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RELATIVISTIC FORM FACTOR OF THE TRITON AND THE ALPHA PARTICLE*

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ABSTRACT: Using the methods of group theory, we obtain an explicit expression for the form factor of a few-particle system, which is valid at very high momentum transfers. The formalism is applied to obtain the form factors of the three and four-nucleon systems. An analysis of the error made in the non-relativistic approximation is performed.

I. INTRODUCTION

In the usual approximation to first order in the electromagnetic interaction, the differential cross section for electron scattering is expressed in terms of the charge form factor for spinless targets and in terms of the charge and magnetic form factors for targets with spin ½. Since the interpretation of the magnetic form factor results is uncertain, at least for nuclei, one usually deals only with the charge form-factor data, which provide the most reliable information for the system under analysis. In this paper we

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shall also restrict our considerations to the charge form factor of a few-body-system, and propose an expression for it, useful at high-momentum transfers. Since experimental results for electron scattering at very high momentum transfers are now available for the alpha particle[1] and for the proton[2] and, in a somewhat more limited range of values of the momentum transfer for the triton and ^3He[3] the method we propose can be applied in these cases.

To first order in the electromagnetic interaction, the electron scattering amplitude A is given by

$$A = e\,\overline{U}(k')\,\gamma^{\mu}\,U(k)\,\frac{1}{q^2}\,\langle p'\beta \,|\, J_{\mu} \,|\, p\,\alpha \rangle \tag{1.1}$$

corresponding to the Feynman diagram of fig. 1. In (1.1) p and p' are the initial and final four momentum of the target, k and k' are the corresponding momenta for the electron, and q is the momentum transfer,

$$q_{\mu} = p'_{\mu} - p_{\mu} = k_{\mu} - k'_{\mu}\,. \tag{1.2}$$

The labels α and β stand for all other quantum numbers characterizing the target initial and final states, J_{μ} is the electric current operator and U_k is the electron Dirac spinor.

If we consider elastic collisions and restrict ourselves to the electric form factor, J_{μ} is equivalent to the operator

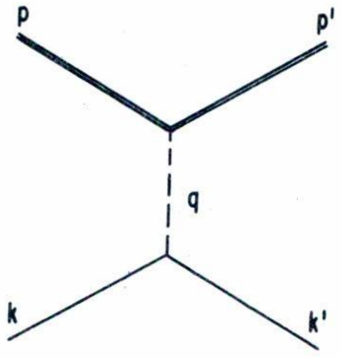
$$\underset{\sim}{Q}\,\frac{\overleftrightarrow{\partial}}{\partial x_{\mu}}\,I$$

with I the unit matrix and Q the target charge; hence, the matrix element appearing in (1.1) becomes

$$e\,(p'_{\mu} + p_{\mu})\,\langle p'\beta \,|\, p\,\alpha \rangle\,\delta_{\alpha\beta}\,. \tag{1.3}$$

For scalar point targets this is the only contribution although for extended systems this is not the case.

The electric or charge form factor is, then, proportional to the scalar form factor, defined as the overlap of the wave function of the system when it moves with momentum p, relative to some fixed reference frame, with the wave function of the system when it moves, relative to the same



frame, with momentum $p' = p + q$.

In the non-relativistic approximation, taking this overlap is equivalent to applying a finite Galilean transformation to $\psi(p)$ and then taking the overlap with this wave function $\psi(p)$. In momentum space one applies the operator $\exp(iq \cdot r)$ to $\psi(p)$ and the scalar form factor then becomes the Fourier transform of $|\psi(p)|^2$, which is proportional to the density. This is the usual interpretation of the charge form factor.

In the relativistic case (i.e. for high values of q^2) one should apply to $\psi(p)$ the finite Lorentz transformation operator $\exp(iL(q))$, the scalar form factor F then being given by

$$F = \langle \psi(p') | \psi(p) \rangle = \langle \psi(p) | e^{iL(q)} | \psi(p) \rangle \qquad (1.4)$$

where

$$L = \sum_{i=1}^{3} L_{0i}\, q_i$$

and L_{0i} stands for the generators of the Lorentz group.

Our purpose in this paper is to obtain an expression for F defined in (1.4), when $\psi(p)$ refers to an extended system formed by several particles. It is indeed possible to obtain an algebraic expression for F if we use the methods of group theory and classify the state vectors of the system by the irreducible representations of a chain of groups. Since to calculate (1.4) the transformation properties of ψ under the Lorentz group $O(3,1)$ must be specified, the chain of groups must include at least a non-compact group that contains as a subgroup an isomorphic group to $O(3,1)$.

This approach has been used to describe the relativistic internal motion of a two-body system[4]. The set of kinematical variables obey a certain algebra and this gives rise to the "group of relativistic motion" G, that contains the Lorentz group as a subgroup. Using unitary representations of G one can then obtain a basis in terms of which the internal motion can be described. This is equivalent to using, for the relative motion, a representation in which the Casimir operators of G are diagonal. In fact, the states of the basis are labelled by the quantum numbers provided by the little group G_p, which is the subgroup of G that leaves the four momentum p of the center of mass of the pair of particles invariant.

One can generalize this approach for a few-body system by describing the internal motion in terms of relative coordinates for each pair of particles and then, as in the case above, use an algebraic basis to describe the internal relative motion of each pair. One encounters two difficulties with this method First, the pairs are not independent of each other and second, the center of mass of each pair move relatively to each other and to the center of mass of the whole system.

In group-theoretical language the second difficulty can be expressed by saying that each little group of internal motion of the pair (ij), $G_{p_{ij}}^{(ij)}$ which provides the algebraic basis for this pair, corresponds to a different four momentum p_{ij}. One can solve this problem under the following assumption: *The relative motion of one pair with respect to another is non-relativistic.* In that case, to transform from the center of mass of each pair to the center of mass of the system, a Galilean transformation may be used. Since such a transformation does not alter the relative momentum, our assumption means that all the internal little groups $G_{p_{ii}}^{(ij)}$ correspond to the same momentum P, that of the center of mass.

Regarding the first problem we mentioned above, it can be solved by using harmonic-oscillator states. This has indeed been done for the 3 and 4-body problem by Moshinsky[5] for the non-relativistic case, using Jacobi coordinates. Specifically, he has obtained a linear expression for the scalar form factor of these systems, in terms of single-particle matrix elements, in which the wave functions are classified by the group $U(3)$ the symmetry group of the oscillator.

In order to obtain a relativistic generalization for the form factor, we assume that the common dynamical group is equal to the non-compact group $U(3,1)$[6] and, using our main assumption, employ the same expression for the scalar form factor as Moshinsky[5], except that we replace the single-particle matrix elements by those obtained with respect to a basis, whose wave functions are classified by the irreducible representations of the chain of groups

$$U(3,1) \supset U(3) \supset O(3)\,, \tag{1.6}$$

where the last group is the three-dimensional rotation group.

In the next section we analyze the main points in the group-theoretical approach to the problem and in section III we obtain an explicit algebraic expression for the form factor of a few-body system. We then apply the formulation to the triton and the α particle and compare our results with the experimental values.

II. CLASSIFICATION CHAIN USING NON-COMPACT GROUPS.

In this section we discuss the application of non-compact groups to the problem of evaluating the form factor of an extended system, with internal degrees of freedom. We shall first indicate how one is forced to introduce non-compact groups if a covariant formulation of the problem is desired.

Let us first consider the description of the intrinsic properties of the system; this implies using a reference frame fixed in its center of mass. Our purpose is to specify the state of the system by a dynamical chain of groups, which would allow us to obtain an algebraic expression for the scalar form factor and at the same time be of some physical interest. Since among the quantum numbers characterizing the state we would like to include the spin, the classification chain should contain the angular momentum group $O(3)$. Among those groups containing $O(3)$ that have been fruitfully used in particle and nuclear physics one can think of $U(3)$ the symmetry group of the harmonic oscillator; this will lead to the simplest formulation, although other classification chains are possible.

Therefore, in the rest frame, one could replace the state vector of a system by a linear combination of states classified by the chain of groups

$$U(3) \supset O(3)\,, \tag{2.1}$$

which then provides the algebraic basis mentioned in the introduction.

The generators of $U(3)$ are C_{ij} , $i, j = 1,2,3$ with the usual commutation rules

$$[C_{ij}\,, C_{kl}] = C_{il}\,\delta_{jk} - C_{kj}\,\delta_{il} \tag{2.2a}$$

and those of $O(3)$, S_{ij}, are given by

$$S_{ij} = C_{ij} - C_{ji} . \tag{2.2b}$$

Let us now look at the system from a reference frame with respect to which it moves with four momentum p_μ, $\mu = 0,1,2,3$ and connected with the rest frame through a Lorentz transformation. Since the operators C_{ij} have tensorial properties under the Lorentz group one is forced to consider, in order to obtain a covariant formulation, the operators $C_{\mu\nu}$, obeying the covariant commutation rules,

$$[C_{\mu\nu}, C_{\lambda\rho}] = -g_{\nu\lambda} C_{\mu\rho} + g_{\rho\mu} C_{\lambda\nu}, \tag{2.3}$$

with the metric tensor $g_{\mu\nu}$ given by $g_{00} = 1$ and $g_{ij} = -\delta_{ij}$, $i,j = 1,2,3$. Equation (2.3) defines the Lie algebra of the non-compact group $U(3,1)$, which has been previously discussed in connection with current algebra techniques in elementary particle physics[7] and with direct reactions in Nuclear Theory[6].

Mathematical details concerning $U(3,1)$ can be found in references (6) and (7). We simply mention here that the completely symmetrical states, corresponding to a single-rowed Young diagram, are labelled by a single quantum number N, real and negative for unitary representations, related to the mean square radius of the system[8].

In order to complete the classification of the state one has to analyze how is the chain (2.1) altered when referred to a frame different from the rest frame. In particular, one should impose the condition that the state vectors have a well defined four-momentum p_μ. We shall proceed by constructing linear combination of the generators $C_{\mu\nu}$ such that the Casimir operators formed with them are Lorentz invariants, on the one hand, and on the other, reduce to the Casimir operators of the chain (2.1) when evaluated in the rest reference frame. It has been shown[6] that the desired linear combination is given by

$$\hat{C}_{\mu\nu} = \sum_{\lambda\rho} \theta_\mu^\lambda \, \theta_\nu^\rho \, C_{\lambda\rho} \tag{2.4}$$

with

$$\theta_\mu^\nu \equiv \delta_\mu^\nu - v_\mu v^\nu \tag{2.5}$$

and where $v_\mu = p_\mu / \sqrt{|p|^2}$ is the four-velocity. Using the definition (2.5) one can readily prove that $\hat{C}_{\mu\nu}$ reduce to C_{ij} when the three velocity $\mathbf{v} = 0$, i.e. when referred to the rest frame. One can construct the Casimir operator Γ

$$\Gamma = \sum_\mu \hat{C}^\mu_\mu \tag{2.6}$$

which commutes with all $\hat{C}_{\mu\nu}$ and is obviously a Lorentz invariant. Therefore, if a wave function is eigenstate of Γ, the eigenvalues (say γ) are defined in an invariant way and coincide with the number of quanta in the rest frame. Using linear combinations of C_{ij} one can define the generators of $O(3)$; with the same linear combinations of $\hat{C}_{\mu\nu}$ one defines in an invariant way the eigenvalues of the invariant Casimir operator of $O(3)$.

We call the group generated by (2.4) and whose first order Casimir operator is Γ, $U(3)_p$ and the corresponding orthogonal subgroup $O(3)_p$.

The eigenvector of the classification chain will now be denoted by

$$\psi_{Nnl}(p) \tag{2.7}$$

where N specifies the irreducible representation of $U(3,1)$ and n and l those of $U(3)_p$ and $O(3)_p$.

Let us now look at a two body problem and indicate how the basis (2.7) can be used as mentioned in the introduction. We assume that the system moves with four-momentum p_μ. The wave function Ψ describing the relative motion can now be expanded, in momentum space, as a linear combination of the states (2.7),

$$\Psi(p) = \sum_{Nnl} a_{Nnl}\, \psi_{Nnl}(p) \tag{2.8}$$

The form factor, defined as in eq. (2.4), will now be expressed as a linear combination of the following scalar products

$$\langle \psi_{Nn'l'}(p+q) | \psi_{Nnl}(p) \rangle \tag{2.9}$$

for which closed algebraic expressions can be obtained, as shown in the appendix.

In other words, using the classification chain introduced above, one can obtain an algebraic expression for the form factor of the two-body system valid in the relativistic limit, i.e. for very high values of the momentum transfer q^2.

We now want to generalize this approach for more than two bodies. The difficulty one encounters immediately is that there are several four momenta coming in, one for each pair of particles. This introduces the problem that different groups $U(3)_p$ should be used, one for each relative coordinate. We assume now that the relative motion within the system is non-relativistic, in such a way that to transform from the center of mass of one pair of particles to the center of mass of any other pair, a Galilean transformation is enough. This has the advantage that the transformation does not alter the relative momentum, therefore allowing us to use a single group $U(3)_p$, where p is the total momentum of the whole system. Furthermore, the treatment of the different relative coordinates can be done as in the non-relativistic case. This has been done for the three and four-body problem by Moshinsky[5] using Jacobi coordinates and harmonic-oscillator states, i.e. states classified by the chain of groups (2.1).

Combining our discussion of the two body system with the technique developed by Moshinsky and co-workers and using the assumption mentioned above, one can obtain a relativistic expression for the form factor of a few-body system. In the next section we review the non-relativistic theory of the form factor and indicate how the relativistic generalization is done. We finally apply our formulation to discuss the body form factors of the three nucleon system and the α particle.

III. REVIEW AND GENERALIZATION OF THE NON-RELATIVISTIC THEORY.

We shall first review the main points in the approach of Moshinsky et al by which they obtain the form factor of the n-nucleon system[5,9]. We then propose a possible relativistic generalization to this theory.

Moshinsky et al start off by definning the probability density operator of finding either a proton ($\nu = 0$) or a neutron ($\nu = 1$) at a definite point $\mathbf{x}$, meassured with respect to the center-of-mass coordinate $\mathbf{X}$,

$$\hat{\Pi}_\nu(\mathbf{x}) = \sum_{s=1}^{n} \delta[\mathbf{x} - (\mathbf{x}^s - \mathbf{X})] \left[\frac{1}{2} + (-1)^\nu t_0^s\right] \tag{3.1}$$

where t_0^s is the third component of the single-particle isospin. The expectation value of (3.1) with respect to the ground-state function will give the probability density at the point x.

The ground state is written as

$$|\pi JM, TM_T> = \sum_{\alpha\gamma} \sum_{fLs} a(\alpha Lf\gamma s) \sum_r \frac{(-1)^r}{\sqrt{d_f}} [|\alpha\pi Lfr>|\gamma STM_T \tilde{f}\tilde{r}>]_{JM} \tag{3.2}$$

where the only good quantum numbers assumed are the parity π, the total angular momentum and isospin J and T and its projections. With the coefficients $a(\alpha Lf\gamma s)$ available (and they could be calculated were the effective interaction known) one expresses the ground state as a linear combination of the orbital states $|\alpha\pi Lfr>$ coupled to the spin-isospin states $|\gamma STM_T \tilde{f}\tilde{r}>$, which have a well defined permutational symmetry, specified by the Young diagram f and the Yamanouchi symbol r. The orbital states are built from single-particle oscillator states coupled to a total orbital angular momentum L. In (2.2) α and γ stand for all other quantum numbers needed to complete the classification.

The matrix element of (2.1) with respect to the state (2.2) gives the density $\rho_\nu(x)$, whose Fourier transform $F_\nu(q^2)$ is equal to the body form factor, which can then be written as,

$$F_\nu(q^2) = \frac{1}{Z} \frac{1}{\sqrt{d_f d_f'}} \sum_{\substack{ff' \\ rr'}} \sum_{Ls} \sum_{\substack{\alpha\alpha' \\ \gamma}} (-1)^{r+r'} \left[\frac{1+(-1)^\nu}{2} n - (-1)^\nu Z \right]$$

$$\times <\alpha'\pi Lf'r' \left| \frac{\sin q|\mathrm{x}|}{q|\mathrm{x}|} \right| \alpha\pi Lfr>$$

$$\times\ a^*(\alpha' Lf'\gamma' s)\, a(\alpha Lf\gamma s), \tag{3.3}$$

where Z is equal to the number of protons in the system and d_f is the dimensionality of the irreducible representation of the symmetric group, characterized by the partition f.

The expression, combined with the fact that neutrons and protons are not point particles, but have a charge density of their own, gives the final result for the charge form factor,

$$F(q^2) = f_1(q^2)\, F_1(q^2) + f_0(q^2)\, F_0(q^2) \qquad (3.4)$$

where f_1 and f_0 refer to the form factors of the neutron and proton, respectively, which are known experimentally[27].

For the particular cases of 3 and 4 nucleus Moshinsky et al. have derived explicit expressions for the matrix element appearing in eq. (3.3). In this manner, the charge form factor for these systems is expressed as a linear combination of the single-particle matrix elements

$$\left\langle nl \left| \frac{\sin q\,|\mathbf{x}|}{q\,|\mathbf{x}|} \right| n'l' \right\rangle \qquad (3.5)$$

In the next section we give the formulae for the cases of 3 and 4 particles, using the most symmetrical partition, to which we shall restrict the numerical analysis, for simplicity.

We now look for a relativistic generalization of this formula assuming that the relative motion remains non-relativistic. This implies using the same formula as before, but changing the single particle matrix elements (3.5). Instead of these latter we use the ones obtained with the relativistic-harmonic oscillator, i.e. with states clasified by the chain of groups (1.6). An explicit expression for these matrix elements is given in the appendix, for all cases needed in the applications we shall discuss.

As can be seen from the formulae in the appendix, the form factor behaves, for very high values of q^2, as $(q^2)^{-|N|}$, N being the label for the I.R. of $U(3,1)$. This is consistent with the theoretical lower limit, obtained either by assuming analiticity in the cut q^2-plane,[10] or by assuming that the axioms of local quantum-field theory hold[11], in which case it can be shown that the form factor cannot decrease faster than $\exp(-\beta\sqrt{|q^2|})$. On the other hand, a truncated oscillator basis always produces a form factor which asymptotically decreases as $\exp(-\alpha q^2)$, violating the lower limit. One may question, therefore, the usefulness of the oscillator basis for very high transfers and think that the relativistic correction might be substantial.

We shall now compare the results given by both approaches for the three nucleon system and for the α particle.

IV. SCALAR FORM FACTOR OF THREE AND FOUR NUCLEON SYSTEMS.

In this section we shall apply the formalism we have developed, to calculate the scalar form factors of the three-nucleon system and of the alpha particle, both in the relativistic and non-relativistic cases and compare the results with the experimental values.

In both applications we shall restrict the analysis to two many-body harmonic oscillator states, one of them being the zero-quantum state and the other one corresponding to four quanta, both states being classified by the most symmetrical irreducible representation of the symmetric group. This approximation has proved to be reasonable in the non-relativistic analysis of the α particle form factor[9]. In the three-particle case these are the most important states obtained in the diagonalization of an effective hamiltonian[12].

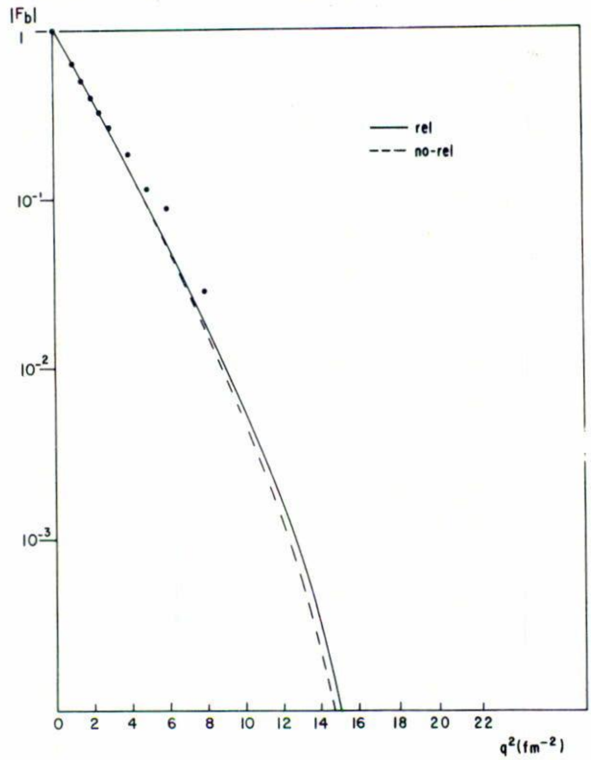


Figure 2. Comparison of relativistic (solid line) and non-relativistic (dotted line) body form factors for the three-nucleon system. The value of $\gamma = -0.05$ has been used, which implies that the amplitude of the zero-quantum state is larger than the amplitude of the four-quantum state.

Using eq. (3.3) and denoting by $\cos\gamma$ and $\sin\gamma$ the amplitudes of the zero and four-quantum states, respectively, we find, for the three-particle system, the following expression for the relativistic body form factor

$$F_B(q^2) = \cos^2\gamma\, I_N(000|000) + \sqrt{2}\sin\gamma\cos\gamma\,[I_N(000|101) + I_N(000|020)] + \frac{\sin^2\gamma}{2}\,[I_N(101|101) + 2I_N(101|020) + I_N(020|020)]\,, \tag{4.1}$$

where

$$I_N(n_1 l_1 n_2 | n_1' l_1' n_2') = \sum_{\substack{\bar{l}_1 \\ (\text{even})}} \sum_{\substack{\bar{n}_1 \bar{n}_2 \\ \bar{n}_2'}} \langle \bar{n}_1 \bar{l}_1 \bar{n}_2 \bar{l}_1 0 | n_1 l_1 n_2 l_1 0\rangle \times \langle \bar{n}_1 \bar{l}_1 \bar{n}_2' \bar{l}_1 0 | n_1' l_1' n_2' l_1' 0\rangle \langle N \bar{n}_2' \bar{l}_1 | e^{iL(q)} | N \bar{n}_2 l_1\rangle \tag{4.2}$$

and $\langle \bar{n}_1 \bar{l}_1 \bar{n}_2 \bar{l}_1 0 | n_1 l_1 n_2 l_1 0\rangle$ is the Brody-Moshinsky bracket[13] and the matrix element

$$\langle N \bar{n}_2' \bar{l}_1 | e^{iL(q)} | N \bar{n}_2 \bar{l}_1 \rangle \tag{4.3}$$

is evaluated explicitly in the appendix. The states $|N \bar{n}_2 \bar{l}_1\rangle$ are single-particle states labelled by the chain of groups (1.6).

In these equations, only states with orbital angular momentum $L = 0$ and partition [3] have been considered; these states represent about 90% of the triton ground state, as can be shown from a realistic-interaction calculation[14].

The non-relativistic formulae for the form factor can be obtained by replacing (4.3) by its non-relativistic equivalent

$$< \overline{n}'_2\, \overline{l}_1 \left| \frac{\sin q\,|\mathbf{x}|}{q\,|\mathbf{x}|} \right| \overline{n}_2 \overline{l}_1 > \ ,$$

as is discussed in the appendix.

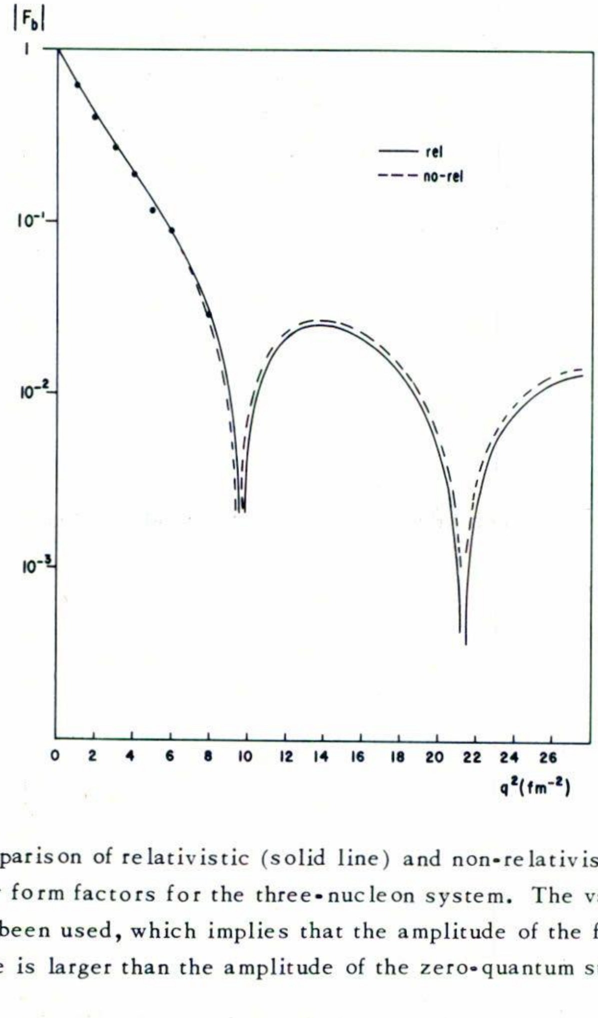


Figure 3. Comparison of relativistic (solid line) and non-relativistic (dotted line) body form factors for the three-nucleon system. The value of $\gamma = -1.03$ has been used, which implies that the amplitude of the four-quantum state is larger than the amplitude of the zero-quantum state

Before discussing the results of the calculation, we note that we have two free parameters, N and γ. They are not independent however, if we fix the root-mean radius, which is rather well known experimentally[3]. Using the fact that the root-mean radius is proportional to the derivative of the form factor at $q^2 = 0$ we obtain the relation

$$N = \frac{N_0}{1 + \frac{4}{3}\sin^2\gamma} \qquad (4.4)$$

where N_0 is calculated by adjusting the root-mean radius with the zero-quantum state only.

In figure 2 we show the results for the value $\gamma = -0.05$ both for the relativistic and non-relativistic cases and compare with the experimental points[15], which are subject to rather large uncertainties. As seen from the figure a minimum is predicted around $q^2 \sim 15\, fm^{-2}$ and the relativistic correction is of the order of 10% at $q^2 \sim 10\, fm^{-2}$ and even larger for higher values of q^2. We then use a value of $\gamma = -1.03$ which implies that the amplitude of the four-quantum state is larger than that of the zero-quantum state.

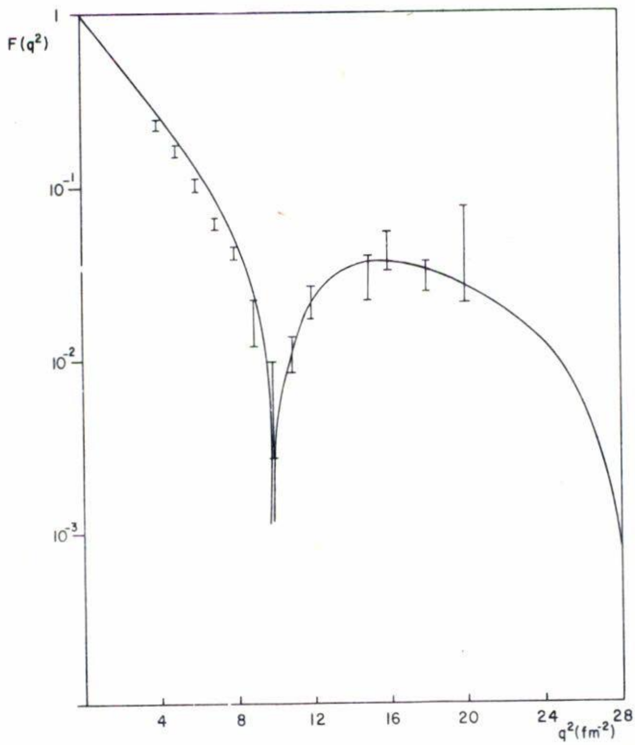


Figure 4. Relativistic (solid line) and non-relativistic (dotted line) body form factors for the α particle. The amplitude of the zero-quantum state is larger than the amplitude of the four-quantum state; the value of $\gamma = -0.3$ has been used

The results are given in fig. 3. Two diffraction minima are now predicted and the relativistic corrections are smaller than in the previous case, as seen from the figures. In view of our approximations, we think that the fit to the experimental values is rather good for both values of γ, although it is somewhat better for $\gamma = -1.03$.

For the α particle, we again restrict the analysis to two symmetrical states belonging to the partition [4] using, as in reference (9), a zero and a four-quantum states. Again, the values of N and γ are not independent of

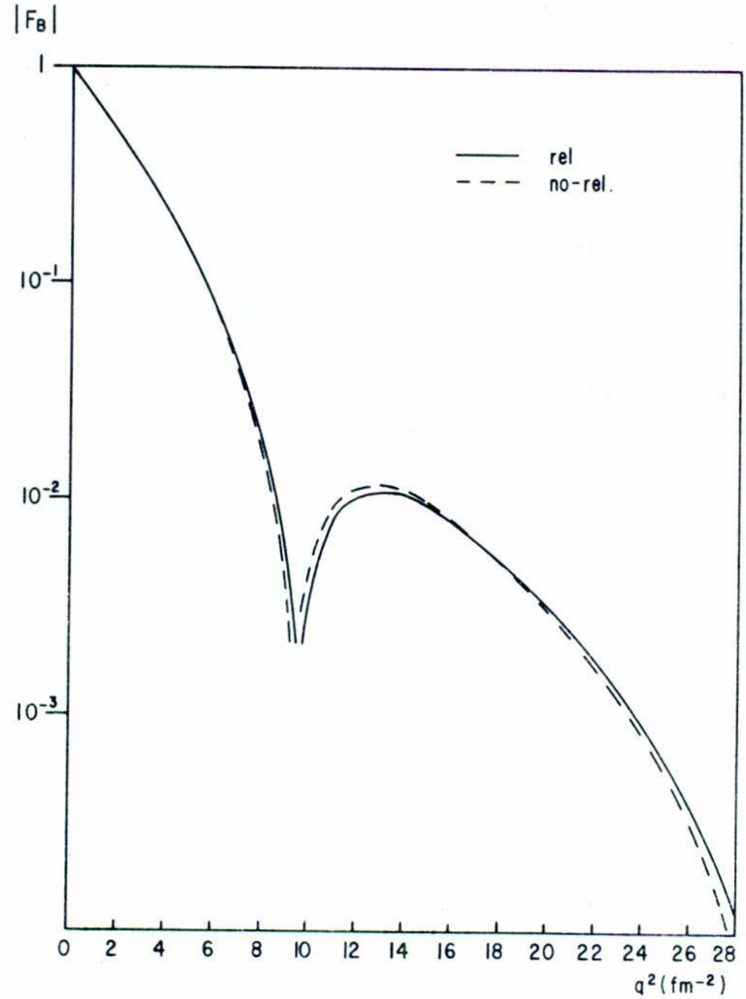


Figure 5. Comparison between the relativistic body form factor and the experimental results for the α particle. The value of $\gamma = -1.2$ has been used

each other if we fix the root-mean radius. In fig. 4 we give the comparison of the relativistic form factor with the experimental values[1]. We take $\gamma = -1.2$ and a reasonable fit is obtained. Two diffraction minima are predicted, the first around $10 fm^{-2}$ and a second one around $q^2 \sim 28 fm^{-2}$. In this case the relativistic corrections are smaller than in the three particle calculation, as indeed one would expect since the α particle is heavier than the triton; the error is now of the order of 3% around $q^2 \sim 16 fm^{-2}$, the position of the second maximum. We now perform the comparison with the value $\gamma = -0.3$; the error in the non-relativistic approximation is larger than for $\gamma = -1.2$, being of the order of 6% at $q^2 = 12 fm^{-2}$ the position of the second maximum, which is predicted too low, however. For the α particle, in contrast to what happened for the triton, the experimental values do distinguish between the two values of γ.

From the analysis we can conclude that the relativistic corrections for the form factor are sizeable in some cases and could be important, when better experimental data are available. Since the asymptotic limit for very high transfers is different in the relativistic and the non-relativistic cases, as we mentioned at the end of section III, the corrections will also be very important when experimental points for larger q values are obtained. We also note that the correction is model dependent being larger the smaller the absolute value of the mixing parameter.

APPENDIX

In this appendix we shall obtain an expression for the scalar product

$$F(q^2) = \langle \psi_{Nn'l}(p+q) | \psi_{Nnl}(p) \rangle \qquad \text{(A.1)}$$

where $\psi_{Nnl}(p)$ is the basis for the irreducible representation of the chain of groups

$$U(3,1) \supset U(3)_p \supset O(3)_p \qquad \text{(A.2)}$$

and corresponds to a four-momentum p.

As discussed in reference (6) one can write (A.1) as the contraction of two generalized tensors, in the form

$$\frac{1}{2l+1} \sum_{l_z} \sum_{\mu_i} \bar{\psi}^{\mu_1 \ldots \mu_N}_{n' l l_z} \psi^{n l l_z}_{\mu_1 \ldots \mu_N} =$$

$$= \frac{1}{2l+1} \sum_{l_z} \sum_{\mu_i \mu'_i} \sum_{\nu_i \nu'_i} \sum_{k} c_k^{N,\sigma,\tau} \bar{\tilde{\psi}}^{\mu_1 \ldots \mu_k \mu'_{k+1} \ldots \mu'_\sigma}_{n' l l_z}(v')$$

$$\times \tilde{\psi}^{n l l_z}_{\mu_1 \ldots \mu_k \nu_{k+1} \ldots \nu_\tau}(v)\, v_{\mu'_{k+1}} \ldots v_{\mu'_\sigma}\, v'^{\nu_{k+1}} \ldots v'^{\nu_\tau}$$

$$\times (v' \cdot v)^{N-\tau-\sigma+k} . \qquad \text{(A.3)}$$

where l_z is the projection of l, v and v' are the four-velocities corresponding to p and $p+q$, respectively, and $(v' \cdot v)$ is the scalar product of the two four-vectors. The normalization coefficient $c_k^{N,\sigma,\tau}$ is given by

$$c_k^{N,\sigma,\tau} = \frac{\sqrt{\sigma!\, \tau!\, (N-\tau)!\, (N-\sigma)!}}{k!\, (\sigma-k)!\, (\tau-k)!\, (N-\sigma-\tau+k)!} \qquad \text{(A.4)}$$

and

$$\sigma = 2n' + l, \quad \tau = 2n + l \tag{A.5}$$

give the total number of oscillator quanta in the bra and ket, respectively.

The state vector $\tilde{\psi}^{nll_z}_{\mu_1 \ldots \mu_\tau}(v)$ is defined as

$$\tilde{\psi}^{nll_z}_{\mu_1 \ldots \mu_\tau}(v) = \hat{\psi}_{\mu_1 \ldots \mu_l}(v)\ \theta_{\mu_{l+1}, \mu_{l+2}} \ldots \theta_{\mu_{\tau-1}, \mu_\tau} \tag{A.6}$$

where $\theta_{\mu\nu}$ is given by (2.5) and

$$\hat{\psi}_{\mu_1 \ldots \mu_l}(v_0)\, \bar{v}^{\mu_1} \ldots \bar{v}^{\mu_l} = \mathcal{Y}_{l, l_z}(\mathbf{v}) \tag{A.7}$$

where $v_0 = (1, 0, 0, 0)$ and $\bar{v} = (0, v)$, $\mathcal{Y}_{l, l_z}(\mathbf{v})$ being the solid spherical harmonic.

Using techniques similar to those employed in the appendix of reference (6) one can now compute expression (A.3). The resultant expression for $F(q^2)$ can now be written as

$$F(q^2) = \mathfrak{N}^{-\frac{1}{2}}_{n', l}\, \mathfrak{N}^{-\frac{1}{2}}_{n, l} \left(1 + \frac{q^2}{2m^2}\right)^{-|N| - 2(n' + n + l)}$$

$$\times \sum_{jk} (-1)^{k+l} A_n(n, n', l, k)\, A_{n'}(n, n', l, k)$$

$$\times \frac{(2n + l - j)!\,(2n' + l - j)!}{(k - j)!\,(2n + l - k)!\,(2n' + l - k)!}\ D(n, l, n', j)$$

$$\times \left(\frac{|N| q^2}{m^2}\right)^{n + n' + l - k} \left(1 + \frac{q^2}{2m^2}\right)^{2k - j} \left(1 + \frac{q^2}{4m^2}\right)^{n + n' + l - k} \tag{A.8}$$

with

$$A_p^2(n,n',l,k) = \left[\left(1 + \frac{2n + 2n' + 2l - k - 1}{|N|}\right)\cdots\left(1 + \frac{2p + 2l + 1}{|N|}\right)\left(1 + \frac{2p + 2l}{|N|}\right)\right] \qquad (A.9)$$

and

$$D(n,l,n',i) = \sum_{st} (-1)^t 4^s \frac{(n+t)!\,(n'+t)!}{(n+t-s)!\,(n'+t-s)!} \times \frac{(i+1)!\,(2l-2t-1)!}{t!\,(2s+1)!\,(i-2s)!\,(l-t-1)!\,(l-2t+2s-i)!}\,. \qquad (A.10)$$

The normalization coefficient $\eta_{n,l}$ is given by

$$\eta_{n,l}^2 = A_n^2(n,n,l,2n+l) \sum_i (2n+l-i)!\,D(n,l,n,i)\,.$$

In eq. (A.8) m stands for the rest mass of the system. Formula (A.8) can be checked by taking the non-relativistic limit and comparing with the expression valid for the non-relativistic oscillator given in ref. (9):

$$F_{n.r.}(q^2) = \frac{e^{-\alpha q^2}}{2} \sum_{st} B(n,l,n',l,t)\ \frac{2^{2t+1}}{2^{2s}}\ \frac{\Gamma\left(s+\frac{1}{2}\right)}{\Gamma\left(t+\frac{3}{2}\right)}\ (-\alpha q^2)^{t-s} \qquad (A.11)$$

where the coefficients $B(n,l,n',l,t)$ are tabulated[13]. The limit of (A.8) is obtained by taking $q^2/2m^2 \ll 1$ and $|N| \to \infty$ with $|N|q^2/m^2$ finite. Expression (A.8) then becomes,

$$F_{n.r.}(q^2) = e^{-\frac{|N| q^2}{m^2}} \cdot \eta_{n,l}^{-\frac{1}{2}} \eta_{n',l}^{-\frac{1}{2}}$$

$$\times \sum_{ik} \frac{(-1)^{k+l} (2n+l-i)!\,(2n'+l-i)!}{(k-i)!\,(2n+l-k)!\,(2n'+l-k)!} D(n,l,n',i) \left(\frac{|N| q^2}{m^2} \right)^{n+n'+l-k} \quad \text{(A.12)}$$

which coincides with (A.11) if α is taken to be $|N|/2m^2$. The formula (A.12) gives the non-relativistic single particle form factor in a more convenient way that the alternative expression (A.11), once the coefficients $D(n,l,n',i)$ are known. We, therefore, give a table of these coefficients, for all possible values of the quantum numbers up to 4 quanta. In the table we have made use of the obvious symmetry relation

$$D(n,l,n',i) = D(n',l,n,i)\,. \quad \text{(A.13)}$$

Table I

Coefficients $D(n, l, n', j)$

n	l	n'	j	D	n	l	n'	j	D
0	0	0	0	1	0	0	2	0	1
0	1	0	0	1	1	0	1	0	1
0	1	0	1	2	1	0	1	1	0
0	2	0	0	2	1	0	1	2	4
0	2	0	1	12	1	1	1	0	1
0	2	0	2	14	1	1	1	1	2
0	3	0	0	4	1	1	1	2	4
0	3	0	1	48	1	1	1	3	16
0	3	0	2	156	1	2	1	0	2
0	3	0	3	144	1	2	1	1	12
0	4	0	0	8	1	2	1	2	14
0	4	0	1	160	1	2	1	3	96
0	4	0	2	1008	1	2	1	4	176
0	4	0	3	2400	1	0	2	0	1
0	4	0	4	1992	1	0	2	1	0
0	0	1	0	1	1	0	2	2	8
0	1	1	0	1	2	0	2	0	1
0	1	1	1	2	2	0	2	1	0
0	2	1	0	2	2	0	2	2	16
0	2	1	1	12	2	0	2	3	0
0	2	1	2	10	2	0	2	4	64

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RESUMEN

Se emplean los métodos de la Teoría de Grupos para obtener una expresión explícita para el factor de forma de un sistema de pocas partículas. La expresión es válida para valores muy grandes de la transferencia de momento. El formalismo se aplica para obtener los factores de forma de sistemas de tres y cuatro nucleones y se discute el orden de magnitud del error que se comete cuando se calcula dentro de la aproximación relativista.