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BASES FOR REPRESENTATIONS OF THE UNITARY GROUP IN THE ROTATION GROUP CHAIN

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ABSTRACT: **The highest weight polynomials of the various irreducible representations of $R(n)$ contained in the reduction of the $R(n)$ part of an irreducible representation (k_1, k_2, k_3) of $U(n)$ are obtained using Littlewood's Theorem for the reduction.**

I. INTRODUCTION

A procedure was developed in reference (1) to obtain polynomial bases for any given irreducible integral representation (IR) of $U(3)$ (where $U(n)$ stands for the unitary group in n-dimensions) such that, with respect to that basis, a given $R(3)$ subgroup ($R(n)$ stands for the rotation group in n-dimensions) is explicitly reduced into blocks. The procedure makes use of a theorem of Littlewood[2] for the reduction into irreducible parts with respect to $R(n)$ of an IR of $U(n)$ when restricted to $R(n)$. In what follows each of these irreducible parts will be called an IR of $R(n)$ contained in an IR of $U(n)$. In this article we obtain, by the same procedure, similar bases for any given IR of $U(4)$ so that a given $R(4)$ subgroup is explicitly reduced with respect to that basis. The construction of such polynomial basis enables one to evaluate fractional parentage coefficients for the spin-isospin states in supermultiplet theory[3], since these coefficients are given by scalar products of these polynomials. The spin-isospin operators can be expressed in terms of the generators of an $SU(2) \times SU(2)$ group which is homomorphic to the $R(4)$ group. The procedure is generalized to the case of the chain $U(n) \supset R(n)$ for IRs (k_1, k_2, k_3) of $U(n)$, with $n > 6$.

II. Equations determining the highest weight polynomials (h.w.p.)

The infinitesimal generators of $U(4)$ can be realized in terms of boson creation and annihilation operators, a^s_m and a^{*s}_m as

$$C^{m'}_{m} = \sum_{s=1}^{4} a^s_m a^{*s}_{m'} , \quad m, m' = 1, .., 4 \tag{i}$$

and the following linear combinations

$$\begin{gathered} C^1_1 - C^4_4, \ C^2_2 - C^3_3, \ C^2_1 - C^4_3, \\ C^3_1 - C^4_2, \ C^1_2 - C^3_4, \ C^1_3 - C^2_4, \end{gathered} \tag{ii}$$

have the same commutation relations as the infinitesimal generators of the $R(4)$ subgroup. Any basis of the IR of $U(4)$ corresponding to the partition $(k) = (k_1, k_2, k_3, 0)$ then consists of polynomials $P(a^s_m)$ acting on the vacuum ket $|0>$ where $P(a^s_m)$ are the solutions of the following equations,[4]

$$C_{11} P = k_1 P, \ C_{22} P = k_2 P, \ C_{33} P = k_3 P, \ C_{12} P = C_{13} P = C_{23} P = 0, \tag{II.1}$$

where

$$C_{ss'} = \sum_{m=1}^{4} a^s_m a^{*s'}_m$$

and the a^{*s}_m are equivalent to the differential operators $\partial / \partial a^s_m$ when acting on polynomials $P(a^s_m)$.

A polynomial P belonging to a basis of the IR (k) of $U(4)$ is of highest weight (λ_1, λ_2) with respect to the group $R(4)$, whose generators are given in (ii), if and only if it is further a solution of

$$\left(C^1_1 - C^4_4\right) P = \lambda_1 P, \tag{II.2}$$

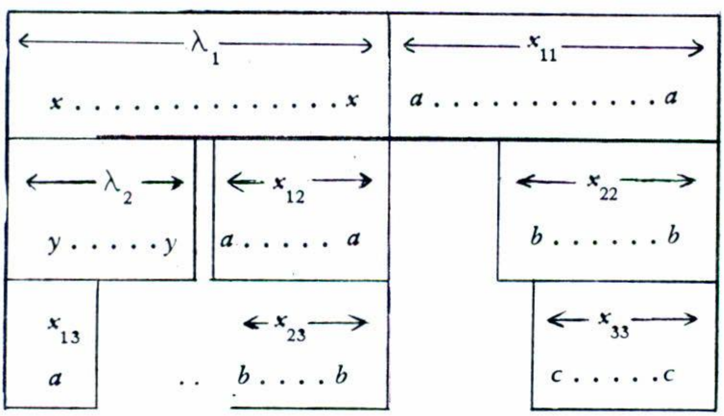
$$\left(C^2_2 - C^3_3\right) P = \lambda_2 P, \tag{II.3}$$

$$\left(C^2_1 - C^4_3\right) P = 0, \tag{II.4}$$

$$\left(C^3_1 - C^4_2\right) P = 0. \tag{II.5}$$

III. Elementary Permissible Diagrams

We use the same theorem quoted in reference 1 for the reduction with respect to $O(4)$ of an IR (k) of $U(4)$ when restricted to $O(4)$, and rearrange the Littlewood diagrams obtained in that connection, just as in reference 1. We find that to each diagram



(D′)

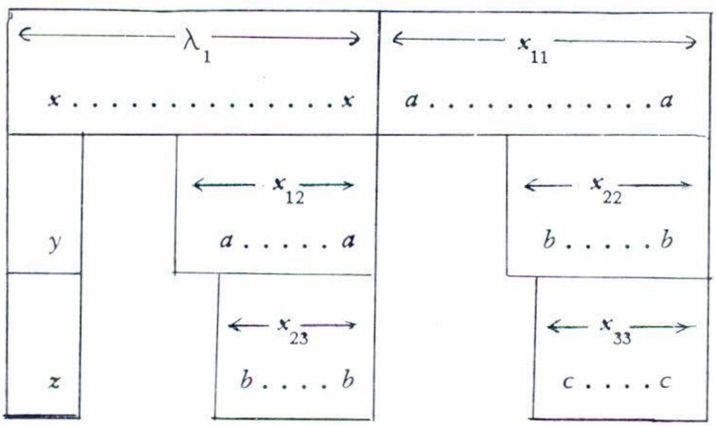
corresponds an IR (λ_1, λ_2) of $O(4)$ contained in the IR (k) of $U(4)$. Whe $\lambda_2 > 0$ the two IRs $(\lambda_1, \lambda_2), (\lambda_1, -\lambda_2)$ of $R(4)$ contained in the IR (k) correspond to D$'$. In D$'$ the numbers $\lambda_1, \lambda_2, x_{11}, x_{12} \ldots x_{23}, x_{33}$ of symbc 'x', 'y', 'a', 'b' are shown in the different rows of the diagram and satisf; the following relations

$$\lambda_1 + x_{11} = k_1, \; \lambda_2 + x_{12} + x_{22} = k_2,$$

$$x_{13} + x_{23} + x_{33} = k_3, \; \lambda_1 \geqslant \lambda_2 + x_{12} \geqslant x_{13} + x_{23}, \tag{III.1}$$

$$\lambda_2 \geqslant x_{13}, x_{11} \geqslant x_{22} \geqslant x_{33}, x_{11} + x_{12} \geqslant x_{22} + x_{23}, x_{13} \leqslant 1,$$

Similarly to the diagram



(D''

where

$$\lambda_1 + x_{11} = k_1, \; \lambda_2 + x_{22} + x_{12} = k_2,$$

$$1 + x_{23} + x_{33} = k_3, \; \lambda_1 \geqslant x_{12} + 1 \geqslant x_{23} + 1, \quad \text{(III.}$$

$$x_{11} \geqslant x_{22} \geqslant x_{33}, \; x_{11} + x_{12} \geqslant x_{22} + x_{23}$$

corresponds the IR $(\lambda_1, 0)$ of $R(4)$ contained in the IR (k) of $U(4)$.

The diagrams D′ and D″ can be split columnwise into what are called elementary permissible diagrams (e.p.d.). A diagram is permissible when it is of the form D′ or D″ with particular values for λ_1, λ_2, x_{ij}, k_i satisfying the conditions (III.1) or (III.2) as the case may be, and is an e.p.d. if it cannot be further split columnwise into two permissible diagrams. The following are all the possible e.p.d.:

$$e_1 = x, \qquad e_2 = \begin{matrix} x \\ y \end{matrix}, \qquad e_3 = \begin{matrix} x & x \\ y & a \\ a & \end{matrix},$$

$$e_4 = \begin{matrix} x & a \\ y & \\ a & \end{matrix}, \qquad e_5 = \begin{matrix} x & x & a & a \\ y & y & & \\ b & b & & \end{matrix}, \qquad e_6 = \begin{matrix} x & x & a \\ y & a & \\ b & b & \end{matrix},$$

$$e_7 = \begin{matrix} x & a & a \\ y & b & \\ b & & \end{matrix}, \qquad e_8 = \begin{matrix} x & x \\ a & a \end{matrix}, \qquad e_9 = \begin{matrix} x & a \\ a & \end{matrix},$$

$$e_{10} = \begin{matrix} x & x \\ a & a \\ b & b \end{matrix}, \qquad e_{11} = \begin{matrix} x & a \\ a & b \\ b & \end{matrix}, \qquad e_{12} = a\ a,$$

$$e_{13} = \begin{matrix} a & a \\ b & b \end{matrix}, \qquad e_{14} = \begin{matrix} a & a \\ b & b \\ c & c \end{matrix}, \qquad e_{15} = \begin{matrix} x \\ y \\ z \end{matrix}.$$

Splitting the diagrams D′ or D″ into e.p.d. can be done in several ways, but we split them in a unique way by making it a convention to include a non-permissible column in an e.p.d. by adjoining to the column the possible column nearest to it as we go from left to right in the diagram. For example, the diagram

$$\begin{matrix} x & x & a & a \\ a & a & & \end{matrix}$$

is split as

$$\begin{matrix} x\,x & . & a\,a \\ a\,a & & \end{matrix}$$

and not as

$$\begin{matrix} x\,a & . & x\,a \\ a & & a \end{matrix}$$

The diagram D$'$ splits into q_1 e.p.d. of the form e_1, q_2 e.p.d. of the form e_2 and so on where

$$\begin{aligned} & q_1 = \lambda_1 - \lambda_2 - x_{12},\ q_2 = \lambda_2 - m_3 - x_{13},\ q_3 = \min(x_{13}, m_2), \\ & q_4 = x_{13} - q_3,\ q_5 = \left[\frac{1}{2} m_3\right],\ q_6 = \min(m_3 - 2q_5, m_1), \\ & q_7 = m_3 - 2q_5 - q_6,\ q_8 = \left[\frac{1}{2}(m_2 - q_3)\right],\ q_9 = m_2 - 2q_8 - q_3, \\ & q_{10} = \left[\frac{1}{2}(m_1 - q_6)\right],\ q_{11} = m_1 - 2q_{10} - q_6, \\ & q_{12} = \left[\frac{1}{2}(x_{11} - x_{22} - 2q_5 - q_9 - q_4 - q_6 - q_7)\right], \\ & q_{13} = \left[\frac{1}{2}(x_{22} - x_{33} - q_7 - q_{11})\right],\ q_{14} = \frac{1}{2} x_{33} \end{aligned} \tag{III.3}$$

where

$$m_1 = \min(x_{12}, x_{23}),\ m_2 = x_{12} - m_1,\ m_3 = x_{23} - m_1$$

and $[q]$ denotes the integral part of the number q. The diagram D$''$ splits into q'_1 e.p.d. of the form e_1, q'_2 e.p.d. of the form e_2 and so on where q'_i is obtained by setting in the expression for q_i in (III.3) $x_{13} = 0$, $m_1 = x_{23}$ and $q'_{15} = 1$.