

THE ROLE OF THE MASS IN THE THEORY OF FERMIONS

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(Recibido: Septiembre 17, 1969)

RESUMEN

En este artículo se analiza la posibilidad de considerar a la masa como un operador en la ecuación de Dirac para fermiones libres.

Se obtiene la transformación de masa inducida por transformaciones generales de coordenadas que dejan invariante a la ecuación de Dirac, y que incluyen inversión de la masa. La invariancia bajo el grupo de Poincaré impone fuertes restricciones sobre la transformación de la masa, pero aparentemente permite la inversión de la masa.

ABSTRACT

We investigate the possibility of regarding the mass as an operator in Dirac's equation for free fermions.

The mass transformation induced by general coordinate transformations which leave invariant Dirac's equation is obtained and includes mass-reversal. Invariance under the Poincaré group imposes strong restrictions on the mass-transformation but seems to allow mass-reversal.

I. GENERAL TRANSFORMATION OF FREE SPINORS WITH VARIABLE MASS

Let us represent a Dirac spinor field which describes a free particle with mass m by $\psi(x, m)$, where x is a point in space-time. The corresponding Dirac equation will be written:

$$\left\{ i\gamma^\mu \frac{\partial}{\partial x^\mu} - m \right\} \psi(x, m) = 0 \quad (1)$$

in the well-known convention; \hbar and c are units of action and velocity respectively and the scalar product of two vectors is given by

$$a_\mu b^\mu = a^\mu b^\nu g_{\mu\nu}$$

$$g_{\mu\nu} = 0, \text{ for } \mu \neq \nu; g_{00} = -g_{11} = -g_{22} = -g_{33} = 1.$$

Equation (1) is obtained by an observer in a reference frame R .

Let us assume that a passive transformation is carried out to the new frame R' . We make the assumption that the free-field under consideration is now described by the spinor $\psi'(x', m')$ where, in general, the geometrical coordinates x' are connected to the former ones, x , by the relations:

$$x'^{\mu} = f^{\mu}(x), \quad (2a)$$

which can be inverted,

$$x^{\alpha} = g^{\alpha}(x'). \quad (2b)$$

The new point is that we allow the mass, as measured in the new frame, to be different from m . We now impose the condition that Dirac's equation (1) be invariant,

$$\left\{ i\gamma^{\mu} \frac{\partial}{\partial x^{\mu}} - m' \right\} \psi'(x', m') = 0. \quad (3)$$

The transformations (2) induce a correlation between the fields $\psi(x, m)$ and $\psi'(x', m')$:

$$\psi'(x', m') = S(x', m'; x, m) \psi(x, m), \quad (4)$$

the operator S being non-singular.

The conditions for the invariance of Dirac's equation are now obtained in the usual way, and can be written as

$$S^{-1} m' S - i S^{-1} \gamma^{\mu} \frac{\partial x^{\nu}}{\partial x'^{\mu}} \frac{\partial \psi}{\partial x^{\nu}} = \pm m. \quad (5)$$

The operator S , besides being a matrix which acts on the spinor index of the fields ψ , is assumed to depend on the points x and x' and on the masses m and m' .

In the particular case of a Poincaré transformation (a, L) , where x' is given in terms of x , as

$$x' = a + Lx,$$

we have

$$\frac{\partial x^\nu}{\partial x'^\mu} = (L^{-1})^\nu{}_\mu, \quad (6)$$

and the conditions (5) are reduced to the well-known relations:

$$\gamma^\mu (L^{-1})^\nu{}_\mu = S \gamma^\nu S^{-1}$$

$$\frac{\partial \mathcal{S}}{\partial x^\nu} = \frac{\partial \mathcal{S}}{\partial x'^\mu} = 0; \quad m = m'. \quad (7)$$

II. MASS GENERATED BY GAUGE FUNCTIONS

It is well known that the field ψ may undergo phase transformations of the type

$$\psi'(x) = e^{i\alpha} \psi(x), \quad (8)$$

where α is a real constant.

When, however, the phase transformation is of the first kind:

$$\psi'(x) = e^{ie\varphi(x)} \psi(x), \quad (9)$$

that is, when the phase is a function $\varphi(x)$ of the coordinates x , it is usually stated that the invariance of Dirac's equation is broken for free fields and restored if the field ψ interacts with an electromagnetic field $A^\mu(x)$ according to the equation

$$\left\{ \gamma^\mu \left(i \frac{\partial}{\partial x^\mu} - e A_\mu \right) - m \right\} \psi(x) = 0, \quad (10)$$

in which case the field A^μ undergoes a gauge transformation of the second kind induced by equation (9):

$$A'_\mu(x) = A_\mu(x) - \frac{\partial \varphi(x)}{\partial x^\mu}. \quad (11)$$

This type of gauge transformation rests on the assumption that the field ψ describes charged particles with constant mass.

We now consider the case of free spinor fields and assume the invariance of Dirac's equation under the transformation (9). In this case, the mass cannot be constant. Indeed if one sets

$$\begin{aligned} x' &= x \\ S(x, x'; m, m') &= e^{i\Lambda(x)} \end{aligned} \quad (12)$$

where $\Lambda(x)$ is a real function of x , conditions (5) reduce to

$$m = m' + \gamma^\alpha \frac{\partial \Lambda(x)}{\partial x^\alpha}. \quad (13)$$

The gauge function $\Lambda(x)$ appears as a kind of internal field which generates a mass which, according to relation (13), is a matrix depending on x :

$$m_{\alpha\beta} = m'_{\alpha\beta} + (\gamma^\mu)_{\alpha\beta} \frac{\partial \Lambda}{\partial x^\mu}, \quad (14)$$

if in some reference frame S' the mass is a constant matrix.

Under this view, the gauge transformation (12) is a mass-transformation.

Let us now assume that the field $\psi(x, m)$ describes particles with charge e which is revealed through interaction with an electromagnetic field $A^\mu(x)$. In this case we let the field ψ undergo the phase transformation

$$\psi'(x, m') = e^{i[\Lambda(x) + e\varphi(x)]} \psi(x, m) \quad (15)$$

and assume the invariance of the equation:

$$\left[\gamma^\mu \left(i \frac{\partial}{\partial x^\mu} - eA'_\mu(x) \right) - m' \right] \psi'(x, m') = 0 \quad (16)$$

Note that the charge e multiplies $\varphi(x)$ only, in equation (15). The assumed invariance will be satisfied by equations (11) and (14).

III. IMPROPER-MASS-TRANSFORMATIONS

The components of the spinor $\psi'(x', m')$ transform under equation (4) according to the relation:

$$\psi'_\alpha(x', m') = S_{\alpha\beta}(x', m'; x, m) \psi_\beta(x, m) \quad (17)$$

An interesting example of such a matrix is one which has the γ^5 -matrix as a factor

$$S(x', m'; x, m) = \gamma^5 F(x', m'; x, m) \quad (18)$$

In this case, the invariance condition imposed on equation (3) leads to the relations,

$$F^{-1} \gamma^5 \gamma^\mu \frac{\partial x^\nu}{\partial x'^\mu} \gamma^5 F = \pm \gamma^\nu$$

$$F^{-1} \gamma^5 m' \gamma^5 F - i F^{-1} \gamma^5 \gamma^\mu \frac{\partial x^\nu}{\partial x'^\mu} \gamma^5 \frac{\partial F}{\partial x^\nu} = \pm m, \quad (19)$$

which results from substituting (18) into eqs. (5).

One must clearly, be careful and regard m and m' as functions of x and of the matrices γ^μ .

A particular case is the mass-reversal transformation¹, for which F is the identity. Equation (3), written now as

$$\left\{ i \gamma^\mu \frac{\partial}{\partial x^\mu} - m' \right\} \psi'(x, m) = 0$$

then becomes

$$\left\{ i \gamma^\mu \frac{\partial}{\partial x^\mu} - m' \right\} \gamma^5 \psi(x, m) = 0 .$$

Or, equivalently

$$\left\{ -i \gamma^\mu \frac{\partial}{\partial x^\mu} - \gamma^5 m' \gamma^5 \right\} \psi(x, m) = 0 .$$

One must therefore have

$$\gamma^5 m' \gamma^5 = -m . \quad (20)$$

The γ^5 -transformation therefore belongs to the type of transformation (19) where the minus sign is to be taken on the right-hand side.

In general, transformations (5) may be written:

$$S^{-1} \gamma^\mu \frac{\partial x^\nu}{\partial x'^\mu} S = e^{i\theta} \gamma^\mu$$

$$S^{-1} m' S - i S^{-1} \gamma^\mu \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial S}{\partial x^\nu} = e^{i\theta} m \quad (21)$$

where θ is a constant. The constant phase factor $e^{i\theta}$ may be absorbed into the field ψ . We shall call proper-mass transformations those for which $\theta = 0$ and improper-mass transformations those for which $\theta = \pi$. Mass-reversal transformation (20) is the simplest improper-mass transformation.

IV. THE RESTRICTIONS IMPOSED BY THE POINCARÉ GROUP

The hermicity conditions that equations (5) must obey, under the choice

$$(\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0, \quad (22)$$

are the following:

$$\gamma^0 S^\dagger \gamma^0 = S^{-1}$$

$$\gamma^\mu \left(\frac{\partial S}{\partial x'^\mu} S^{-1} \right) = \left(\frac{\partial S}{\partial x'^\mu} S^{-1} \right) \gamma^\mu, \quad (23)$$

provided that the hermitian conjugate of the mass is of the form

$$m^\dagger = \gamma^0 m \gamma^0. \quad (23a)$$

The first of eqs. (23) is well known. The second equation (23) is satisfied by the ansatz

$$S(x, x') = f(x, x') G \quad (24)$$

where $f(x, x')$ is an ordinary function of modulus one

$$f^* f = 1 \quad (25)$$

and G is a matrix in the spinor space.

The usual treatment of Dirac's equation is a consequence of the theory of the representations of the Poincaré group. This theory requires that the operator $P^\mu P_\mu$ be a multiple of the identity, where P^μ are the generators of space-time translations:

$$P^\mu P_\mu = m_0^2 I, \quad (26)$$

where m_0 is a number and I is the identity.

In this case, the term which contains the space-time derivatives of the transformation operator S in equation (5) must vanish. Equations (5) reduce to:

$$S^{-1} \gamma^\mu (L^{-1})^\nu_\mu S = \pm \gamma^\nu$$

$$S^{-1} m' S = \pm m, \quad (27)$$

provided that it is assumed that

$$m'^2 = m^2 = m_0^2 I.$$

It is clear that, in general, formulae (5) for the mass-operator would violate condition (26).

If equation (26) holds, the gauge transformations (12) cannot give rise to equation (13). The Λ -function would have to fulfill the condition

$$g^{\alpha\beta} \frac{\partial \Lambda}{\partial x^\alpha} \frac{\partial \Lambda}{\partial x^\beta} = 0 \quad (28)$$

and the mass operator would have to be such that

$$m' \gamma^\alpha + \gamma^\alpha m' = 0, \quad (29)$$

i.e. the mass would have to be proportional to γ^5 , which would violate the hermiticity requirement (23a). Thus invariance of Dirac's equation both under the Poincaré and the gauge group (12) requires a vector field transforming like in equation (11).

We see that mass-reversal (equation (20)) is a special case of formulae (27), where

$$S = \gamma^5,$$

$$(L^{-1})^\nu{}_\mu \rightarrow \delta^\nu{}_\mu$$

and the minus sign holds in the right hand side of eqs. (27).

Mass reversal would be satisfied by an ansatz of the form:

$$m = m_0 a_\mu \gamma^\mu$$

where a_μ is a time-like vector (for instance, normal to a space-like surface at every point) and m_0 is a real number. In other words,

$$a_\mu a^\mu = 1,$$

so that

$$m^2 = m_0^2$$

and

$$m^+ = \gamma^0 m \gamma^0.$$

Thus an equation invariant under mass-reversal

$$\gamma^\mu \left\{ i \frac{\partial}{\partial x^\mu} - m_0 a_\mu(x) \right\} \psi(x) = 0$$

is similar to the equation of a massless charged particle in interaction with an electromagnetic field.

REFERENCE

1. Tomno, J. - Nuovo Cimento 1, 226 (1955).

