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A NEW FORMULATION OF STOCHASTIC THEORY AND QUANTUM MECHANICS
III. LAGRANGIAN FORMULATION AND ELECTROMAGNETIC COUPLING

L. de la Peña-Auerbach*
Instituto de Física, Universidad Nacional de México
Ana M. Cetto**
Facultad de Ciencias, Universidad Nacional de México
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RESUMEN

Los propósitos fundamentales de este trabajo son los siguientes:

a) Demostrar que, a partir de un principio de D'Alembert generalizado, es posible expresar en forma lagrangiana las ecuaciones fundamentales de una teoría estocástica presentada recientemente por uno de los autores.

b) A partir de estas ecuaciones de Lagrange generalizadas, obtener una expresión para la fuerza electromagnética que actúa sobre una partícula estocástica en presencia de un campo electromagnético externo: sumados a la fuerza de

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* Consultant, Comisión Nacional de Energía Nuclear, México.
** Fellow of the Ford Foundation, Facultad de Ciencias, UNAM, México.

Lorentz, aparecen nuevos términos que se reducen a cero en el límite clásico.

c) Bajo la aproximación usual de la mecánica cuántica, con esta fuerza substituída en las ecuaciones estocásticas, deducir la ecuación de Schrödinger con acoplamiento electromagnético minimal.

ABSTRACT

The purpose of this note is threefold:

a) To show that, starting from a generalized D'Alembert principle, it is possible to express in Lagrangian form the basic equations of a stochastic theory recently proposed by one of us.

b) To obtain, using these generalized Lagrange equations, an expression for the electromagnetic force acting on a stochastic particle in an external electromagnetic field, in the non-relativistic approximation: besides the Lorentz force, additional terms arise, which go to zero in the Newtonian limit.

c) To show that, under the usual quantum-mechanical approximation, the substitution of this electromagnetic force in the fundamental stochastic equations leads to Schrödinger's equation with minimal electromagnetic coupling.

I. INTRODUCTION

In a previous paper[1] devoted to some developments of the stochastic theory recently proposed by one of us[2], the possibility was shown of integrating the fundamental equations for a particle under the action of an external electromagnetic field, thus obtaining Schrödinger's equation with minimal electromagnetic coupling. However, this derivation was carried out for the special case in which $\nabla \times \mathbf{H} = 0$*. Clearly, to obtain a complete agreement with the usual quantum theory, it is necessary to eliminate this restriction. A more thorough analysis of

*We regret that this restriction was not explicitly stated in ref. 1, but it immediately appears when carrying out the algebra.

this problem allowed us to trace the difficulty in the postulated form of the external force. In fact, in our earlier work[1], the expression

$$\mathbf{F} = e\,[\,\mathbf{E} + \frac{1}{c}(\mathrm{v} + u) \times \mathbf{H}\,] \tag{1}$$

was proposed for the Lorentz force, the total velocity $\mathbf{c} = \mathbf{v} + u$ being the sum of a systematic and a stochastic term respectively.

If we introduce eq. (1) into the system of fundamental stochastic equations used in refs. (1) and (2) and demand that these equations be explicitly integrable, then the condition $\nabla \times \mathbf{H} = 0$ follows immediately. However, equation (1) is taken directly from classical electrodynamics, where the motion of the particle is assumed to be purely systematic, i.e., no diffusion processes occur. Now, if the particle's motion has a stochastic component, then, at least in principle, additional terms may be required in the expression for the acting force. The purpose of this paper is to show that in fact this is the case, eq. (1) being consistent with the fundamental stochastic equations only when $\nabla \times \mathbf{H} = 0$.

In order to obtain the correct form for the force acting on the stochastic particle subject to the action of an external electromagnetic field, we proceed – in analogy to classical mechanics – to derive the Lagrangian equations for our stochastic problem, using D'Alembert's principle as a starting point*. If in these equations we introduce the Lagrangian of a particle in an electromagnetic field, then we are able to derive the explicit form for the external force acting on the particle. This result approximated to second order (which is the order of approximation corresponding to usual quantum mechanics[1,2]) shows that we must add to eq. (1) a term proportional to the diffusion coefficient and which consequently is zero in classical mechanics. The integration of the basic equations can now be performed readily and thus Schrödinger's equation is obtained without any further restriction.

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II. THE LAGRANGE EQUATIONS OF THE STCCHASTIC THEORY

For consistency, we shall use the notation appearing in previous papers. In particular, the external force per unit mass, as derived in ref. (2), may be written in the form:

$$\mathbf{f}_0 = \mathcal{D}\mathbf{c} - (1+\lambda)\,\mathbf{a}_S \,, \tag{2}$$

where

$$\mathcal{D} = \frac{\partial}{\partial t} + \mathbf{c}\cdot\nabla + D\nabla^2 + \dots \tag{3}$$

$$\mathbf{c} = \mathcal{D}\mathbf{x} = \mathbf{v} + \mathbf{u}\,; \quad \tilde{\mathbf{c}} = \hat{T}\mathbf{c} = \tilde{\mathcal{D}}\mathbf{x} = -\mathbf{v} + \mathbf{u}\,; \tag{4}$$

$$\mathcal{D}_C = \frac{1}{2}(\mathcal{D} - \tilde{\mathcal{D}})\,; \quad \mathcal{D}_S = \frac{1}{2}(\mathcal{D} + \tilde{\mathcal{D}}) \tag{5}$$

and

$$\mathbf{a}_S = \mathcal{D}_S\,\mathbf{u}\,. \tag{6}$$

$\hat{T}$ is the time-inversion operator, the action of which is expressed by a tilde over the function, and λ is a parameter which assumes the value $+1$ in the quantum-mechanical case[2].

Let us consider a virtual displacement δx_i; eq. (2) can be written as

$$[f_{0i} - \mathcal{D}c_i + (1+\lambda)\,a_{Si}]\,\delta x_i = 0 \tag{7}$$

(a summation is understood over repeated indices). We now transform to generalized coordinates q_j, such that

$$x_i = x_i(q_j, t) \tag{8}$$

and since there is no variation of time involved,

$$\delta x_i = \frac{\partial x_i}{\partial q_j}\,\delta q_j \;. \tag{9}$$

Therefore,

$$(\mathcal{D}c_i)\,\delta x_i = (\mathcal{D}^2 x_i)\,\frac{\partial x_i}{\partial q_j}\,\delta q_j \tag{10}$$

A straightforward calculation shows that

$$\mathcal{D}\left[(\mathcal{D}x_i)\,\frac{\partial x_i}{\partial q_j}\right] = (\mathcal{D}^2 x_i)\,\frac{\partial x_i}{\partial q_j} + (\mathcal{D}x_i)\left(\mathcal{D}\,\frac{\partial x_i}{\partial q_j}\right) \;, \tag{11}$$

further terms of this expression being all equal to zero. Since from (3) and (4) it also follows that

$$\mathcal{D}\,\frac{\partial x_i}{\partial q_j} = \frac{\partial}{\partial q_j}\,\mathcal{D}x_i = \frac{\partial}{\partial q_j}\,c_i \quad \text{and} \quad \frac{\partial c_i}{\partial \dot{q}_j} = \frac{\partial x_i}{\partial q_j} \;,$$

equation (11) becomes

$$\mathcal{D}\left[(\mathcal{D}x_i)\,\frac{\partial c_i}{\partial \dot{q}_j}\right] = (\mathcal{D}^2 x_i)\,\frac{\partial x_i}{\partial q_j} + c_i\,\frac{\partial c_i}{\partial q_j} \;. \tag{12}$$

With the aid of this result, eq. (10) can be written as

$$(\mathcal{D}c_i)\,\delta x_i = \left[\mathcal{D}\left(c_i\,\frac{\partial c_i}{\partial \dot{q}_j}\right) - \frac{1}{2}\,\frac{\partial c_i^2}{\partial q_j}\right]\delta q_j \;. \tag{13}$$

Upon substitution of eqs. (9) and (13) into eq. (7) , there results

$$\left[f_{0i} \frac{\partial x_i}{\partial q_j} - \mathcal{D} \frac{\partial T}{\partial \dot{q}_j} + \frac{\partial T}{\partial q_j} + (1 + \lambda)\, a_{Si} \frac{\partial x_i}{\partial q_j} \right] \delta q_j = 0\,, \tag{14}$$

where $T = \frac{1}{2} c^2$. Since the only way for (14) to hold in general is for the separate coefficients of δq_j to vanish, we obtain the following set of dynamical equations:

$$\mathcal{D} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} - f_{0i} \frac{\partial x_i}{\partial q_j} = (1 + \lambda)\, a_{Si} \frac{\partial x_i}{\partial q_j}\,. \tag{15}$$

Let f_0 be a conservative force:

$$f_{0i} \frac{\partial x_i}{\partial q_j} = - \frac{\partial V}{\partial q_j}\,. \tag{16}$$

Defining the Lagrangian as is customary,

$$\mathcal{L} = T - V\,, \tag{17}$$

equations (15) become

$$\mathcal{D} \frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \frac{\partial \mathcal{L}}{\partial q_j} = (1 + \lambda)\, a_{Si} \frac{\partial x_i}{\partial q_j} \equiv (1 + \lambda)\, Q^Q_{Sj}\,. \tag{18}$$

These are the Lagrange equations of our stochastic theory. Upon comparison with the classical Lagrange equations, in which d/dt appears instead of the total derivative operator $\mathcal{D}$, one sees that the generalized force for the stochastic problem is given by

$$Q = (1 + \lambda)\, a_{Sj}^{Q} \quad . \tag{19}$$

It can be readily shown that application of the time-inversion operator to eq. (2) yields after a procedure analogous to the foregoing,

$$\tilde{\mathcal{D}} \frac{\partial \tilde{\mathcal{L}}}{\partial \tilde{\dot{q}}_j} - \frac{\partial \tilde{\mathcal{L}}}{\partial \tilde{q}_j} = (1 + \lambda)\, \tilde{a}_{Sj}^{Q} \quad . \tag{20}$$

To return to the original coordinate system, we make

$$q_i = \delta_{ij} x_j \; ,$$

equations (18) and (20) then become

$$\mathcal{D} \frac{\partial \mathcal{L}}{\partial c_i} - \frac{\partial \mathcal{L}}{\partial x_i} = (1 + \lambda)\, a_{Si} \; , \tag{21a}$$

$$\tilde{\mathcal{D}} \frac{\partial \tilde{\mathcal{L}}}{\partial \tilde{c}_i} - \frac{\partial \tilde{\mathcal{L}}}{\partial x_i} = (1 + \lambda)\, a_{Si} \; , \tag{21b}$$

since a_{Si} is invariant under time inversion[2]. For a conservative force, therefore,

$$\mathcal{D} \frac{\partial \mathcal{L}}{\partial c_i} = \tilde{\mathcal{D}} \frac{\partial \tilde{\mathcal{L}}}{\partial \tilde{c}_i} \; ,$$

which means

$$\mathcal{D}\mathbf{c} = \tilde{\mathcal{D}}\tilde{\mathbf{c}} . \tag{22}$$

Expansion of this equation into systematic and stochastic components leads to

$$\mathcal{D}_C \mathbf{u} + \mathcal{D}_S \mathbf{v} = 0 , \tag{23}$$

which is one of the two basic equations of our theory, for a conservative problem*, the other one being equation (2). Thus the set of fundamental equations can be considered as consisting of eq. (2) and either its $\hat{T}$-transformed or eq. (23). Hence, eqs. (18) and (20) represent the complete set of equations of motion in Lagrangian form.

III. THE ELECTROMAGNETIC CASE

Let us now introduce an arbitrary external electromagnetic field, described by the potentials $\mathbf{A}$ and ϕ. Substitution of the classical Lagrangian for this problem†:

$$\mathcal{L} = \frac{1}{2}\,\mathbf{c}^2 + \frac{e}{mc_0}\,\mathbf{c}\cdot\mathbf{A} - \frac{e}{m}\,\phi \tag{24}$$

in eq. (21a) yields, after some basic operations,

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*In the general case, we must write f_0^- instead of zero in eq. (23), f_0^- being that part of the external force which changes sign under $\hat{T}$. See ref (1).

†In order to avoid confusion between the total velocity $\mathbf{c}$ and the velocity of light, a nought has been added to the latter.

$$\mathcal{D}c_i + \frac{e}{mc_0}\,\mathcal{D}A_i - \frac{e}{mc_0}(\mathbf{c}\cdot\nabla)\,A_i - \frac{e}{mc_0}(\mathbf{c}\times\mathbf{H})_i = -\frac{e}{m}\,\partial_i\phi + (1+\lambda)\,a_{Si}\;. \tag{25}$$

Recalling eq. (2), we obtain for the external force (per unit mass)

$$f_0 = -\frac{e}{mc_0}(\mathcal{D}_c + \mathcal{D}_S)\,\mathbf{A} + \frac{e}{mc_0}\left[(\mathbf{u}+\mathbf{v})\cdot\nabla\right]A + \frac{e}{mc_0}(\mathbf{u}+\mathbf{v})\times\mathbf{H} - \frac{e}{m}\nabla\phi\;. \tag{26}$$

This force can be written as a sum of a $\hat{T}$-invariant and a $\hat{T}$-reversible term, which are respectively:

$$\begin{aligned} f_0^+ &= -\frac{e}{mc_0}\,\mathcal{D}_c\,\mathbf{A} + \frac{e}{mc_0}(\mathbf{v}\cdot\nabla)\,\mathbf{A} + \frac{e}{mc_0}\,\mathbf{v}\times\mathbf{H} - \frac{e}{m}\nabla\phi\;, \\ f_0^- &= -\frac{e}{mc_0}\,\mathcal{D}_S\,\mathbf{A} + \frac{e}{mc_0}(\mathbf{u}\cdot\nabla)\,\mathbf{A} + \frac{e}{mc_0}\,\mathbf{u}\times\mathbf{H}\;. \end{aligned} \tag{27}$$

To second order in the moments these expressions reduce to

$$\begin{aligned} f_0^{+(2)} &= \frac{e}{m}\left[\mathbf{E} + \frac{1}{c_0}\,\mathbf{v}\times\mathbf{H}\right], \\ f_0^{-(2)} &= \frac{e}{mc_0}\,\mathbf{u}\times\mathbf{H} - \frac{eD}{mc_0}\nabla^2\mathbf{A}\;. \end{aligned} \tag{28}$$

and the total force is simply:

$$f_0^{(2)} = \frac{e}{m}\left[\mathbf{E} + \frac{1}{c_0}(\mathbf{u}+\mathbf{v})\times\mathbf{H}\right] - \frac{eD}{mc_0}\nabla^2\mathbf{A}\;. \tag{29}$$

Thus we see that an additional term $-(eD/mc_0)\nabla^2 \mathbf{A} = -(eD/mc_0)(\nabla\nabla\cdot\mathbf{A} - \nabla\times\mathbf{H})$ must be added to the external force given by eq. (1), in order to obtain consistency with the general equations of motion. Since we can work in the Coulomb gauge $\nabla\cdot\mathbf{A} = 0$ (or Lorentz gauge if $\partial\phi/\partial t = 0$), this term may be simply written as $(eD/mc_0)\nabla\times\mathbf{H}$, with $D = \hbar/2m$ for the quantum-mechanical case[2]. It is also possible to obtain directly this last form for the extra term, by adding to the Lagrangian (eq. 24) the physically irrelevant term $(eD/mc_0)\nabla\cdot\mathbf{A}$, in order to eliminate the unwanted term $\nabla\cdot\mathbf{A}$ belonging to $\nabla^2\mathbf{A}$ in eq. (29). Since both procedures are equivalent in physical content, we can write the external force in the following final form:

$$f_0 = f_0^+ + f_0^- = \frac{e}{m}\left[\mathbf{E} + \frac{1}{c_0}\mathbf{c}\times\mathbf{H}\right] - \frac{e}{mc_0}\left[\mathcal{D}\mathbf{A} - (\mathbf{c}\cdot\nabla)\mathbf{A} - D\nabla(\nabla\cdot\mathbf{A})\right] \tag{30}$$

or to second order:

$$f_0^{(2)} = f_0^{+(2)} + f_0^{-(2)} = \frac{e}{m}\left[\mathbf{E} + \frac{1}{c_0}\mathbf{c}\times\mathbf{H}\right] + \frac{eD}{mc_0}\nabla\times\mathbf{H}. \tag{31}$$

Those terms in $f_0^{(2)}$ which reverse their sign under time inversion, i.e., $f_0^{-(2)}$, of course disappear in the Newtonian limit (i.e., when $u = 0$ and $D = 0$), leading to the well-known Lorentz force.

When eq. (31) is substituted in the basic set of equations and these are integrated following a procedure analogous to that of the aforementioned papers[1,2], we obtain the Schrödinger equation for a particle in an external electromagnetic field, namely,

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{1}{2m}\left(-i\hbar\nabla - \frac{e}{c_0}\mathbf{A}\right)^2\psi + e\phi\psi, \tag{32}$$

without any restriction on the field. Clearly, we can follow the same procedure to integrate the fundamental equations to all orders, using eq. (30) for the external force and a method entirely similar to that presented in ref. (1). With the definitions used there, namely,

$$\mathcal{D}_Q = \mathcal{D}_S + i\mathcal{D}_C ,$$

$$w = R + iS , \tag{33}$$

we obtain to all orders in the derivatives contained in $\mathcal{D}_Q$:

$$2D\mathcal{D}_Q w - 2D^2(\nabla w)^2 = \frac{e\phi}{m} + \frac{ie}{mc_0}\left[D\nabla\cdot\mathbf{A} - \frac{ie}{2mc_0}\mathbf{A}^2\right] , \tag{34}$$

where we have restricted ourselves to the usual quantum-mechanical case, by making $D_0 = D = D_+$, $D_- = 0$ and $\lambda = 1$. Eq. (34) represents the first integral of the fundamental stochastic equations and may be given the usual Schrödinger form by means of the change of variable $\nabla w = \nabla ln\,\psi$:

$$2iD\frac{\partial\psi}{\partial t} = 2D^2\left(-i\nabla - \frac{e}{2Dmc_0}\mathbf{A}\right)^2\psi + \frac{e\phi}{m}\psi - 2D\psi\,\hat{\mathbf{L}}_Q\cdot\nabla ln\psi . \tag{35}$$

The last term of the right-hand side includes the derivatives in $\mathcal{D}_Q$ of order > 2. Comparing with the previous results[1], we see that the extra terms of f_0 lead to a simplification of the final results.

IV. CONCLUSIONS

In summary, the main results of this note are:

a) It is possible to reformulate the basic stochastic equations of our theory in "Lagrangian" form, with the operator $\mathbb{D}$ playing the role of $\partial/\partial t$ in classical theory;

b) The Lorentz force does not account for all the electromagnetic force acting on the stochastic particle under the action of the external field, as can be seen from eqs. (30) and (31);

c) Substitution of this force in the stochastic equation leads, in the usual approximation, to Schrödinger's equation with minimal electromagnetic coupling without any restriction. In other words, the new terms added to the Lorentz force, to second approximation, are being taken into account automatically when writing down Schrödinger's equation.

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