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# AN ELECTROSTATIC ANALOGUE OF THE RELATION BETWEEN $R$- AND $S$- MATRICES*

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ABSTRACT: A quantitative analysis of the poles and residues of the $S$-matrix in terms of a given $R$-matrix can be carried out by diagonalizing a complex symmetric matrix. In the present paper we develop an electrostatic analogue for the one channel problem, consisting of a set of parallel infinitely-long charged wires; from the intuitive behaviour of the electric field one can gain a simple qualitative insight on some of the properties of the $S$-matrix parameters in terms of those of the $R$-matrix, for the single-channel case.

## INTRODUCTION

A detailed study of the $S$-matrix parameters which appear in a general resonance-pole expansion (Mittag-Leffler expansion) usually involves the difficulty of explicitly enforcing unitarity of the $S$-matrix. On the other hand, the well-known[1] $R$-matrix formalism, in terms of which the $S$-matrix

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may be written, expressly guarantees unitarity: the main advantage of this formalism is then that one may parametrize the $S$-matrix through proposed models for the $R$-matrix without ever violating probability flux conservation.

In recent years Moldauer has proposed very simple $R$-matrix models whereby he has been able to study general properties of the ensuing $S$-matrix poles and residues. In particular, he has proposed[2] an $R$-matrix consisting of an infinite number of equidistant poles with identical residues ("infinite picket-fence") which turns out to have an exact, analytic solution. Also he has considered several cases of a "finite picket-fence" model (with and without a smooth background term) which can be solved numerically, and furthermore he has studied more realistic models with specific distribution properties for the poles and residues.

Our purpose here will be to give an **electrostatic analogue** of the problem which allows one to visualize, graphically and in a qualitative way, some $S$-matrix properties resulting from a given $R$-matrix model for the single-channel case.

## II. ELECTROSTATIC ANALOGUE OF THE RELATION BETWEEN $R$- AND $S$- MATRICES.

In the one-channel case, the relation between the Wigner $R$-matrix, for $n$ levels,

$$R = \sum_{\lambda=1}^{n} \frac{\gamma_\lambda^2}{E_\lambda - E} \tag{2.1}$$

and the $S$-matrix, for a given partial-wave, is

$$S = \exp\left[(2i(\phi - \omega)\right] \frac{1 - R(L^0)^*}{1 - RL^0} . \tag{2.2}$$

The reduced-widths $\gamma_\lambda^2$ and the $R$-matrix poles $E_\lambda$ are real. The quantities $\phi$ and $\omega$ refer respectively to hard-sphere and Coulomb phase-shifts; $L^0 \equiv L - B$, with $L = \mathcal{S} + iP$ being the logarithmic derivative of the outgoing wave evaluated at the channel-radius $a$ and $B$ the boundary-condition for the logarithmic derivative of the $R$-matrix wave functions at radius $a$. The functions $\mathcal{S}$ and $P$ are real; $P$ is the penetrability. The $S$-matrix as given by Eq. (2.2) is manifestly unitary for real values of $E$ for arbitrary but real values of the $R$-matrix parameters $\gamma_\lambda^2$ and $E_\lambda$.

For a given set of $R$-matrix parameters, the problem of finding the corresponding $S$-matrix poles is equivalent to finding the complex energies

$$\mathcal{E}_i \equiv E_i - i\,\frac{\Gamma_i}{2} \tag{2.3a}$$

which satisfy

$$1 - RL^0 = 0 \qquad \text{or} \qquad \sum_{\lambda=1}^{n} \frac{L^0 \gamma_\lambda^2}{E_\lambda - \mathcal{E}} = 1\ . \tag{2.3b}$$

Clearly, the roots $\mathcal{E}_i$ of this equation are complex because $L^0$ is complex.

The linear, fractional transformation Eq. (2.2) has been studied by Wigner[3] for the special case

$$f = \frac{a_1 R + b_1}{a_2 R + b_2} \tag{2.4}$$

with **real** coefficients $a$ and $b$; he finds that the integrated cross-section $\sigma(E)$ can be written in terms of the real function $f$ in such a way that when $f$ has **poles**, $\sigma(E)$ has **maxima**. The poles of $f$ are real and are the solutions of

$$a_2 R + b_2 = 0 \qquad \text{or} \qquad \sum_{\lambda=1}^{n} \frac{\gamma_\lambda^2}{E_\lambda - E} = -\, b_2/a_2 \equiv \alpha \tag{2.5}$$

which can be solved graphically as shown in Fig. 1 for the special case of $n = 4$ levels. We see that, with the exception of the first pole, there is always a pole of $f$ between two successive poles of $R$. Therefore the average distance of a sample of $n$ poles of $R$ not including the first one will be the same, for $n$ big enough, as the average distance of the corresponding $n$ poles of $f$.

Now, the problem of finding the $S$-matrix poles is more general than this since one must deal with Eqs. (2.2) and (2.3) which involve *complex* quantities so that a simple graphical solution, like that of Fig. 1, is no longer possible. There exists, however, a simple electrostatic problem

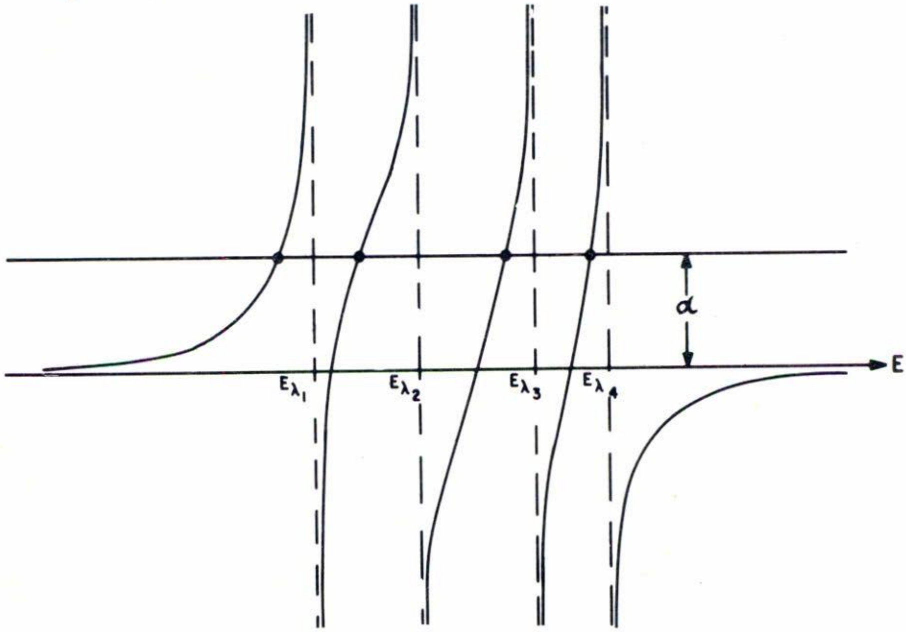


Fig. 1. Graphical solution for the poles of $f$ (Eq. 2.4). The dots indicate the solutions and $\alpha \equiv - b_2/a_2$.

which can give valuable insights in the behaviour of the $S$-matrix poles and residues. Consider a set of $n$ infinitely long wires which, in a given cartesian coordinate system, lie in the plane $y = 0$ and are all parallel to the $z$-axis; the $x$-$y$ plane projection is shown in Fig. 2 where the wire positions are given by $x_\lambda$ ($\lambda = 1, 2, \ldots, n$); assume that the wires have positive uniform charge per unit length $\tau_\lambda$. The electric field $\boldsymbol{\epsilon}(\mathbf{r})$ at an arbitrary point $\mathbf{r} = (x, y)$ is then given by

$$\boldsymbol{\epsilon}(\mathbf{r}) = \sum_{\lambda=1}^{n} \frac{2\tau_\lambda}{|\mathbf{r} - \mathbf{r}_\lambda|} \frac{\mathbf{r} - \mathbf{r}_\lambda}{|\mathbf{r} - \mathbf{r}_\lambda|} . \tag{2.6}$$

To express vectors in the complex $(x, y)$-plane, associate

$$\mathbf{r} \rightarrow z = x + iy , \qquad \mathbf{r}_\lambda \rightarrow z_\lambda ,$$

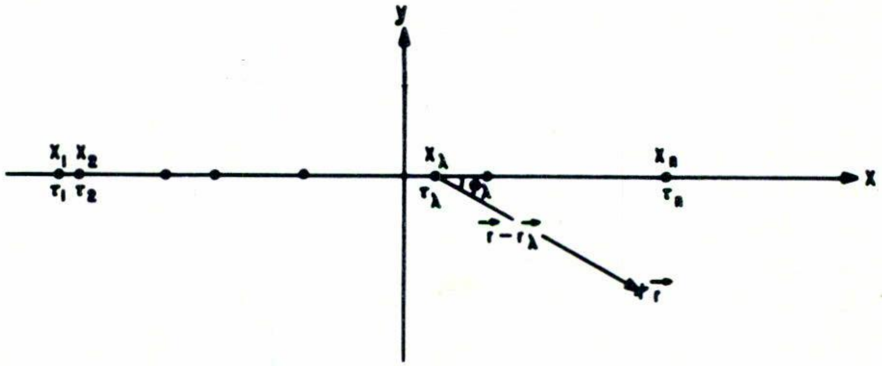


Fig. 2. The projection on the $xy$-plane of the electrostatic analogue of the $R$-$S$ matrix problem. The $xy$-plane corresponds to the complex energy plane in the $R$-matrix problem; the dots at $x = x_\lambda$ correspond to the poles $E_\lambda$ of $R$ and are the projection of infinitely long wires perpendicular to the page and with charge $\tau_\lambda$ per unit length.

so that

$$r - r_\lambda \to |z - z_\lambda| e^{i\phi_\lambda}, \tag{2.7}$$

where $\phi_\lambda$ is the angle shown in Fig. 2 . Hence, Eq. (2.6) becomes the complex function

$$\epsilon(z) = \sum_{\lambda=1}^{n} \frac{2\tau_\lambda}{|z - z_\lambda|} e^{i\phi_\lambda} \tag{2.8}$$

$$\epsilon(z) = -\left[\sum_{\lambda=1}^{n} \frac{2\tau_\lambda}{z_\lambda - z}\right]^* , \qquad (2.9)$$

so that $-\epsilon^*(z)$ is an $R$-function[3]. If we associate the $x$-$y$ plane with the complex energy plane, then from the electric field of the electrostatic problem just described, we can deduce the $R$-function whose poles $E_\lambda$'s are associated with the $x_\lambda$'s of the electrostatic problem and its residues with $-2\tau_\lambda$.

As the prime objective is to obtain the $S$-matrix poles and residues, we are interested in the complex roots $\mathcal{E}_i$ of Eq. (2.3b) in which $L^0$ is, strictly speaking, a function of the complex energy $\mathcal{E}$; however, we shall expect this dependence to be *weak* if the $\gamma^2_{\lambda j}$'s are much smaller[6] than the $E_\lambda$'s span an energy interval much smaller than their average distance from threshold. Under this assumption we shall neglect it in what follows and $L^0$ will be taken as a given constant vector in the complex plane. Moreover, choosing for convenience the logarithmic derivative boundary condition $B$ equal to $\mathcal{S}$, we have

$$L^0 \equiv \mathcal{S} + iP - B = iP , \qquad (2.10)$$

so that the equation to be solved becomes

$$\sum_{\lambda=1}^{n} \frac{\Gamma^0_\lambda/2}{E_\lambda - \mathcal{E}} = -i \qquad (2.11)$$

$$\Gamma^0_\lambda \equiv 2P\gamma^2_\lambda . \qquad (2.12)$$

Now suppose that we construct an electrostatic problem by associating the $x_\lambda$'s with the $E_\lambda$'s of (2.11) and $\tau_\lambda$ with $\Gamma^0_\lambda/4$. The problem of solving Eq. (2.11) can then be written as

$$\epsilon(z) = -i \qquad (2.13)$$

and this is equivalent to plotting the electrostatic field produced by the $n$ wires and selecting only those points where the field is *vertical, directed downwards* and of *unit magnitude*. This is evidently fulfilled *only* in the lower-half of the $xy$-plane and is equivalent to the familiar result that the

$S$-matrix has poles only in the lower-half energy plane. The locus of points in the $xy$-plane where the field is vertical is obtained from Eq. (2.6) with zero horizontal component of $\epsilon(r)$ i.e.,

$$\sum_{\lambda=1}^{n} \tau_\lambda \frac{x - x_\lambda}{(x - x_\lambda)^2 + y^2} = 0 . \tag{2.14}$$

As regards the *residues* of the $S$-matrix, Eq. (2.2) becomes, for $L^0 = iP$,

$$S = e^{2i(\phi - \omega)} W , \tag{2.15a}$$

$$W \equiv \frac{1 + iPR}{1 - iPR} \tag{2.15b}$$

and, to conform to the usual notation of the literature, we define the quantities

$$g_i^2 \equiv \lim_{\mathcal{E} \to \mathcal{E}_i} [(\mathcal{E} - \mathcal{E}_i)\, W(\mathcal{E})] \tag{2.16}$$

with $\mathcal{E}_i$ a pole of $S$, i.e.,

$$1 - iPR(\mathcal{E}_i) = 0 . \tag{2.17}$$

Now, the limit in Eq. (2.16) is easily evaluated and gives

$$g_i^2 = -2 / \left[\frac{dPR}{d\mathcal{E}}\right]_{\mathcal{E} = \mathcal{E}_i} \tag{2.18}$$

or, as $PR$ was associated with $-\epsilon^*$ [ Eqs. (2.11), (2.12) and (2.13) ] , we have

$$g_i^2 = 2 / \left[ \frac{d\epsilon^*}{dz} \right]_{z = z_i} , \tag{2.19}$$

where $z_i$ is the point in the $xy$ plane corresponding to the pole $\mathcal{E}_i$ in the energy plane.

It is probably evident that the "electrostatic" approach to solving for the $S$-matrix poles and residues, from a given $R$-matrix of $n$ levels, would not be as *practical*, from a calculational standpoint, as that, e.g., used by Moldauer[4] based on diagonalizing a complex, symmetric $n \times n$ matrix[1], a method first introduced by Wigner. But it will become clear that the electrostatic analogue provides certain *qualitative* features of the $S$-matrix which are muddled in the matrix-diagonalization method.

## III. APPLICATIONS

The simplest possible example is the trivial case of a single wire (one $R$-matrix level) in which, of course, the electric field lines emerge radially, Fig. 3, and the *only* solution (i.e., the *single* $S$-matrix pole) must lie on the downward, vertical line of force (lower half-plane) at the point where the field magnitude is unity

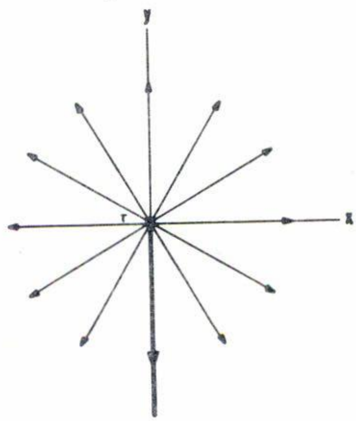


Fig. 3. The radial electric field of a single wire. The thick line is the locus of points where the field is vertical and directed downwards. The $S$-matrix pole should be on the point of this line where the magnitude of the electric field is unity.

$$1 = |\epsilon(z)| = \frac{2\tau}{|z|} = \frac{\Gamma^0/2}{|z|} \tag{3.1}$$

$$|z| = y = \Gamma^0/2 \, . \tag{3.2}$$

Using notation (2.3a) one gets for the single pole of $S$

$$\Gamma = \Gamma^0 \tag{3.3}$$

i.e., the $S$-matrix width $\Gamma$ coincides with the $\Gamma^0$ defined in Eq. (2.12) in terms of $R$-matrix quantities. Finally, applying Eq. (2.19) for the pole residue one gets

$$g_i^2 = \Gamma^0 \, ,$$

in accordance with the well-known single-level (Breit-Wigner) resonance formula result.

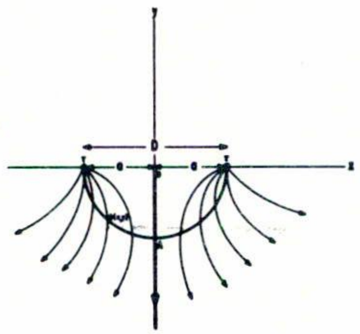

Fig. 4. The electric field (drawn only for $y < 0$) for two wires with the same charge density. Again the thick lines are the locus of points where the field is vertical and directed downwards.

For *two* wires with *equal charge densities* $\tau$ the lines of force are those shown in Fig. 4, where the thick curves are the locus of points where the field is vertical and directed downwards. It is clear from the figure that the $S$-matrix solutions will be exclusively in the lower-half plane and that the real part of the two $S$-matrix poles will ***always*** lie between the two $R$-matrix poles (which of course coincide with the wires); because of symmetry with respect to the central vertical, the $S$-matrix poles will be reflections of one another relative to that vertical. For very small values of $\Gamma^0$ (very small charge densities), field magnitudes of unity are realizable only at a very small distance from each wire and, as $\Gamma^0$ is increased, the two solutions will move *away* from the wires along the semi-circumference until they "collide" at the point A. At this point the electric field has the magnitude

$$2\tau/a = \Gamma^0/2a \ ,$$

so that the collision occurs when

$$\Gamma^0 = 2a \ , \qquad (3.4)$$

that is, when the "$R$-matrix width" $\Gamma^0$, as defined by Eq. (2.12), becomes equal to the distance between the two $R$-matrix poles. As $\Gamma^0$ (or, charges) is further increased, the solutions will move along the directions where the electric field ***decreases*** until the points with $|\epsilon| = 1$ are reached: this is satisfied by letting the poles move away from A along the vertical thick line, one towards the point O and the other toward $-\infty$, as these are points where the field obviously vanishes. This behaviour agrees with the results found by McVoy[5] and others on "colliding $S$-matrix poles". At an arbitrary point $Q(x, y)$ on the semi-circumference of Fig. 4, it is easy to verify that the field magnitude is simply

$$2\tau/y = \Gamma^0/2y \ ,$$

so that for a fixed $\Gamma^0$ the $S$-matrix poles will occur at $y = \Gamma^0/2$ and hence

$$\Gamma = \Gamma^0 \ , \qquad (3.5)$$

showing that for the two-level case with $\Gamma_1^0 = \Gamma_2^0$, the $S$- and $R$-matrix widths coincide also.

We now generalize the previous two-wire example to different charge densities: $\tau_2 = \alpha\tau_1$, say, with $\alpha > 0$ so that $\Gamma_2^0 = \alpha\Gamma_1^0$ (since we always have $\tau_\lambda \rightarrow \Gamma_\lambda^0/4$). This corresponds to a two-level $R$-matrix with differ-

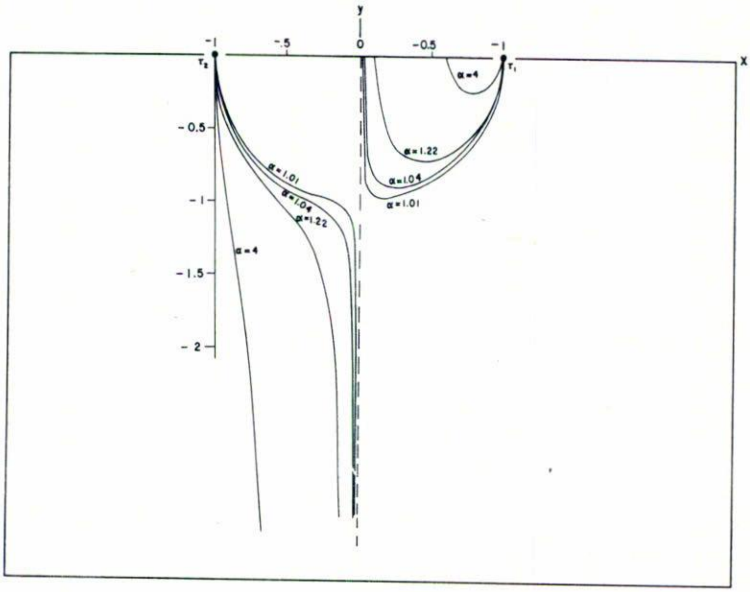


Fig. 5. The locus of points where the field is vertical and directed downwards, for the case of two wires with different charge densities: $\tau_2 = \alpha\tau_1$. Notice how in the limit $\alpha \to 1$, the locus tends to the semicircumference plus vertical line of Fig. 4.

ent reduced widths $\gamma_\lambda^2$. Finding the locus of points along which the electric field is vertical implies solving Eq. (2.14) for $n = 2$ and in Fig. 5 are plotted the different loci for $\alpha = 1.01, 1.04, 1.22$ and 4. Thus one clearly sees how as $\alpha \to 1$, the loci approach the semi-circle-and-vertical-line solution of Fig. 4 and as $\alpha \to \infty$ the solution approached is the single-wire solution of Fig. 3 for the left wire and a solution that shrinks to a point for the right wire. From an electrostatic standpoint, the behaviour shown in Fig. 5 is *intuitively* transparent since, as the charge on the left wire increases, the lines of force emanating from it will deviate from perfect radial lines, due to the repulsion provided by the right wire, *farther* away from the wire as compared to the case of equal charges (Fig. 4); the points where the field becomes vertical are thus removed farther from the circle of Fig. 4. By contrast, for the weaker-charged *right* wire the deflection of the electric field occurs *closer* to the wire so that the locus of points where the field is vertical will be more localized around this wire than for the stronger-charged left wire. It is immediately evident that the corresponding $S$-matrix poles will *always* have projections on the real axis *between* the two $R$-matrix poles (wires) regardless of the relative values $\Gamma_2^0/\Gamma_1^0 = \alpha$ of the two $R$-matrix widths since the electric field never becomes vertical for values of $x$ outside the spacing of the two wires. Again, we observe that for small $\Gamma_\lambda^0$'s the two $S$-matrix poles will respectively lie in the vicinities of the $R$-matrix poles and that as $\Gamma_\lambda^0$'s are increased they will move *away* from the wires along the curves so that in the limit $\alpha \to 1$ we in fact understand the phenomenon of "colliding poles".

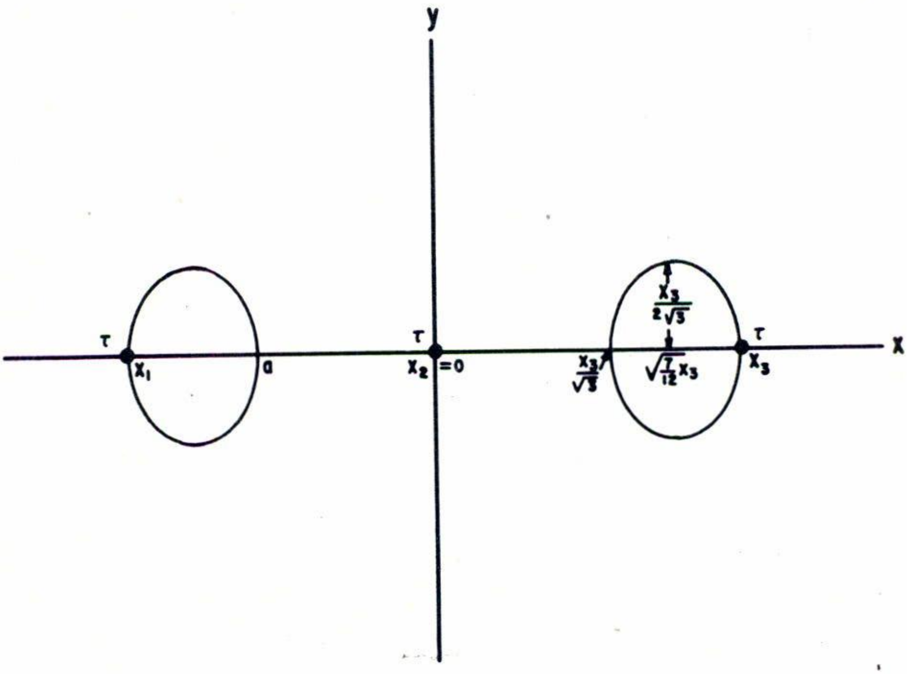


Fig. 6. The locus of points where the field is vertical in the case of 3 equally spaced wires with the same charge density.

Passing now to the case of three equally-spaced wires of equal charges, the points along which the field is vertical are given by Eq. (2.14) for $n = 3$. Explicitly, we have (see Fig. 6)

$$x\,[3y^4 + 6x^2y^2 + (x^2 - x_3^2)(3x^2 - x_3^2)] = 0, \qquad (3.6)$$

from which one immediately sees that a possible solution is $x = 0$, corresponding to the vertical line passing through wire 2 of Fig. 6 and which is a solution obvious from symmetry considerations. A simple analysis of Eq. (3.6) shows in addition the following: a) the loci also cross the $x$-axis at $x = \pm x_3$ and $\pm x_3/\sqrt{3}$, b) the slope of the solution becomes zero at $x = \pm\sqrt{7/12}\,x_3$ and at each of these points $y$ takes the double value $\pm x_3/(2\sqrt{3})$, c) there are no real solutions for $-x_3/\sqrt{3} \leqslant x \leqslant x_3/\sqrt{3}$. These results are all graphed in Fig. 6. The fact that the loci cross the $x$-axis at each wire is obvious on considering that in a vicinity sufficiently near to a wire, the effect of neighbouring wires may be neglected so that the field is approximately radial, as in the single-wire case of Fig. 3; hence, from each wire there is always a line of force emerging vertically and this result is clearly independent of the number of wires, their charges and separations.

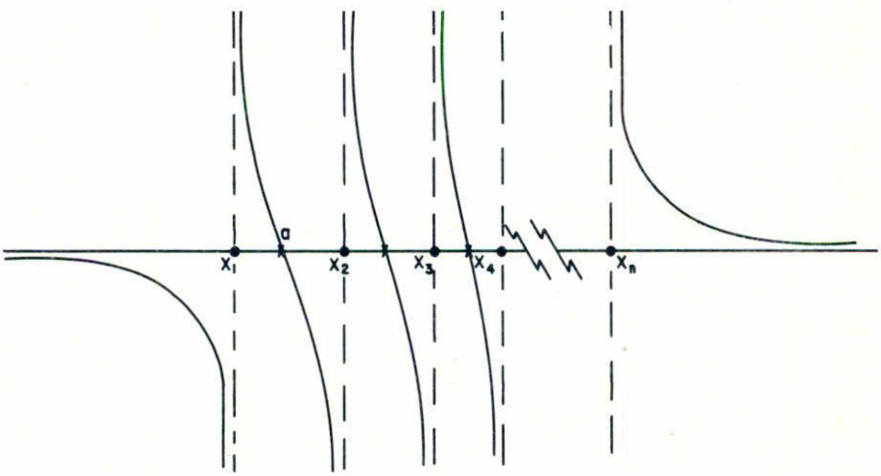


Fig. 7. A graphical solution of Eq. (3.7) giving the points on the $x$-axis where the field vanishes.

Another result which is independent of these latter conditions is the fact that there is one and only one solution crossing the $x$-axis between two adjacent wires, as can be seen from Eq. (2.14) with $y = 0$ and $n$ arbitrary:

$$\sum_{\lambda=1}^{n} \frac{\tau_\lambda}{x - x_\lambda} = 0, \qquad (3.7)$$

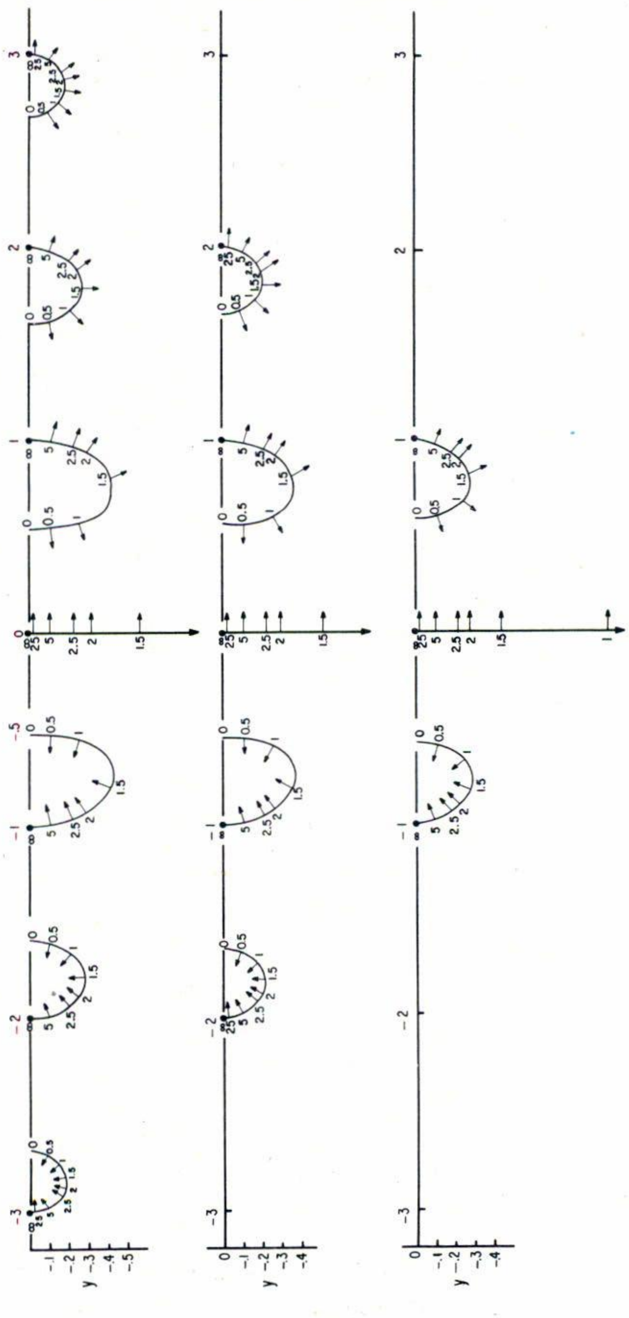

Fig. 8. The locus of points where the field is vertical and directed downwards for the case of 3, 5 and 7 wires. The numbers on the loci indicate the magnitude of the electric field at that point and the arrows indicate the direction of the complex quantity $g_i^2$ of Eq. (2.18). The length of the arrows is not significative.

whose left-hand side evidently has negative-definite slope and is graphed in Fig. 7. where the crosses indicate the solutions for $y = 0$, excluding of course, those at $x = x_\lambda$, that cannot be contained in Eq. (3.7). Furthermore, it is clear in general that a line of force to the left of the left-most wire or, to the right of the right-most wire, will *never* become vertical since in those two regions the electric field contributions will always point away from the (positively-charged) wires. Moreover, the lines of force emerging from *either* extreme wire will deviate from radial lines sooner, the larger the total charge density of the *other* wires. Thus, the locus of points where the field is vertical becomes more and more localized around the extreme wire if we add more and more wires, all in complete analogy with the weaker-charged, right wire behaviour in the two-wire case discussed before. For a wire not on either extreme it is difficult to make precise predictions regarding the shape of the neighboring electric field without actually solving (e.g., numerically) the electrostatic equation (2.12).

Numerical calculations were carried out for $n$ equally-spaced and-charged wires, $\tau_\lambda = 1/4$ and $|x_\lambda - x_{\lambda-1}| \equiv D = 1$, for $n = 2, 3, 4, 5, 6, 7$. Since $\tau_\lambda$ corresponds to $\Gamma_\lambda^0/4$, we have $\Gamma_\lambda^0 = 1$. The odd-number cases appear in Fig. 8 and the even number ones in Fig. 9. As mentioned before, it is in fact apparent from Figs. 8 and 9 that the extreme lobes become more localized, around their respective wire, as $n$ increases. Therefore, if one is concerned with an $R$-matrix having a large number of poles, the real part of the $S$-matrix poles not too removed from the extremes will tend to line up with the $R$-matrix poles at these extremes so that, at least as far as extremes are concerned, we may have *the same pole-density* for both $R$- and $S$-matrices if the number of poles is large. If one defines an *overall* density as the *sum* of all the separations divided by the *number* of separations, this overall density will also be the same in the limit of a large number of $R$-matrix poles. This is analogous to Wigner's result discussed in Sec. 1. From Figs. 8 and 9 one also observes the fact that, for fixed $n$, the lobes *expand* as one moves *away* from an extreme wire. Further, we note that as $n$ increases, the lobe associated with a *given wire* becomes *wider*, but the width approaches the limit given by the half-distance between two successive wires, as can be directly seen from Eq. (3.7), for $n \to \infty$, $\tau_\lambda = \tau$, $x_\lambda = \lambda D$:

$$\frac{\pi \tau}{D} \cot \frac{\pi x}{D} = 0 \qquad (3.8)$$

which is solved for

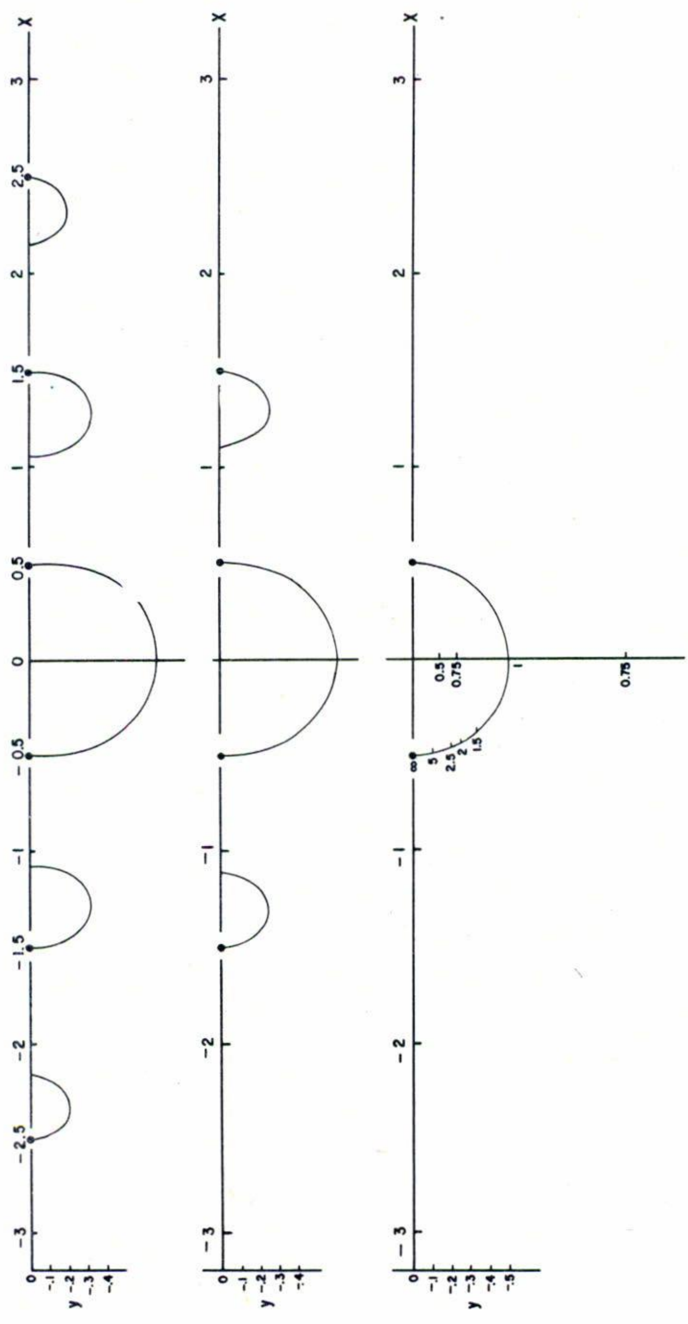

Fig. 9. The locus of points where the field is vertical and directed downwards for the case of 2, 4 and 6 wires. The numbers on the locus for $n = 2$ indicate the magnitude of the electric field at that point.

$$x = \frac{2m+1}{2} D, \quad m = 0,1,2,\ldots \quad . \tag{3.9}$$

It is also apparent from Figs. 8 and 9 that as $n$ increases, a *given lobe* becomes *deeper*; in fact, as $n \to \infty$ from Eq. (2.14) one deduces that the depth of a given lobe becomes infinite so that in this limit a lobe degenerates into two vertical lines.

This is in agreement with Moldauer's infinite picket-fence model in which as $\Gamma^0/D$ goes from 0 to $2/\pi$, the $S$-matrix pole directly underneath and $R$-matrix pole (wire) moves *vertically* away from the latter to $-\infty$ and reappears at $-\infty$ but *displaced* a horizontal distance $D/2$, so that as $\Gamma^0/D$ is further increased beyond $2/\pi$, the $S$-matrix pole moves vertically upward approaching the real axis as $\Gamma^0/D \to \infty$. (Ref. 2, Eqs. (16d) and (16e).

Also shown in Figs. 8 and 9 are some values of the electric field indicated on the loci of points where the field is vertical and directed downwards. The points with $|\epsilon| = 1$ represent, in the energy plane, the $S$-matrix poles corresponding to the $R$-matrix with $\Gamma_\lambda^0 = 1$ and $D = 1$. For an $R$-matrix with $\Gamma_\lambda^0/D = 1/2$ and 2 say, the corresponding $S$-matrix poles would be the points in Fig. 8 given by $|\epsilon| = 2$ and 1/1 respectively. Evidently as $\Gamma_\lambda^0/D \to 0$ for the $R$-matrix, then the $S$-matrix poles approach the position of the wires (case of extremely narrow and well-separated resonances. As $\Gamma_\lambda^0/D$ is increased, all the $S$-matrix poles move *away* from the respective wires, along the corresponding loci. For $n$ odd the central pole moves away to $-\infty$ as $\Gamma_\lambda^0/D \to \infty$ while all the others approach the real axis again, on the side of the lobe opposite to the wire. For $n$ even (Fig. 9), the $S$-matrix poles associated with all but the two central wires behave similarly to the odd-wire case; as $\Gamma_\lambda^0/D$ is increased the two central $S$-matrix poles collide and then repel each other, as in the two wire case discussed before, so that only one of these two poles returns to the $x$-axis.

The fact that lobes deepen as one moves from wire to wire toward the center, as seen e.g., in Fig. 8, explains qualitatively the *increase* in $S$-matrix pole-widths as one goes from either extreme to the center. To be more precise, consider a given value of $|\epsilon|$ in Fig. 8 and follow it from any extreme lobe towards the center: this illustrates a result of Moldauer with the finite picket-fence model (Ref. 4, Fig. 1).

Finally we make some observations regarding the $S$-matrix *residues* using Eq. (2.19). Since $\epsilon^*(z)$ is a meromorphic function,and in fact by Eq. (2.9) is a Wigner $R$-function, the derivative $d\epsilon^*/dz$ will be the same in any direction at every point not on a wire. To calculate the derivative required in Eq. (2.19) it will be convenient to choose the direction defined by the tangent to the lobe at $z = z_i$ . Define a vector $\Delta z$ in this direction and pointing *away* from the corresponding wire. It is then easy to verify that $g_i^2$ is a vector perpendicular to the lobe at $z = z_i$ and pointing $90^\circ$ counter-clockwise to $\Delta z$ : these are shown as arrows in Fig. 8 for several points on the lobes, i.e., for the $S$-matrix poles associated with several $\Gamma^0/D$ values. Take for example a lobe in Fig. 8. As $\Gamma^0/D$ varies from 0 to $\infty$, then $g_i^2$ rotates in the complex-plane, from a real positive value to a real negative value, passing through a purely imaginary value at the bottom of the lobe. But for the central pole of Fig. 8 the residue is always real and positive. As mentioned before, a given lobe deepens as $n$ increases until eventually it degenerates into two vertical branch lines; hence the pole on the left branch has always in this limit a real, positive $g_i^2$ but for the right branch it is again real but ***negative***, this being in agreement with Moldauer's explicit result for the infinite picket-fence (Ref. 2, Eq. 16c).

## IV. CONCLUSIONS

We have seen that the problem of finding the $S$-matrix poles for a given $n$-level $R$-matrix in the single-channel case is equivalent to constructing an electrostatic system of $n$ charged infinitely long wires and looking, in the $x$-$y$ plane perpendicular to the wires, for those points where the electric field is in the $-y$ direction and has magnitude unity. One can deduce in a simple intuitive way some properties of the poles when one goes from the extremes to the center of the distribution of poles, both as a function of the $R$-matrix parameters and the number $n$ of poles.

The residues of the $S$-matrix turn out to be related to the derivative of the electric field and some properties are also derived in a simple intuitive fashion.

The previous properties coincide with the numerical results obtained by Moldauer in finite and infinite picket-fences by diagonalizing complex symmetric matrices.

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## RESUMEN

Un análisis cuantitativo de los polos y residuos de la matriz $S$ en términos de una matriz $R$ dada se puede llevar a cabo diagonalizando una matriz simétrica compleja.

En el presente trabajo se desarrolla una analogía electrostática del problema de un solo canal, consistente en un conjunto de alambres cargados, paralelos, e infinitamente largos; de la intuición que se tiene del comportamiento del campo eléctrico se puede obtener una idea cualitativa de algunas de las propiedades de los parámetros de la matriz $S$ en función de los de la matriz $R$, para el caso de un solo canal.