

NEW FORMULATION OF STOCHASTIC THEORY
AND QUANTUM MECHANICS
IV. THE TWO-PARTICLE SYSTEM[†]

Luis de la Peña-Auerbach^{*}

Instituto de Física, Universidad Nacional de México

A. M. Cetto^{**}

Facultad de Ciencias, Universidad Nacional de México

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RESUMEN

El propósito de este trabajo es mostrar que la teoría cuántica estocástica comprende también el problema de dos (o, en general, de N) cuerpos. El método utilizado muestra que las dos partículas interactúan mutuamente, aún en el caso en que el potencial clásico de interacción se suponga nulo. Se discute asimismo el modelo de dos partículas independientes a partir de la interpretación estocástica.

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^{*} Consultant, Comisión Nacional de Energía Nuclear, México.

^{**} Fellow of the Ford Foundation, Fac. de Ciencias, UNAM, México.

ca; en esta forma, se recuperan varios resultados conocidos y se obtienen otros interesantes respecto a la interacción entre las partículas.

ABSTRACT

The purpose of this paper is to show that the quantum stochastic theory covers the two-body (and in general, the N-body) problem. The method shows that the two particles interact with each other, even when the external classical interaction is assumed to be zero. The two-independent particle model is discussed from the stochastic point of view; in this form, we are able to recover some well-known results and obtain other interesting ones about the interaction between particles.

I. INTRODUCTION

The purpose of this paper is to show how the stochastic theory of quantum mechanics which we have discussed in previous papers^{1,2} may be directly extended to cover the two-particle case. The procedure is straightforward and may be easily extended to cover the N-body problem. The method used in our derivation (Sec. II) is as follows: starting from the system of dynamical equations obeyed by the two particles, we obtain the equations of motion in terms of center-of-mass and relative coordinates (which will be referred to as the C.M. system). The result is what one may expect in advance: if we speak in terms of two-quasi-particles, one associated with the C.M. coordinate with mass $M = m_1 + m_2$ and the other with the relative coordinate with reduced mass $\mu = m_1 m_2 / M$, then the motion of each of these "particles" is described by the same dynamical equations, written for the corresponding external forces.

At this point we introduce the only additional postulate, which establishes that the C.M. velocities depend only on the C.M. coordinates, while the relative velocities depend only on the relative coordinates. This postulate is familiar from classical mechanics and represents the most economic one in con-

nection with our problem. Introducing this restriction and following the method of integration discussed in previous papers^{1,2}, we arrive at a couple of Schrödinger equations for the C.M. and the relative coordinates, which yield directly the single Schrödinger equation for two particles in its usual form, with an external potential separable in the C.M. system. In this form, our theory justifies the assumptions usually made in quantum mechanics when dealing with the two-body problem.

Two comments are worth making about the method employed here to establish Schrödinger's equation for two particles. In the first place, our starting point is a particular case of the fundamental stochastic equations, which means that the resulting two-body Schrödinger equation represents a mathematically simple, but physically non-immediate way of describing the very complex motion of two - in general, interacting - stochastic particles. Secondly, from the very beginning, the equations show that the two particles are interacting with each other, even when the external classical interaction force is assumed to be zero. In other words, our dynamical equations are such that when applied to a system of particles they automatically exhibit the dependence of the motion of one particle on the others motions.

In Sec. III we demonstrate that, under the above-mentioned postulate about the velocities, our original system of dynamical equations is also a direct consequence of Schrödinger's equation. This demonstration is included only for completeness, but should not be considered as a proof of the validity of the dynamical equations, since in the form they are written down they correspond only to a particular case and their physical sense is lost, or at least not immediate.

In Sec. IV we apply the previous results to a usual case, the so-called independent-particle model (or approximation, according to the specific situation); in this simple model we explicitly show how the very construction of the probability amplitude implies that the particles are not independent, but strongly influence one another (which, incidentally, shows that the current name is not appropriate to the situation). We close this section with a discussion of the possibility of interpreting the usual results from a stochastic point of view; well-known results emerge as direct consequences of this interpretation. For example, the "two-independent-particle" amplitude is consistent with a stochastic interpretation only

when it is symmetric or antisymmetric, and the one-particle amplitudes must belong to an orthonormal family of functions. This description is interesting in that it allows to state clearly the effects of one particle on the other; for example, the probability of one particle to be in some given state depends on the state which is simultaneously occupied by the other particle. All these results are almost trivial, but this part of the paper is written within the spirit of the well-known statement by Feynman³, that "there is a pleasure in recognizing old things from a new point of view". In fact, the whole series to which this paper belongs is written in this spirit.

II. THE TWO-PARTICLE SYSTEM

As shown in reference (1), a stochastic particle of mass m , subject to the action of the external force $\mathbf{F}_0 = m\mathbf{f}_0$, has a motion characterized by the systematic and stochastic velocities \mathbf{v} and \mathbf{u} , respectively, which satisfy the system of equations

$$\mathcal{D}_C \mathbf{v} - \mathcal{D}_S \mathbf{u} = \frac{\mathbf{F}_0}{m}, \quad (1)$$

$$\mathcal{D}_S \mathbf{v} + \mathcal{D}_C \mathbf{u} = 0.$$

\mathcal{D}_C and \mathcal{D}_S are the systematic (or current) and stochastic derivative operators previously defined¹. System (1) is written for the particular case in which the constant defined earlier $\lambda = 1$ and the external force \mathbf{f}_0 is invariant under time-inversion²; furthermore, we shall restrict ourselves to the second-order approximation, when explicitly writing down the operators \mathcal{D}_C and \mathcal{D}_S . As has been demonstrated^{1,2}, all these assumptions are consistent with the quantum-mechanical description of the particle's motion. Presently we shall deal with two particles; hence we must write down the corresponding set of equations. Since Eqs. (1) correspond to the stochastic generalization of Newton's second law, we shall write the same system for each particle, i.e., instead of Eqs. (1) we have

$$\mathbb{D}_C v_i - \mathbb{D}_S u_i = \frac{F_{0i}}{m_i} \quad (2)$$

$$\mathbb{D}_S v_i + \mathbb{D}_C u_i = 0 ,$$

where i may take the values 1 and 2, corresponding to the two particles acted on by the external forces F_{0i} . Our purpose is to show that the first integral of Eqs. (2) leads to Schrödinger's equation. To achieve this in the simplest possible way, let us transform to center-of-mass and relative coordinates:

$$R = \alpha_2 r_1 + \alpha_1 r_2 , \quad (3)$$

$$r = r_2 - r_1 ,$$

where

$$\alpha_1 = \frac{m_2}{M} = \frac{\mu}{m_1} ; \quad \alpha_2 = \frac{m_1}{M} = \frac{\mu}{m_2} ; \quad (4)$$

$$M = m_1 + m_2 ; \quad \mu = \frac{m_1 m_2}{M} .$$

The above definitions imply the relations

$$\alpha_1 + \alpha_2 = 1 ; \quad \alpha_1 \alpha_2 = \frac{\mu}{M} , \quad (5)$$

which will be frequently used in the following. For further convenience, let us introduce the definitions:

$$D_r = \frac{\hbar}{2\mu} = D_1 + D_2 ; \quad (6)$$

$$D_R = \frac{\hbar}{2M} = \frac{D_1 D_2}{D_r} .$$

D_r is the diffusion coefficient associated with the motion of the quasi-particle of mass μ equal to the reduced mass of particles 1 and 2; likewise, D_R is the diffusion coefficient associated with the motion of the quasi-particle of mass $M = m_1 + m_2$. The inverse of (3) is

$$\begin{aligned} r_1 &= R - \alpha_1 r, \\ r_2 &= R + \alpha_2 r. \end{aligned} \tag{7}$$

Let us apply to (7) the operators $\mathcal{D} = \mathcal{D}_C + \mathcal{D}_S$ and $\tilde{\mathcal{D}} = -\mathcal{D}_C + \mathcal{D}_S$ in succession. From the usual definitions¹ and Eq. (7), above, we have

$$\begin{aligned} \mathcal{D}r_1 &= v_1 + u_1 = V + U - \alpha_1(v + u) \\ \mathcal{D}r_2 &= v_2 + u_2 = V + U + \alpha_2(v + u) \\ \tilde{\mathcal{D}}r_1 &= -v_1 + u_1 = -V + U - \alpha_1(-v + u) \\ \tilde{\mathcal{D}}r_2 &= -v_2 + u_2 = -V + U + \alpha_2(-v + u), \end{aligned} \tag{8}$$

where V and U are the systematic and stochastic velocities of the C.M. and v and u refer, in the same order, to the relative motion. Hence we obtain

$$\begin{aligned} v_1 &= V - \alpha_1 v \\ v_2 &= V + \alpha_2 v \\ u_1 &= U - \alpha_1 u \\ u_2 &= U + \alpha_2 u \end{aligned} \tag{9}$$

and correspondingly:

$$\begin{aligned}
 \mathbf{v} &= \mathbf{v}_2 - \mathbf{v}_1 \\
 \mathbf{u} &= \mathbf{u}_2 - \mathbf{u}_1 \\
 \mathbf{V} &= \alpha_1 \mathbf{v}_2 + \alpha_2 \mathbf{v}_1 \\
 \mathbf{U} &= \alpha_1 \mathbf{u}_2 + \alpha_2 \mathbf{u}_1 .
 \end{aligned} \tag{10}$$

Under the transformation defined by Eqs. (5'), the system (1) assumes the following form:

$$\left. \begin{aligned}
 \mathbb{D}_C \mathbf{v} - \mathbb{D}_S \mathbf{u} &= \frac{\mathbf{f}_0}{\mu} \\
 \mathbb{D}_S \mathbf{v} + \mathbb{D}_C \mathbf{u} &= 0
 \end{aligned} \right\} \tag{11}$$

$$\left. \begin{aligned}
 \mathbb{D}_C \mathbf{V} - \mathbb{D}_S \mathbf{U} &= \frac{\mathbf{F}_0}{M} \\
 \mathbb{D}_S \mathbf{V} + \mathbb{D}_C \mathbf{U} &= 0 ,
 \end{aligned} \right\} \tag{12}$$

where we have introduced the abbreviations:

$$\frac{\mathbf{f}_0}{\mu} = \frac{\mathbf{F}_{02}}{m_2} - \frac{\mathbf{F}_{01}}{m_1} , \tag{13}$$

$$\mathbf{F}_0 = \mathbf{F}_{01} + \mathbf{F}_{02} .$$

Clearly, \mathbf{f}_0 is the relative force between particles and \mathbf{F}_0 represents the total force acting on the C.M. It should be emphasized that \mathbf{f}_0 and \mathbf{F}_0 refers *only* to the classical forces which come directly from \mathbf{F}_{0i} and do not include any additional

force which may arise from effective interactions between the particles or between any given particle and the vacuum. In fact, since the operators \mathcal{D}_C and \mathcal{D}_S depend on the velocities \mathbf{v}_i and \mathbf{u}_i , we see from Eqs. (2) that the motions of the two particles influence each other in a generally complex way. In other words \mathbf{f}_0 and \mathbf{F}_0 are the *classical* relative and C.M. forces, respectively.

Eqs. (11) and (12) represent a first result of interest: they indicate that the laws of motion of the "particles" of masses M and μ are equal to the law of motion of a single particle. The separation into relative and C.M. motions corresponds closely to that usual in classical mechanics. However, this separation is still not complete, because in general the operators \mathcal{D}_C and \mathcal{D}_S involve both coordinates through the four velocities \mathbf{v}_i and \mathbf{u}_i . In fact, upon substitution of (7) in their definition¹, we obtain

$$\mathcal{D}_C = \frac{\partial}{\partial t} + \mathbf{v}_1 \cdot \nabla_1 + \mathbf{v}_2 \cdot \nabla_2 = \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla_R + \mathbf{v} \cdot \nabla_r \quad (14a)$$

and

$$\begin{aligned} \mathcal{D}_S &= \mathbf{u}_1 \cdot \nabla_1 + \mathbf{u}_2 \cdot \nabla_2 + D_1 \nabla_1^2 + D_2 \nabla_2^2 = \\ &= \mathbf{U} \cdot \nabla_R + \mathbf{u} \cdot \nabla_r + D_R \nabla_R^2 + D_r \nabla_r^2. \end{aligned} \quad (14b)$$

Introduction of (14) into Eqs. (11) and (12) explicitly shows the presence of crossed terms. But both from the physical and the mathematical point of view, we may assume that the C.M. velocities \mathbf{V} and \mathbf{U} do not depend on r , while the relative velocities \mathbf{v} and \mathbf{u} do not depend on \mathbf{R} . This postulate will be referred to as (P) in what follows. With it, Eqs. (11) and (12) separate completely to give:

$$\left. \begin{aligned} \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla_r \mathbf{v} - \mathbf{u} \cdot \nabla_r \mathbf{u} - D_r \nabla_r^2 \mathbf{u} &= \frac{f_0}{\mu} \\ \frac{\partial \mathbf{u}}{\partial t} + \mathbf{v} \cdot \nabla_r \mathbf{u} + \mathbf{u} \cdot \nabla_r \mathbf{v} + D_r \nabla_r^2 \mathbf{v} &= 0 \end{aligned} \right\} \quad (15)$$

$$\left. \begin{aligned} \frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla_R \mathbf{V} - \mathbf{U} \cdot \nabla_R \mathbf{U} - D_R \nabla_R^2 \mathbf{U} &= \frac{F_0}{M} \\ \frac{\partial \mathbf{U}}{\partial t} + \mathbf{V} \cdot \nabla_R \mathbf{U} + \mathbf{U} \cdot \nabla_R \mathbf{V} + D_R \nabla_R^2 \mathbf{V} &= 0 \end{aligned} \right\} \quad (16)$$

These sets of equations refer to two independent stochastic particles. As known from previous treatments^{1,2}, each one of these sets give rise upon integration and simple mathematical transformations, to a corresponding Schrödinger equation, namely,

$$2i\mu D_r \frac{\partial \psi_r}{\partial t} = -2\mu D_r^2 \nabla_r^2 \psi_r + V_r \psi_r, \quad (17)$$

$$2iM D_R \frac{\partial \psi_R}{\partial t} = -2M D_R^2 \nabla_R^2 \psi_R + V_R \psi_R$$

where $\psi_r = \psi_r(\mathbf{r})$, $\psi_R = \psi_R(\mathbf{R})$, $f_0 = -\nabla_r V_r$ and $F_0 = -\nabla_R V_R$. From Eqs. (17) it immediately follows that the function

$$\psi = \psi_R \psi_r \quad (18)$$

satisfies the equation:

$$i\hbar \frac{\partial \psi}{\partial t} = -\hbar D_1 \nabla_1^2 \psi - \hbar D_2 \nabla_2^2 \psi + V \psi, \quad (19)$$

i.e., Schrödinger's equation for a potential V , which may be expressed as the sum of an external, C.M. potential V_R and an interaction potential V_r . (Clearly, in Eq. (19) these two potentials are to be considered functions of r_1 and r_2). Thus we have achieved our first purpose, namely, to show that the usual Schrödinger equation for two particles is a direct consequence of the system of dynamical equations (2).

III. RECOVERY OF THE FUNDAMENTAL EQUATIONS

In this section we proceed to recover the system of equations (2), starting from Schrödinger's equation (19) and using (P) as the only additional postulate. Although the result will not give anything fundamentally new, this demonstration is profitable in that it yields useful relations as byproducts (which, incidentally, could have been obtained from the expressions appearing in Sec. II). Furthermore, it allows us to see clearly that the treatment can be generalized to N particles without any further complication.

As suggested by the procedure used earlier^{1,2} we write the amplitude ψ in the usual form

$$\psi = \exp(R + iS) . \quad (20)$$

Clearly, the most general form for the functions R and S is

$$\begin{aligned} R &= R_0(1,2) + R_1(1) + R_2(2) , \\ S &= S_0(1,2) + S_1(1) + S_2(2) . \end{aligned} \quad (21)$$

(The time-dependence is omitted for brevity). R_0 and S_0 are the terms responsible for the interference between particles, while $R_i(i) = R_i(r_i)$ and $S_i(i) = S_i(r_i)$ correspond to independent-particle motions. As usual, the velocities of the particles are:

$$\begin{aligned} \mathbf{v}_i &= 2D_i \nabla_i S = 2D_i \nabla_i (S_0 + S_i) , \\ \mathbf{u}_i &= 2D_i \nabla_i R = 2D_i \nabla_i (R_0 + R_i) , \end{aligned} \quad (22)$$

and must therefore satisfy the relation

$$\nabla_2 \cdot (m_1 \mathbf{v}_1) = \nabla_1 \cdot (m_2 \mathbf{v}_2)$$

and a similar relation for \mathbf{u}_i . To see the meaning of these restrictions, we transform to the C.M. system, thus obtaining

$$\begin{aligned} \nabla_R \cdot (\mu \mathbf{v}) &= \nabla_r \cdot (M \mathbf{V}) , \\ \nabla_R \cdot (\mu \mathbf{u}) &= \nabla_r \cdot (M \mathbf{U}) . \end{aligned} \quad (23)$$

Now it is evident that under (P), Eqs. (23), and hence (22), are valid*. We proceed further by introducing Eqs. (20) and (22) in (19), and separating real and imaginary parts;

$$2 \frac{\partial}{\partial t} (R_0 + R_1 + R_2) + \nabla_1 \cdot \mathbf{v}_1 + \nabla_2 \cdot \mathbf{v}_2 + D_1^{-1} \mathbf{u}_1 \cdot \mathbf{v}_1 + D_2^{-1} \mathbf{u}_2 \cdot \mathbf{v}_2 = 0 , \quad (24)$$

$$2 \frac{\partial}{\partial t} (S_0 + S_1 + S_2) - \nabla_1 \cdot \mathbf{u}_1 - \nabla_2 \cdot \mathbf{u}_2 + (2D_1)^{-1} (\mathbf{v}_1^2 - \mathbf{u}_1^2) + (2D_2)^{-1} (\mathbf{v}_2^2 - \mathbf{u}_2^2) = - \frac{2V}{\hbar}$$

It is a simple though somewhat lengthy task to show that the two equations (24) represent the first integral of the four equations (2). To accomplish this, we apply the operators ∇_1 and ∇_2 to both equations in (24). Let us briefly sketch

* (P) is somewhat stronger than the conditions expressed in Eqs. (23); later on we shall make full use of it to recover system (2).

the procedure on an example, by applying ∇_1 to the first of Eqs. (24). From the previous definitions, we have that

$$2\nabla_1 \frac{\partial R}{\partial t} = D_1^{-1} \frac{\partial \mathbf{u}_1}{\partial t},$$

$$\frac{\partial}{\partial t} + \mathbf{v}_1 \cdot \nabla_1 = \mathbb{D}_C - \mathbf{v}_2 \cdot \nabla_2,$$

$$\mathbf{u}_1 \cdot \nabla_1 + D_1 \nabla_1^2 = \mathbb{D}_S - \mathbf{u}_2 \cdot \nabla_2 - D_2 \nabla_2^2$$

and hence the result may be written in the form:

$$\begin{aligned} & \mathbb{D}_C \mathbf{u}_1 + \mathbb{D}_S \mathbf{v}_1 + D_2^{-1} \{ D_1 (\mathbf{v}_2 \cdot \nabla_1) \mathbf{u}_2 - D_2 (\mathbf{v}_2 \cdot \nabla_2) \mathbf{u}_1 \} + \\ & + D_2^{-1} \{ D_1 (\mathbf{u}_2 \cdot \nabla_1) \mathbf{v}_2 - D_2 (\mathbf{u}_2 \cdot \nabla_2) \mathbf{v}_1 \} + \{ D_1 \nabla_1 (\nabla_2 \cdot \mathbf{v}_2) - D_2 \nabla_2 (\nabla_2 \cdot \mathbf{v}_1) \} = 0. \end{aligned}$$

It is easy to show that upon use of (P), each one of the expressions enclosed by curly brackets vanishes identically, thus yielding as a final result

$$\mathbb{D}_C \mathbf{u}_1 + \mathbb{D}_S \mathbf{v}_1 = 0,$$

which corresponds to one of the equations in (2). Let us transform, for example, the first bracket:

$$\begin{aligned} & D_1 (\mathbf{v}_2 \cdot \nabla_1) \mathbf{u}_2 - D_2 (\mathbf{v}_2 \cdot \nabla_2) \mathbf{u}_1 = \\ & = D_1 [(\mathbf{V} + \alpha_2 \mathbf{v}) \cdot (-\nabla_r + \alpha_2 \nabla_R)] (\mathbf{U} + \alpha_2 \mathbf{u}) - D_2 [(\mathbf{V} + \alpha_2 \mathbf{v}) \cdot (\nabla_r + \alpha_1 \nabla_R)] (\mathbf{U} - \alpha_1 \mathbf{u}) = \\ & = -(D_1 \alpha_2 - D_2 \alpha_1) [\mathbf{V} \cdot \nabla_r \mathbf{u} - \mathbf{V} \cdot \nabla_R \mathbf{U} + \alpha_2 \mathbf{v} \cdot \nabla_r \mathbf{u} - \alpha_2 \mathbf{v} \cdot \nabla_R \mathbf{U}] = 0; \end{aligned}$$

since

$$D_1 \alpha_2 - D_2 \alpha_1 = \frac{h}{2m_1} \frac{\mu}{m_2} - \frac{h}{2m_2} \frac{\mu}{m_1} = 0 .$$

Following an analogous procedure with the remaining equations, we finally obtain:

$$\begin{aligned} D_C u_i + D_S v_i &= 0 , \\ D_C v_i - D_S u_i &= - \frac{\nabla_i V}{m_i} , \end{aligned}$$

which is just the original system (2).

Now that the equivalence between Schrödinger's equation for two particles and the corresponding system of stochastic equations has been established, we may go further and introduce the dynamical operators, in order to assign a physical meaning to the first-integral system (24). The procedure is similar to that followed for the case of one particle¹. In the present case, it can be readily shown that the first equation in (24) assumes the form of a Fokker-Planck equation for the probability $\rho = \psi^* \psi$:

$$\frac{\partial \rho}{\partial t} + \nabla_1 \cdot (c_1 \rho) + \nabla_2 \cdot (c_2 \rho) - D_1 \nabla_1^2 \rho - D_2 \nabla_2^2 \rho = 0 , \quad (25a)$$

where, as usual, $c_i = v_i + u_i$. In terms of C.M. and relative coordinates, we have:

$$\frac{\partial \rho}{\partial t} + \nabla_r \cdot (c_r \rho) + \nabla_R \cdot (c_R \rho) - D_r \nabla_r^2 \rho - D_R \nabla_R^2 \rho = 0 . \quad (25b)$$

From the discussion included in ref. (1), it is evident that the second equation in (24) is the energy law of the problem; from this we infer that the kinetic energy

due to the stochastic motion is simply the sum of the contributions of each particle, i.e., all the interference effects are due to the dependence of each velocity on the coordinates of both particles.

IV. THE TWO-INDEPENDENT-PARTICLE SYSTEM

From Eqs. (20) and (21) we have:

$$\rho = \psi^* \psi = e^{2R} = \rho_1(1) \rho_2(2) \rho_0(1,2) \quad , \quad (26)$$

where $\rho_i(i) = \rho_i(r_i)$, $i = 1, 2$, may be considered one-particle densities, while $\rho_0(1,2) = \exp [2R_0(1,2)]$ is a two-particle factor, which measures the effect of interference between particles upon the probability density. Clearly, if we put $\rho_0 = 1$ the two particles become independent and at the same time, the Fokker-Planck equation separates into two independent equations*. When the potential can be written as $V = V_1(1) + V_2(2)$, this separation is in general permissible and causes any interference between particles to vanish, since the probability amplitude becomes $\psi = \psi_1(1) \psi_2(2)$, and hence $S_0 = R_0 = 0$.

From a stochastic point of view, this situation is physically acceptable only for very distant particles, i.e., when in fact we are dealing with two stochastically independent systems. Since we are considering that each particle interacts with the vacuum, we are obliged to accept the existence of a correlation between the motions of the particles, induced by their motion in a common medium. In other words, the two particles and the vacuum must be treated as a single system.

This lack of independence between the motions of classically independent identical particles is usually taken into account in quantum mechanics by introducing a very simple but extremely far-reaching idea: Pauli's exclusion principle. In the text-book language of quantum mechanics we say that being there two

* This is equivalent to writing for two events a and b , $P(ab) = P(a) P(b)$, i.e., the events are statistically independent.

states a and b^* , the identical particles 1 and 2 may be in either state, due to their indistinguishability, and we therefore write:

$$\Psi = C_1 \psi_a(1) \psi_b(2) + C_2 \psi_b(1) \psi_a(2) , \quad (27)$$

where C_1 and C_2 guarantee the normalization of Ψ and the required symmetry properties according to Pauli's principle^{**}. At first sight, it appears that Eq. (27) refers to independent particles but, as is well known, this is not the case; the simplest and most frequent way to show this consists in constructing the probability density associated to Ψ :

$$\rho = C_1^2 \rho_a(1) \rho_b(2) + C_2^2 \rho_a(2) \rho_b(1) + 2C_1 C_2 \sqrt{\rho_a(1) \rho_b(1) \rho_a(2) \rho_b(2)} \cos \Omega , \quad (28)$$

where

$$\Omega = s_a(2) - s_b(2) + s_b(1) - s_a(1) \quad (29)$$

and the C_i are assumed to be real. Eq. (28) explicitly shows the existence of interference, which means that the amplitude (27) corresponds to two particles with correlated motions. But now the argument breaks down: we constructed Ψ thinking in terms of two indistinguishable independent particles and yet we arrive at the conclusion that the particles are not stochastically independent. However, we also know that in spite of our arguments, Eqs. (27) and (28) are

* For simplicity we shall consider only different, non-degenerate eigenstates, hence $E_a \neq E_b$.

** The text-book arguments are not entirely consistent, at least from the standpoint of a corpuscular theory like ours. This lack of consistency comes about in the following form: First, the particles are labelled and then some correspondence is established between labels and states, thus implying some distinguishability between particles. Next, it is argued that actually the particles are indistinguishable and hence the labels may as well be interchanged, obtaining a second wave function. Since from very general arguments it may be shown that the wave function for two classically non-interacting particles must be symmetric or antisymmetric (ref. 4), the total wave function is constructed from the two previous ones, in order to achieve the required symmetry properties. Similar criticisms have been raised by other authors (see for example ref. 5). Arguments concerning the distinguishability of particles introduce into the physical theory a subjective ingredient related to our capabilities of observation. As long as we do not explicitly introduce into the theory an external perturbation due to our observations, the system must be considered unperturbed and unobserved and evolving according to its own laws, which have nothing to do with our capability to attach labels to the particles.

physically correct. Our purpose in this section is to assign to the above expressions a consistent meaning based on our stochastic point of view. At present we content ourselves with the display of some basic ideas, hoping to extend them in a forthcoming paper.

Let us begin with two remarks, concerning the interference term in Eq. (28) and the general structure of Ψ . First, we know that the interference terms appear when dealing with amplitudes instead of probabilities, and actually their existence is essential for the foundation of quantum mechanics. In stochastic theory, on the other hand, such terms are non-existent. This apparent contradiction might seem to indicate that our stochastic interpretation of quantum mechanics is incorrect. However, a more careful analysis shows that although such interference terms are uncommon in stochastic theory, they are entirely consistent with its fundamental equations. We may construct a theory with or without them; which one to use is to be decided by considerations not properly within the scope of the theory itself. In particular, Eq. (28) is consistent with our basic stochastic equations, and these, as has been previously shown, contain the Fokker-Planck equation. To clarify this point, let us consider the following simple example: if ρ_1 and ρ_2 are two different solutions of the continuity equation (which, with $\mathbf{v} = \mathbf{c} - \mathbf{u}$, gives the Fokker-Planck equation¹), both for the same systematic velocity \mathbf{v} , then ρ_i , $i = 1, 2$, is a solution of

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\mathbf{v}\rho) = 0 \quad . \quad (30)$$

Since this equation is linear, $\rho = \alpha\rho_1 + \beta\rho_2$ is also a solution, but not the most general one that can be constructed with ρ_1 and ρ_2 . To show this, let us introduce the function

$$\rho = \rho_1^p \rho_2^q \quad , \quad (31)$$

with p and q real numbers. Substitution in the continuity equation gives

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\mathbf{v} \rho) = (1 - p - q) \rho \nabla \cdot \dot{\mathbf{v}} .$$

Hence the function (31) is a solution of Eq. (30) if p and q are such that

$$p + q = 1 , \quad (32)$$

which means that from any pair of solutions ρ_1, ρ_2 we may construct an infinity of new solutions as linear combinations of terms of the form $\rho_1 (\rho_2 / \rho_1)^q$, with arbitrary (real) q . Three out of these are particularly important to us, namely, those corresponding to $q = 0, 1$ and $1/2$, which give:

$$\rho = \alpha \rho_1 + \beta \rho_2 + \gamma \sqrt{\rho_1 \rho_2} . \quad (33)$$

Generally, in stochastic theory one selects *a priori* $\gamma = 0$, to eliminate any interference between different ρ_i ; quantum mechanics, however, is a particular stochastic problem in which such interactions play a significant role. Since ρ itself is also to be interpreted as a probability, the constants α, β and γ are not independent of each other. The density ρ has the same form as that given by Eq. (28), except for the factor $\cos \Omega$. As can be easily shown, this factor arises when ρ_1 and ρ_2 are not assumed to correspond to the same velocity \mathbf{v} anymore; in this case, the probability density assumes the more general form given by Eq. (28) and satisfies the continuity equation for a new velocity whose value is obtained from the corresponding amplitude, i.e., Eq. (27). Hence we see that to deal with amplitudes instead of probabilities provides a simple means of introducing interference terms of the particular form illustrated by Eq. (33), which amounts to constructing a stochastic theory more general than classical stochastic theory. However trivial this result may seem, it merits the above discussion, because it points out the fundamental difference between classical and quantum stochastic processes. This difference is so important that it may be used as a starting point for the formulation of the stochastic theory of quantum mechanics. A formal but very interesting approach along these lines has recently been proposed⁶

Our second remark refers to the structure of the amplitude corresponding to two particles. We wish to recall that it can be shown on very general grounds⁵, and without the use of any argument about the indistinguishability of the particles, that a two-particle amplitude constructed from single-particle amplitudes must be either symmetric or antisymmetric. Thus Eq. (27) implies nothing about the indistinguishability of the two particles.

Let us now give some simple arguments to justify from a stochastic point of view the use of two-particle amplitudes of the usual form (27). For this purpose, we recall once more that a product of the type $\psi_a(1)\psi_b(2)$ refers not to a system of two particles, but to two independent systems; furthermore, that the particles must not be labelled to distinguish them, since they are supposed to be identical. Let the system be stationary; if the energies of the particles are E_a and E_b , and these do not interact classically*, the total energy of the system is $E = E_a + E_b$. The probability for the particle with energy E_a to be between r_1 and $r_1 + dr_1$ is $W(1a) dr_1 = \rho_a(1) dr_1$, and so on; clearly, any one of the particles may be in the neighborhood of r_1 with probability density $W(1a)$ or $W(1b)$, depending on its energy state. Now we select two arbitrary points in space and ask for the probability of the particles being in their neighborhoods. There will be two different answers, according to whether we require the particle within dr_1 to be in one of the energy states E_a or E_b , or whether we refer to a particle without regard to its energy. In the first case, there are the two following possible questions:

- a) we may ask for the probability $W(1a; 2b) dr_1 dr_2$ of a particle with energy E_a to be within dr_1 and the other particle with energy E_b to be simultaneously within dr_2 ,
- b) or we may ask for the probability $W(1b; 2a) dr_1 dr_2$ of a particle with energy E_b being within dr_1 and a particle with energy E_a being simultaneously within dr_2 .

Hence the probability per unit volume for two particles to be simultaneously within the ranges dr_1 and dr_2 , irrespective of their energies, is

*That is, if Schrödinger's equation either does not contain an interaction potential or we are neglecting it.

$$W(12; ab) = W(1a; 2b) + W(1b; 2a) \quad (34)$$

If $P(2b | 1a)$ stands for the probability per unit volume that a particle with energy E_b is at r_2 , while the other particle (with energy E_a) is at r_1 , then

$$W(1a; 2b) = W(1a) P(2b | 1a) = W(2b) P(1a | 2b) \quad (35)$$

and therefore,

$$W(12; ab) = W(1a) P(2b | 1a) + W(1b) P(2a | 1b) \quad (36)$$

Stochastic independence implies $P(2b | 1a) = W(2b) = \rho_b(2)$; any departure from this relation is a consequence of the mutual interaction between particles.

We may now identify $W(12; ab)$ with ρ as given by Eq. (28), since it is evident from its construction that such a ρ gives the probability per unit volume that the particles are at r_1 and r_2 , without regard to their specific energy states. For simplicity, let us consider the most common case $\Omega = 0^*$ (this holds, in particular, for stationary, non-degenerate states). Writing $C_1 = C$ and $C_2 = \epsilon C$, we have

$$W(12; ab) = C^2 \left[W(1a) W(2b) + \epsilon^2 W(2a) W(1b) + 2\epsilon \sqrt{W(1a) W(1b) W(2a) W(2b)} \right] \quad (37)$$

From Eqs. (36) and (37) it then follows that

$$P(2b | 1a) = C^2 W(2b) \left[1 + \epsilon \sqrt{\frac{W(1b) W(2a)}{W(1a) W(2b)}} \right] \quad (38a)$$

* Special care must be taken when making $\cos \Omega = 1$, since the sign of \sqrt{W} is then lost. To see this, let us write $S = S' + i\pi n$, where $n = n(\mathbf{r})$ is 0 or 1, according to the sign of G in $\psi = Ge^{iS} = (-)^n e^{\mathbf{R}} + iS'$; then $\sqrt{W} = \sqrt{\psi^* \psi} = (-)^n e^{\mathbf{R}}$. In what follows we shall everywhere incorporate this overall sign $(-)^n$ into \sqrt{W} and take $\cos \Omega = 1$.

$$P(2a | 1b) = C^2 W(2a) \left[\epsilon^2 + \epsilon \sqrt{\frac{W(1a) W(2b)}{W(1b) W(2a)}} \right]. \quad (38b)$$

Since an interchange of a and b in $P(2a | 1b)$ must lead to $P(2b | 1a)$, we necessarily have from (38)

$$\epsilon = \pm 1. \quad (39)$$

The amplitude Ψ in Eq. (27) thus becomes either symmetric or antisymmetric with respect to an interchange of a and b (or r_1 and r_2 , which is equivalent). That this condition is trivial may be seen from Eq. (36) or even from the very definition of the probability density, which requires that $W(12; ab) = W(12; ba) = W(21; ab)$. Eq. (39) simply states that symmetric probabilities can be generated only by symmetric or antisymmetric amplitudes.

Moreover, for consistency, our results must satisfy the following requirements:

a) Since a double integration of $W(12; ab)$ over all space counts the particles twice, we must have

$$\int W(12; ab) dr_1 dr_2 = 2. \quad (40)$$

b) Since energy is conserved,

$$W(1a) = \int W(1a; 2b) dr_2 \quad (41)$$

for any r_1 , and so on.

c) The probability for one particle being in state a is

$$\int W(1a) dr_1 = 1 \quad (42)$$

d) From Eqs. (35) and (41), we also must require that

$$\int P(2b|1a) dr_2 = 1. \quad (43)$$

Eq. (42) is satisfied by construction. Upon introducing condition (43) into Eqs. (38), we obtain

$$C^2 \int W(2b) dr_2 + C^2 \epsilon \sqrt{\frac{W(1b)}{W(1a)}} \int \sqrt{W(2a)W(2b)} dr_2 = 1,$$

which, together with (42), means that for the interpretation to have a meaning, two conditions are to be satisfied, namely,

$$C = \pm 1 \quad (44)$$

and

$$\int \sqrt{W(1a)W(1b)} \int \sqrt{W(2a)W(2b)} dr_2 = 0. \quad (45)$$

This last equation, written in terms of the amplitude, has the only non-trivial solution

$$\int \psi_a^*(r) \psi_b(r) dr = 0 \quad (46)$$

and hence condition (43) demands that the amplitudes belonging to different energy eigenstates be orthogonal. But from the stationary Schrödinger equation we know that this is always the case, and hence Eq. (43) is satisfied. Further, we note from Eq. (44) that in the stochastic interpretation, the normalization is fixed, giving a value of unity to the coefficient C instead of the quantum-mechanical value $1/\sqrt{2}$ for $a \neq b$. Now, it is easy to check that if both conditions (44) and (46) are fulfilled, then $W(12; ab)$ as given by Eq. (37) and $W(1a; 2b)$ automatically satisfy the corresponding requirements (40) and (41).

Thus we conclude that a system of two classically non-interacting particles may be described with the aid of single-particle amplitudes, expressing the total amplitude as a symmetric or antisymmetric combination of them; that with this amplitude we may associate the conditional probabilities

$$P(2b | 1a) = W(2b) [1 + \epsilon f_{ab}(1,2)] \quad (47)$$

and that, since $P(2b | 1a) \neq W(2b)$, this construction automatically takes into account the mutual interaction between the stochastic particles. In Eq. (47), f_{ab} is given by

$$f_{ab}^2(1,2) = \frac{W(1b)W(2a)}{W(1a)W(2b)}$$

and has the following interesting property:

$$f_{ab}(2,1) = f_{ba}(1,2) = f_{ab}^{-1}(1,2) \quad .$$

Now we proceed to show that, in the stationary state, the amplitude (27) corresponds to an effective attractive state between particles when $\epsilon = +1$, and to a repulsive state when $\epsilon = -1$. The first step consists in calculating the relative velocity of the particles described by Eq. (27). The systematic velocities v_i may be set equal to zero, since for non-degenerate states we may take $S'_a(1) = -E_a t$, and so on⁷; furthermore, as was mentioned above, we may write $\Omega = 0$ for stationary states. On the other hand, a direct calculation gives for the stochastic velocities at the points r_i , $i = 1, 2$:

$$u_i = \frac{1}{2} [u_a(i) + u_b(i)] + (-)^{i-1} \frac{1}{2} [u_a(i) - u_b(i)] F, \quad (48)$$

with

$$F = [\rho_a(1)\rho_b(2) - \rho_b(1)\rho_a(2)]/\rho. \quad (49)$$

The factor $(-)^{i-1}$ takes care of the change of sign which otherwise would affect F when the space-variables are interchanged. Hence, the relative velocity

$$u_r = u_2 - u_1 \text{ is}$$

$$u_r = u_r^{(+)} + u_r^{(-)} F, \quad (50)$$

with

$$u_r^{(\pm)} = \frac{1}{2} [u_b(2) \mp u_b(1) \pm u_a(2) - u_a(1)]. \quad (51)$$

A few algebraic manipulations allow us to rewrite Eq. (49) in the particularly simple form:

$$F = (\text{tanh } \Delta R)^\epsilon = \begin{cases} \tanh \Delta R & \text{for } \epsilon = +1 \\ \coth \Delta R & \text{for } \epsilon = -1 \end{cases} \quad (52)$$

where

$$\Delta R = \frac{1}{2} [R_b(2) - R_b(1) - R_a(2) + R_a(1)]. \quad (53)$$

Let us now consider the special situation $r_1 = r_2$. We see from Eq. (51) that in this case $u_r^{(+)} = 0$, but $u_r^{(-)} \neq 0$; from Eq. (53), we have also $\Delta R = 0$ for $r_1 = r_2$ and hence, $F = 0$ for $\epsilon = 1$, but $F = \infty$ for $\epsilon = -1$. Upon introduction of these values into Eq. (50) we obtain:

$$u_r(r_1 = r_2) = \begin{cases} 0 & \text{if } \epsilon = +1 \\ \infty & \text{if } \epsilon = -1 \end{cases}. \quad (54)$$

In other words, a symmetric amplitude implies that when the particles are very near each other, their relative velocity goes to zero and they will continue moving together; this result is what we are interpreting as an effective attraction between particles in the symmetric state. On the other hand, for $\epsilon = -1$ we have the opposite situation: if the particles happen to come close to each other, they acquire an unbounded relative velocity, which causes a violent separation; this is the result we are interpreting as an effective repulsion between particles in the antisymmetric state. Clearly, the value of ϵ must be selected according to the physical situation. For example, when the spin of the electrons is introduced in the usual quantum mechanics, ϵ is selected according to the relative spin orientations, in such a form as to guarantee attraction ($\epsilon = +1$) for antiparallel spins and repulsion ($\epsilon = -1$) for parallel spins; in other words, it is selected such that the total amplitude (the product of spin and configuration factors) be antisymmetric. This gives us an interesting insight into the exclusion principle from elementary considerations.

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