_i &= \frac{1}{2} \Sigma_{ijk} S_{jk} && (i,j,k = 1,2,3) \\
N_i &= S_{4i} \\
A_i &= S_{0i} \\
B &= S_{40} \\
\Gamma_a &= S_{a5} && (a = 0,1,2,3,4)
\end{aligned}
\tag{1.2}
$$

From (1.1) and (1.2) one sees that the $M_i$ are angular momentum operators. The $N_i$ is the Lenz vector. $A_i$ are the Lorentz boosters. $B$ is the so-called tilt operator and finally, since

$$
[S_{ab}, \Gamma_c] = - i (g_{ac} \Gamma_b - g_{bc} \Gamma_a) \tag{1.3}
$$

the $\Gamma_a$ is a vector operator w.r.t. the O(4,1) group generated by the $S_{ab}$.

Incidentally, one notes that the $M_i$ and $N_i$ generate a O(4) sub-algebra and that the $M_i$ and $A_i$ generate a O(3,1) Lorentz sub-algebra of O(4,2).

Since O(4,2) has rank 3 its irreducible representations are characterized by 3 labels associated to three Casimir invariants. Unitary representations satisfy the hermiticity condition $S_{AB} = S^+_{AB}$ in the representation space. In the chain

$$
O(4,2) \supset O(4,1) \supset O(4) \supset O(3), \tag{1.4}
$$

the so-called canonical chain, a basis for an irreducible representation can be written as

$$|\tau, \tau', (j_0 n)\, jm > \tag{1.5}$$

where $\tau$ and $\tau'$ denote, collectively, the indices associated with the O(4,2) and its O(4,1) subgroup, and the labels $(j_0, n)$ and $(j, m)$ are the O(4) and O(3) ones. As is known and as will become clear in the following, there exist particular classes of UIR of the O(4,2) group - the ladder representations - which are multiplicity-free under (1.4), that is, they are irreducible w.r.t O(4,1) and decompose w.r.t O(4) in such a way that each IR of O(4) appears once and only once. So, for the ladder representations, the $\tau'$ indices are determined by the $\tau$ alone and for brevity of notation, the basis associated with the ladder representation may simply be denoted by

$$|(j_0, n)\, jm > \quad .$$

## 2. MATRIX-ELEMENTS OF THE O(4,2) GENERATORS

The application of the O(4,2) compact generators to the basic states $|(j_0 n)\, jm>$ is well-known from the theory of the UIR of the O(4) group[6] and its derivation will be omitted. We simply, for completeness, give the results:

$$<(j_0 n)\, j(m+1)\, |M_+|\, (j_0 n)\, jm> = [(j+m+1)(j-m)]^{\frac{1}{2}}$$

$$<(j_0 n)\, j(m-1)\, |M_-|\, (j_0 n)\, jm> = [(j-m+1)(j+m)]^{\frac{1}{2}}$$

$$<(j_0 n)\, jm\, |M_3|\, (j_0 n)\, jm> \quad = m \tag{2.1}$$

$$\langle (j_0 n)\, j' m \,|\, N_+ \,|\, (j_0 n)\, jm \rangle = [(j-m)(j-m-1)]^{\frac{1}{2}}\, C_j^{(j_0\, n)}\, \delta_{j',\, j-1}$$

$$+ [(j-m)(j+m+1)]^{\frac{1}{2}}\, A_j^{(j_0,\, n)}\, \delta_{j',\, j}$$

$$- [(j+m+1)(j+m+2)]^{\frac{1}{2}}\, C_{j+1}^{(j_0,\, n)}\, \delta_{j',\, j+1}$$

$$\langle (j_0 n)\, j' m \,|\, N_- \,|\, (j_0 n)\, jm \rangle = -[(j+m)(j+m-1)]^{\frac{1}{2}}\, C_j^{(j_0,\, n)}\, \delta_{j'\, j-1}$$

$$+ [(j+m)(j-m+1)]^{\frac{1}{2}}\, A_j^{(j_0,\, n)}\, \delta_{j',\, j}$$

$$+ [(j-m+1)(j-m+2)]^{\frac{1}{2}}\, C_{j+1}^{(j_0,\, n)}\, \delta_{j',\, j+1}$$

where $M_+ = M_1 \pm i M_2$ , $N_\pm = N_1 \pm i N_2$ and

$$A_j^{(j_0,\, n)} = \frac{j_0 n}{j(j+1)}\,, \quad C_j^{(j_0,\, n)} = \frac{i}{j}\left[\frac{(j^2 - j_0^2)(j^2 - n^2)}{4j^2 - 1}\right]^{\frac{1}{2}} .$$

The matrix elements of the remaining O(4,2) generators (1.2) are derived by a judicious choice of commutators (1.1).

In fact, one has from (1.1)

$$[B, M_i] = 0$$

$$[[N_3, B], N_3] = -B \tag{2.2}$$

which allows, by standard algebraic procedures[6], to derive the matrix elements of $B$, from (2.1) alone:

$$<(j_0 n')\, j' m' \,|\, B \,|\, (j_0 n)\, jm> \;=\; \frac{1}{2}\Big( [(n-j)(n+j+1)]^{\frac{1}{2}}\, \delta_{n',\, n+1}$$

$$+\, [(n-j)(n-j-1)]^{\frac{1}{2}}\, \delta_{n',\, n-1}\Big)\, \delta_{jj'}\, \delta_{mm'} \qquad (2.3)$$

The matrix-elements of the Lorentz-boosters are then easily determined from (2.3) and (2.1) by means of

$$[B,\, N_i] = iA_i \qquad (2.4)$$

One gets, for the non-vanishing ones,

$$<(j_0, n+1)\, j-1, m \,|\, A_3 \,|\, (j_0 n)\, jm>$$

$$= -\frac{i}{2j}\left[\frac{(j^2 - j_0^2)(j^2 - m^2)(n-j)(n-j+1)}{4j^2 - 1}\right]^{\frac{1}{2}}$$

$$<(j_0, n+1)\, jm \,|\, A_3 \,|\, (j_0 n)\, jm> \;=\; \frac{i}{2j}\,\frac{j_0}{j+1}\; m\, [(n-j)(n+j+1)]^{\frac{1}{2}}$$

$$(2.5)$$

$$<(j_0, n+1)\, j+1, m \,|\, A_3 \,|\, (j_0 n)\, jm> \;=$$

$$= -\frac{i}{2(j+1)}\left[\frac{[(j+1)^2 - j_0^2]\,[(j+1)^2 - m^2]\,(n+j+1)(n+j+2)}{4(j+1)^2 - 1}\right]^{\frac{1}{2}}$$

$$<(j_0, n-1)\, j-1, m \,|\, A_3 \,|\, (j_0 n)\, jm>$$

$$= \frac{i}{2j} \left[ \frac{(j^2 - j_0^2)(j^2 - m^2)(n+j)(n+j-1)}{4j^2 - 1} \right]^{\frac{1}{2}}$$

$$<(j_0, n-1)\, jm \,|\, A_3 \,|\, (j_0 n)\, jm>$$

$$= - \frac{i}{2j} \frac{j_0}{j+1} m \left[ (n-j-1)(n+j) \right]^{\frac{1}{2}} \tag{2.5}$$

$$<(j_0, n-1)\, j+1, m \,|\, A_3 \,|\, (j_0 n)\, jm> =$$

$$= \frac{i}{2(j+1)} \left[ \frac{[(j+1)^2 - j_0^2]\,[(j+1)^2 - m^2]\,(n-j-2)(n-j-1)}{4(j+1)^2 - 1} \right]^{\frac{1}{2}}$$

For $A_\pm = A_1 \pm i A_2$, one gets $(A_- = A_+^\dagger)$:

$$<(j_0, n-1)\, j-1, m+1 \,|\, A_+ \,|\, (j_0 n)\, jm> =$$

$$= \frac{i}{2j} \left[ \frac{(j^2 - j_0^2)(j-m)(j-m-1)(n+j)(n+j-1)}{4j^2 - 1} \right]^{\frac{1}{2}}$$

$$<(j_0, n-1)\, j, m+1 \,|\, A_+ \,|\, (j_0, n)\, jm> \tag{2.6}$$

$$= - \frac{i}{2j} \frac{j_0}{j+1} \left[ (j-m)(j+m+1)(n+j)(n-j-1) \right]^{\frac{1}{2}}$$

$$<(j_0, n-1)\, j+1, m+1 \,|A_+|\, (j_0 n)\, jm>$$

$$= \frac{i}{2(j+1)} \left[ \frac{[(j+1)^2 - j_0^2]\,(j+m+1)(j+m+2)(n-j-1)(n-j-2)}{4(j+1)^2 - 1} \right]^{\frac{1}{2}}$$

$$<(j_0, n+1)\, j-1, m+1 \,|A_+|\, (j_0 n)\, jm>$$

$$= -\frac{i}{2j} \left[ \frac{(j^2 - j_0^2)(j-m)(j-m-1)(n-j)(n-j+1)}{4j^2 - 1} \right]^{\frac{1}{2}}$$

$$<(j_0, n+1)\, j, m+1 \,|A_+|\, (j_0 n)\, jm> \tag{2.6}$$

$$= \frac{i}{2j} \frac{j_0}{j+1} \left[ (j-m)(j+m+1)(n-j)(n+j+1) \right]^{\frac{1}{2}}$$

$$<(j_0, n+1)\, j+1, m+1 \,|A_+|\, (j_0 n)\, jm> =$$

$$= -\frac{i}{2(j+1)} \left[ \frac{[(j+1)^2 - j_0^2]\,(j+m+1)(j+m+2)(n+j+1)(n+j+2)}{4(j+1)^2 - 1} \right]^{\frac{1}{2}}$$

The remaining task is the determination of the $\Gamma_a$ generators. For this, we note that $\Gamma_0$ is an O(4)-scalar since

$$[\Gamma_0, M_i] = [\Gamma_0, N_i] = 0 \ . \tag{2.7}$$

Therefore, the matrix element of $\Gamma_0$ is of the form

$$< (j_0 n') \, j' m' \left| \Gamma_0 \right| (j_0 n) \, jm > = f_{j_0}(n) \, \delta_{nn'} \, \delta_{jj'} \, \delta_{mm'} \tag{2.8}$$

The function $f_{j_0}(n)$ can be most easily determined from the commutator

$$[B, [\Gamma_0, B]] = \Gamma_0 \tag{2.9}$$

from which, making use of (2.3) and (2.8), the following relations can be derived:

$$2f_{j_0}(n) - f_{j_0}(n-1) - f_{j_0}(n+1) = 0$$

$$(n-j)(n+j+1)\big(f_{j_0}(n+1) - f_{j_0}(n)\big) + (n+j)(n-j-1)\big(f_{j_0}(n-1) - f_{j_0}(n)\big) = 2f_{j_0}(n) \tag{2.10}$$

These relations admit the solution

$$f_{j_0}(n) = \pm \, n \tag{2.11}$$

The determination of the matrix elements of $\Gamma_4$ and $\Gamma_i$ is then derived from the commutators

$$[B, \Gamma_0] = i\Gamma_4 \tag{2.12}$$

and

$$[A_i, \Gamma_0] = -i\Gamma_i \tag{2.13}$$

with the results

$$<(j_0 n')\, j'm' \,|\, \Gamma_{\pm \text{ or } 3} \,|\, (j_0 n)\, jm> \; = \; \pm\, i\,(n-n') <(j_0 n')\, j'm' \,|\, A_{\pm \text{ or } 3} \,|\, (j_0 n)\, jm>$$

$$<(j_0 n')\, j'm' \,|\, \Gamma_4 \,|\, (j_0 n)\, jm> \; = \mp \frac{i}{2}\,(n-n') <(j_0 n')\, j'm' \,|\, B \,|\, (j_0 n)\, jm> \qquad (2.14)$$

The relations (2.14) complete the determination of the matrix elements of the O(4,2) generators.

From the representation theory of the O(4) group, one knows that

$$j_0 \leq j \leq n-1 \qquad (2.15)$$

and consequently, $n$ is an integer (half-integer) for $j_0$ integer (half-integer).

One recovers, in this way, that the ladder representations, w.r.t the O(4) subgroup, into an infinite sequence of O(4) unitary multiplets, characterized by $n = j_0 + 1,\ j_0 + 2,\ j_0 + 3, \ldots$, exhibiting the form of an H atom spectrum, the actual H-atom case corresponding to $j_0 = 0$.

## 3. REPRESENTATION RELATIONS FOR THE LADDERS

Before entering into the subject of this section, we recall that the ladder representations are characterized by a second-order Casimir invariant

$$Q(\mathrm{O}(4,2)) = \Gamma_0^2 + \mathbf{M}^2 + \mathbf{N}^2 - 2\,(\mathbf{A}^2 + B^2) \qquad (3.1)$$

One can also consider the Casimir operator of the O(4.1) subgroup

$$Q(\mathrm{O}(4,1)) = \mathbf{M}^2 + \mathbf{N}^2 - (\mathbf{A}^2 + B^2) \qquad (3.2)$$

The values of the Casimir operators (3.1) and (3.2) have been first derived by Olzewski in his treatment of the SU(2.2) group, the covering group of O(4,2). From the matrix elements of Section 2, one gets

$$<(j_0 n)\, jm \left| \mathbf{A}^2 + B^2 \right| (j_0 n)\, jm> \; = n^2 + 1 - j_0^2 \tag{3.3}$$

from which, using the well-known value of $\mathbf{M}^2 + \mathbf{N}^2 = j_0^2 + n^2 - 1$, one gets

$$\begin{aligned} Q(\mathrm{O}(4,2)) &= -3(1 - j_0^2) \\ Q(\mathrm{O}(4,1)) &= -2(1 - j_0^2) \end{aligned} \tag{3.4}$$

from which one sees that the ladders of O(4,2) are also UIR of the O(4,1) subgroup.

As an alternative, the ladders may be characterized by certain representation relations, involving anticommutators of the group generators, as discussed by A. Böhm[5] for the H-atom ladders. In the case $j_0 \neq 0$, these relations are

$$\{S_{AB}, S^{A}_{\bullet C}\} = -2g_{BC}(1 - j_0^2) \tag{3.5}$$

where $S^{A}_{\bullet C} = g^{AD} S_{DC}$. Multiplying (3.5) by $g^{CB}$ and summing in the $B$ and $C$ indices, one easily gets

$$Q(\mathrm{O}(4,2)) = \frac{1}{2} S_{AB} S^{AB} = -3(1 - j_0^2) \tag{3.6}$$

in agreement with (3.4).

On the other hand, it follows from (3.5) that

$$\{\Gamma_b, \Gamma_c\} + \{S_{ab}, S^{a}_{\bullet c}\} = -2g_{bc}(1 - j_0^2) \tag{3.7}$$

Multiplying (3.7) by $g^{cb}$ and summing with respect to $b$ and $c$, one derives

$$\{S_{ab}, S^{ab}\} = -10(1-j_0^2) - 2\Gamma_a\Gamma^a = 4Q(O(4,1)) \qquad (3.8)$$

Now, (3.6) can be rewritten as

$$S_{AB}S^{AB} = 2Q(O(4,1)) + 2\Gamma_a\Gamma^a = -6(1-j_0^2) \qquad (3.9)$$

From (3.8) and (3.9) one immediately recovers the value of $Q(O(4,1))$ given by (3.4) and the relation

$$\Gamma_a\Gamma^a = -(1-j_0^2) \qquad (3.10)$$

Finally, it may be noted that from (3.10) and the second (3.4) relation it follows that $\Gamma_0^2 = n^2$, in agreement with the result (2.11) of the previous Section.

## ACKNOWLEDGEMENTS

The author is grateful to Professors Abdus Salam and Paolo Budini and to the IAEA for the hospitality kindly extended to him at the ICTP, Trieste, where this work was performed. He is indebted to Dr. S.R. Komy for useful discussions in the first stage of this work. Thanks are also due to Professor A.O. Barut for stimulating discussions.

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