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COMPARISON BETWEEN TWO FORMULATIONS OF STOCHASTIC QUANTUM MECHANICS*

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RESUMEN

Se revisa la teoría estocástica de la mecánica cuántica propuesta por Kershaw[1], con el objeto de eliminar una inconsistencia interna que contiene; al hacer esta corrección, se recupera la teoría propuesta recientemente por uno de los autores del presente trabajo[2].

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ABSTRACT

The stochastic theory of quantum mechanics proposed by Kershaw[1] *is revised to eliminate an internal inconsistency contained in it; by doing this, one recovers the theory recently proposed by one of the authors*[2].

I. INTRODUCTION

In an interesting paper[1], Kershaw proposed a theory in which the stationary states of a classical system subject to random fluctuations of position, correspond to the stationary states described by Schrödinger's equation; however, the author did not succeed in covering the nonstationary case. At the same time, in a series of recent papers[2] it has been shown that quantum mechanics may be consistently interpreted as a stochastic process, both in the stationary and in the nonstationary case. Furthermore, the set of fundamental equations given in ref. (2) represents a generalization of those previously derived by Nelson[3] on the basis of stochastic (markovian) arguments.

A direct comparison of the theories proposed in refs. (1) and (2) shows, however, that they are not equivalent, one of their fundamental equations being different. It is the purpose of this note to explain the origin of the discrepancy and to eliminate it. With this aim we give first a short account of Kershaw's theory (section II), omitting most of the mathematical details, but clearly stating the essential ideas. We then point out the existence of an internal inconsistency in the theory, the removal of which carries us to the theory proposed in ref. (2) (section III).

II. KERSHAW'S THEORY

Let $w(t, x, dt, dx)$ represent the probability of a stochastic (markovian) particle to be at $x + dx$ at time $t + dt$, if it was at x at time t. The velocity c of the particle at point x and time t is defined by

$$\mathbf{c}(x,t) = \lim_{dt \to 0} \left(\frac{1}{dt}\right) \int dx\, w(t,x,dt,dx)\, d^3(dx) \,. \tag{1}$$

Writing for short

$$W(t,x,dt,dx) = w(t,x,dt,dx + \mathbf{c}dt) \,,$$

we require, from standard arguments[4,5], that

$$W(t,x,dt,dx) = W(dt,dx)$$

and (2)

$$\int (dx)^2\, W(dt,dx)\, d^3 dx = 2D dt \,,$$

where D is a constant[4]. Let us consider now a given time interval $\Delta t = ndt$ and let $n \to \infty$ and $dt \to 0$ in such a way that Δt remains fixed; we also write

$$\Delta x = \sum^{n} dx_i \,.$$

Then in the limit $dt \to 0$ we get for three dimensional motion[5]

$$W(\Delta t, \Delta x) = (4\pi D\Delta t)^{-3/2} \exp\{-(\Delta x)^2/4D\Delta t\} \,.$$

During Δt the total displacement of the particle is δx given by

$$\delta x = \mathbf{c}(x,t)\,\Delta t + \Delta x \,;$$

therefore the probability for the particle to go from x to $x + \delta x$ during Δt is[6]

$$P(\delta x, \Delta t, x, t) = (4\pi D\Delta t)^{-3/2} \exp\{(\delta x - \mathbf{c}\Delta t)^2/4D\Delta t\} \,. \tag{3}$$

From eq. (3) it follows that the probability $\rho(\mathbf{x}, t)$ of finding the particle at the point x at time t satisfies the following equation:[5]

$$\rho(\mathbf{x}, t+\Delta t) = \int \rho(\mathbf{x} - \delta\mathbf{x}, t)\, P(\delta\mathbf{x}, \Delta t, \mathbf{x} - \delta\mathbf{x}, t)\, d^3\delta x \; . \tag{4}$$

This relation may be alternatively written in differential form as follows:

$$\frac{\partial\rho}{\partial t} + \mathrm{div}\,[\rho\mathbf{c} - D\,\mathrm{grad}\,\rho] = 0 \; . \tag{5}$$

All equations written so far, are standard results in the theory of Brownian motion, eq. (5) being the continuity or Fokker-Planck equation and its integral form (4) a particular case of the Chapman-Kolmogorov equation for continuous time. In order to get an additional, dynamical law, Kershaw further assumes that

$$c_i(\mathbf{x}, t+\Delta t) = N^{-1}\int [c_i(\mathbf{x} - \delta\mathbf{x}, t) - \Delta t\, \partial_i V(\mathbf{x} - \delta\mathbf{x})] \times$$
$$\times\, \rho(\mathbf{x} - \delta\mathbf{x}, t)\, P(\delta\mathbf{x}, \Delta t, \mathbf{x} - \delta\mathbf{x}, t)\, d^3\delta x \tag{6}$$

where the normalization constant N given by

$$N = \int \rho(\mathbf{x} - \delta\mathbf{x}, t)\, P(\delta\mathbf{x}, \Delta t, \mathbf{x} - \delta\mathbf{x}, t)\, d^3\delta x \; . \tag{7}$$

In writting eq. (6) Kershaw argues as follows: to calculate the mean velocity of the particle at point x and time $t + \Delta t$ froms its velocity at point $\mathbf{x} - \delta\mathbf{x}$ and time t, we must add to the latter one the velocity increment due to the forces acting on the particle; this increment is assumed to be $-\Delta t\, \partial_i V$, V being the external potential per unit mass. The factor

$$\rho(\mathbf{x} - \delta\mathbf{x}, t)\, P(\delta\mathbf{x}, \Delta t, \mathbf{x} - \delta\mathbf{x}, t)$$

measures the total number of particles arriving at $\mathbf{x}$ from $\mathbf{x}-\delta\mathbf{x}$ in time Δt. Furthermore, a Taylor expansion allows us to transform eq. (6) into differential form

$$\frac{\partial c_i}{\partial t} + (\mathbf{c}\cdot\nabla)\, c_i = -\partial_i V(x) + D\left[\frac{\nabla^2 \rho c_i}{\rho} - c_i\,\frac{\nabla^2\rho}{\rho}\right] . \qquad (8)$$

Eqs. (5) and (8) constitute the fundamental system of equations proposed by Kershaw. For completeness, let us indicate how the stationary Schrödinger equation is derived from this system of equations. The stationary case is defined[1] as that in which the diffusion velocity $D\nabla\rho/\rho \equiv \boldsymbol{u}$ just counter-balances the mean total velocity, i.e., $\mathbf{c} = \boldsymbol{u}$ (in the language of ref. (2), this simply means that we set $\mathbf{v} = \mathbf{c} - \boldsymbol{u} = 0$). The continuity equation implies that in such a case both $\mathbf{c}$ and $\boldsymbol{u}$ are time independent. Some simple algebraic manipulations then allow us to rewrite eq. (8) in the form

$$2D^2\nabla^2\sqrt{\rho} + V\sqrt{\rho} = E m^{-1}\sqrt{\rho} \qquad (9)$$

where

$$E = \int \rho\left[\frac{1}{2}\, m u^2 + V\right] d^3x \, . \qquad (10)$$

Eq. (9) is Schrödinger's equation for a stationary state with the amplitude

$$\psi = \sqrt{\rho}\, \exp\left(-i\,\frac{E}{\hbar}\, t\right) ;$$

clearly, E is the average energy of the particle[2] when $\mathbf{v} = 0$. Let us now introduce the operators defined in ref. (2); in the Markovian case the systematic derivative is

$$\mathcal{D}_C = \frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla \qquad (11)$$

and the stochastic derivative is written as:

$$\mathcal{D}_S = \boldsymbol{u}\cdot\nabla + D\nabla^2 \; . \tag{12}$$

Then, the total (mean) derivative is

$$\mathcal{D} = \mathcal{D}_C + \mathcal{D}_S = \frac{\partial}{\partial t} + \mathbf{c}\cdot\nabla + D\nabla^2 \; , \tag{13}$$

where, as usual, the (mean) total velocity is

$$\mathbf{c} = \mathbf{v} + \boldsymbol{u} \; . \tag{14}$$

In terms of these operators, the gradient of eq. (5) for curlless v is written as

$$\mathcal{D}_S \mathbf{v} + \mathcal{D}_C \boldsymbol{u} = 0 \tag{15}$$

and eq. (8) takes the form

$$\mathcal{D}_C \mathbf{v} - \mathcal{D}_S \boldsymbol{u} = -\nabla V - 2\mathcal{D}_C \boldsymbol{u} \; . \tag{16}$$

It is now possible to compare both theories. With respect to eq. (15) there is not any trouble because it is the same in both theories; however, eq. (16) differs from the corresponding equation in ref. (2), where it lacks the term $2\mathcal{D}_C \boldsymbol{u}$:

$$\mathcal{D}_C \mathbf{v} - \mathcal{D}_S \boldsymbol{u} = -\nabla V \; . \tag{17}$$

To realize the consequences of this difference, let us proceed as follows. Firstly, we recall that from eq. (17), Schrödinger's equation follows in all cases, including the electromagnetic one if we introduce the corresponding external force instead of $-\nabla V$.[2] Secondly, we have just seen that from eq. (16) we may recover the stationary Schrödinger's equation; but in order to be more general, let us apply to this equation the procedure used in ref. (2) for deriving Schrödinger's equation. The result is

$$- 2D^2\nabla^2\psi + V_K \psi = 2iD \frac{\partial\psi}{\partial t} \quad , \tag{18}$$

where we have introduced the definition

$$V_K = V - 4D^2\nabla^2 S - 2D\phi \quad , \tag{19}$$

ϕ being such that

$$D\nabla\phi = (\mathbf{u}\cdot\nabla)\,\mathbf{v} \tag{20}$$

and, as usual, $\psi = \exp(R + iS)$ with $\mathbf{v} = 2D\nabla S$ and $\mathbf{u} = 2D\nabla R$. Clearly, in the general case, the right-hand-side of eq. (20) is not the gradient of a function, i.e., ϕ does not exist and we cannot write eq. (18). Interesting enough, when S does not depend on the coordinates and hence $\mathbf{v} = 0$, eq. (20) is satisfied with $\phi = 0$ and V_K reduces itself to V, eq. (18) going into the stationary Schrödinger equation. Still simpler: in the particular case $\mathbf{v} = 0$, the function $\mathbb{D}_C\,\mathbf{u}$ reduces to zero and hence eqs. (16) and (17) coincide, i.e., the two theories become mathematically equivalent.

However, the derivation of eq. (16) is not free from inconsistencies. In fact when writing down eq. (6) the *classical law* $\Delta \mathbf{c} = m^{-1}\mathbf{f}_0\Delta t$, where $\mathbf{f}_0$ is the external force, is assumed to be valid, although the equation of motion is given in this theory by eq. (16) and not by Newton's second law. This inconsistency leads to incorrect results, as for example eq. (18). Clearly, we can follow Kershaw's method and still obtain the correct result, if instead of considering only the force $\mathbf{f}_0$, we also take into account the effective force acting on the stochastic particle.

III. DERIVATION OF THE EQUATION OF MOTION

From the above discussion it follows that instead of eq. (6) we must write

$$c_i(\mathbf{x}, t+\Delta t) = N^{-1}\int [c_i(\mathbf{x}-\delta\mathbf{x}, t) - \Delta t\,\partial_i V(\mathbf{x}-\delta\mathbf{x}) + \Delta t\, F_i(\mathbf{x}-\delta\mathbf{x}, t)] \times$$

$$\times\, \rho(\mathbf{x}-\delta\mathbf{x}, t)\; P(\delta\mathbf{x}, \Delta t, \mathbf{x}-\delta\mathbf{x}, t)\; d^3\delta x\ , \qquad (21)$$

where $-\partial_i V + F_i$ is the total effective force (per unit mass) acting on the particle; F_i is a force that guarantees the internal consistency of the theory. The differential form of eq. (21) is

$$\mathcal{D}\mathbf{c} = -\nabla V + \mathbf{F} + 2\mathcal{D}_S\,\mathbf{c}\ . \qquad (22)$$

We can write similar relations for v_i and u_i, the velocity increments for the first being $\Delta t\,[-\partial_i V + F_i']$ and for the second $\Delta t\, F_i''$. The results are, in differential form,

$$\mathcal{D}\mathbf{v} = -\nabla V + \mathbf{F}' + 2\mathcal{D}_S\,\mathbf{v}\ , \qquad (23)$$

$$\mathcal{D}\mathbf{u} = \mathbf{F}'' + 2\mathcal{D}_S\,\mathbf{u}\ . \qquad (24)$$

Since, from eq. (14), the forces must satisfy the relation

$$\mathbf{F} = \mathbf{F}' + \mathbf{F}'' \qquad (25)$$

and furthermore, according to eq. (24), $\mathbf{F}''$ is given by

$$\mathbf{F}'' = \mathcal{D}_C\,\mathbf{u} - \mathcal{D}_S\,\mathbf{u}\ , \qquad (26)$$

it follows that only one of $\mathbf{F}$ or $\mathbf{F}'$ remains to be fixed. We fix it by demanding that the dynamical equations be invariant under time reversal ($\hat{T}$-transformation). Let us rewrite eq. (22) in the form

$$\mathcal{D}_C \mathbf{v} - \mathcal{D}_S \boldsymbol{u} = -\nabla V + \boldsymbol{F} + \mathcal{D}_S \mathbf{v} - \mathcal{D}_C \boldsymbol{u} \quad . \tag{27}$$

Then its $\hat{T}$-transform is[2]

$$\mathcal{D}_C \mathbf{v} - \mathcal{D}_S \boldsymbol{u} = -\nabla V + \widetilde{\boldsymbol{F}} - \mathcal{D}_S \mathbf{v} + \mathcal{D}_C \boldsymbol{u} \quad , \tag{28}$$

where $\widetilde{\boldsymbol{F}} = \hat{T}\boldsymbol{F}$. From eqs. (27) and (28) it follows that the simplest choice of $\boldsymbol{F}$ which makes eq. (27) $\hat{T}$-invariant

$$\boldsymbol{F} = -\mathcal{D}_S \mathbf{v} + \mathcal{D}_C \boldsymbol{u} = 2\,\mathcal{D}_C \boldsymbol{u} \tag{29}$$

and hence, that

$$\boldsymbol{F}' = \mathcal{D}\boldsymbol{u} \quad . \tag{30}$$

With $\boldsymbol{F}$ given by eq. (29), eq. (28) reduces itself to eq. (17), i.e., we recover the set of equations proposed in ref. (2). It is a simple task to convince oneself that the proposed selection of $\boldsymbol{F}$, $\boldsymbol{F}'$ and $\boldsymbol{F}''$ does not introduce any further inconsistency. In this form we see that the extra term $-2\mathcal{D}_C \boldsymbol{u}$ in Kershaw's theory (compare eqs. (16) and (17)) is due to his neglecting the force $\boldsymbol{F}$, which just cancels it.

The derivation of the basic equations of the stochastic theory of quantum mechanics presented here is not only less straightforward than the original one, but also more restrictive; nevertheless, it has the advantage of throwing a little light upon the complex nature of the motion of the stochastic particle, as it allows us to explicitely write the effective forces $\boldsymbol{F}$, $\boldsymbol{F}'$ and $\boldsymbol{F}''$; besides, it represents an alternative and more orthodox derivation of the fundamental stochastic equations of quantum mechanics.

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6. A different proof of this result may be found in L. de la Peña-Auerbach and Ana M. Cetto, University of Mexico preprint (to be published).