NEW FORMULATION OF STOCHASTIC THEORY AND QUANTUM MECHANICS. V. THE RELATIVISTIC SPINLESS PARTICLE*

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ABSTRACT: In this paper we develop an approximate relativistic formulation of the stochastic theory of quantum mechanics for spinless particles. The method used parallels as much as possible the non-relativistic derivation, which assumes local equilibrium. This last condition reflects itself in that the theory developed is valid only for times greater than $\hbar/mc^2$.

I. INTRODUCTION

The purpose of this paper is to formulate the stochastic theory of quantum mechanics in relativistic form. Our interest in this problem arises mainly in connection with the study of particles with spin; however, we shall restrict ourselves, for the time being, to the spinless case, deferring to a forthcoming paper the problem of spin.

Some efforts have been made to define and treat the relativistic stochastic process; nevertheless, a study of the pertinent literature seems

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to indicate that actually there does not exist such a thing as *the* relativistic generalization of stochastic theory, but that there are a number of open possibilities, at least in principle. Fortunately, we may simplify the present problem by the following considerations. In the non-relativistic case, Schrödinger's equation was deduced from a stochastic theory in configuration space[1]; we also know that the theory is approximate and can be considered valid only for time intervals larger than a certain minimum time, since the formulation presupposes local equilibrium (this point has been discussed at length in other works[2]). Therefore, when formulating the relativistic theory in space-time, we may be sure that the description will be valid only after local equilibrium has been attained, i.e., after a certain minimum time τ has elapsed. This restriction to space-time enormously simplifies our task, because the essentially non-relativistic interactions may be adequately treated as Markovian processes if only time intervals greater than τ are considered (see, e.g., refs. 3 and 4); moreover, it seems that, for the time being, this equilibrium approximation is sufficient for our purpose, since it leads directly to the usual quantum mechanical theories. In fact, the relativistic theory developed within this approximation will automatically satisfy the uncertainty relationships. The characteristic time τ may be estimated from simple arguments, the result being –as one may foresee – $\tau \sim \hbar/mc^2$; this result is a measure of the limit of applicability of the relativistic-quantum equations under conditions of local equilibrium.

II. THE FUNDAMENTAL RELATIVISTIC EQUATIONS

We begin our treatment by assuming that there exists a four-velocity c_μ, which may be decomposed into the sum of a systematic (v_μ) and a stochastic (u_μ) component, just as in the non-relativistic case ; hence, we write:

$$c_\mu = v_\mu + u_\mu \ . \qquad (1)$$

The four-velocities v_μ and u_μ have, by definition, different behaviour under time reversal (we here adhere to our previous convention and write $\hat{T}Q = \tilde{Q}$ for any quantity Q, $\hat{T}$ being the time reversal operator); the action of $\hat{T}$ on x_μ is defined as follows*:

*Throughout this paper we use the summation convention over repeated greek indexes, with $\mu = 1,2,3,4$ and $x_\mu = (\boldsymbol{x}, ix_0)$.

$$\tilde{x}_\mu = g_{\mu\nu} x_\nu , \tag{2}$$

where $g_{\mu\nu}$ is diagonal with elements (1, 1, 1, -1). We also postulate that the velocities transform under $\hat{T}$ as follows:

$$\begin{aligned} \tilde{v}_\mu &= - g_{\mu\nu} v_\nu \\ \tilde{u}_\mu &= g_{\mu\nu} u_\nu \end{aligned} \tag{3}$$

and hence

$$\begin{aligned} v_\mu &= \frac{1}{2} (c_\mu - g_{\mu\nu} \tilde{c}_\nu) \\ u_\mu &= \frac{1}{2} (c_\mu + g_{\mu\nu} \tilde{c}_\nu) . \end{aligned} \tag{4}$$

We seek differential operators representing a generalization of both the relativistic, non-stochastic d/ds and the stochastic, non-relativistic $\mathbb{D}$, with the help of which we may construct v_μ and u_μ from x_μ. With this aim, let us assume that we may find a parameter s, to be called in this paper the (ensemble) proper time, such that x_μ may be considered a function of it: $x_\mu(s)$, and that for a given s (which implies given initial conditions):

a) there exists an interval δs long enough for a particle to reach a local equilibrium state, but at the same time small enough for

$$\delta x_\mu = x_\mu(s + \delta s) - x_\mu(s)$$

to be small and to contain information about the random environment acting on the particle;

b) there exists a distribution of δx_μ, with the help of which it is possible to define for any function Q pertaining to the ensemble, the mean value of Q, which we write as $E\{Q\}$;

c) s is such that in the non-relativistic limit, $s = ct$, c being the velocity of light in vacuum*.

*For a more complete discussion about the definition and construction of s, we refer the reader to the literature, especially to Ref. (3).

Now let us consider a (well-behaved) function f, depending on s only through x_μ ; we then write

$$\frac{c}{\delta s} E\,\{f(x_\mu + \delta x_\mu) - f(x_\mu)\} = \frac{c}{\delta s} E \left\{ \frac{\partial f}{\partial x_\mu} \delta x_\mu + \frac{1}{2} \frac{\partial^2 f}{\partial x_\mu \partial x_\nu} \delta x_\mu \delta x_\nu + \dots . \right\}$$

In the non-relativistic theory, the process of calculating the corresponding expression in the limit $\delta s \to 0$ is considered permissible in spite of the assumption of local equilibrium[5]. This is no longer the case in the relativistic theory, because as soon as we go to this limit, some internal inconsistencies appear, as will be shown below. Hence we must content ourselves with an approximate expression, obtained by taking the limit as δs tends to a certain minimum called $s_0 = c\tau_0$, which measures the proper time the ensemble needs to attain local equilibrium. Let us suppose, in accordance with assumptions a) and b) above, that s_0 is small enough for the derivatives of f to be considered constant over the interval δs ; then we write

$$\mathbb{D} f \equiv \lim_{\delta s \to s_0} \frac{c}{\delta s} E\,\{f(x + \delta x) - f(x)\} = c_\mu \partial_\mu f + D'_{\mu\nu} \partial^2_{\mu\nu} f + \dots \tag{5}$$

with

$$c_\mu = c \lim_{\delta s \to s_0} E \left\{ \frac{\delta x_\mu}{\delta s} \right\} , \tag{6}$$

$$D'_{\mu\nu} = c \lim_{\delta s \to s_0} E \left\{ \frac{\delta x_\mu \delta x_\nu}{2 \delta s} \right\} \tag{7}$$

and so on.

We have thus obtained a covariant generalization of the total (mean) derivative operator, namely,

$$\mathbb{D} = c_\mu \partial_\mu + D'_{\mu\nu} \partial^2_{\mu\nu} + \dots \tag{8}$$

This operator is such that

$$c_\mu = \mathbb{D} x_\mu$$
$$\tilde{c}_\mu = \tilde{\mathbb{D}} \tilde{x}_\mu \tag{9}$$

or taking into account Eqs. (4),

$$v_\mu = \mathbb{D}_c x_\mu$$
$$u_\mu = \mathbb{D}_s x_\mu \tag{10}$$

with the systematic and stochastic operators defined by

$$\mathbb{D}_c = \frac{1}{2}(\mathbb{D} - \tilde{\mathbb{D}}) = - \tilde{\mathbb{D}}_c$$
$$\mathbb{D}_s = \frac{1}{2}(\mathbb{D} + \tilde{\mathbb{D}}) = \tilde{\mathbb{D}}_s \ , \tag{11}$$

respectively. Of particular interest for us is the case in which $D'_{\mu\nu}$ is proportional to the metric tensor:

$$D'_{\mu\nu} = D g_{\mu\nu} \tag{12}$$

since, if this condition is satisfied, the second order term in Eq. (8) reduces to $D\partial_\mu \partial_\mu$ when written in terms of the tensor

$$D_{\mu\nu} = g_{\mu\lambda} D'_{\lambda\nu} = D\delta_{\mu\nu} \tag{13}$$

$\partial_\mu \partial_\mu$ has precisely the form we are looking for in view of our direct interest in the Klein-Gordon theory. Note that Eq. (12) is valid only under the condition $s_0 \neq 0$, as can be seen by considering the non-relativistic approximation, in which we may identify s with ct; since furthermore $D_{ij} = D\delta_{ij}$, with $D = \hbar/2m$, we obtain from Eqs. (7) and (13):

$$D = c^2 D_{44} \sim \frac{1}{2} c^2 \tau$$

and hence,

$$\tau \sim \frac{2D}{c^2} = \frac{\lambda_c}{c}. \tag{14}$$

τ represents the time needed by the system to reach a state of local equilibrium and hence represents the minimum time interval for the theory to have any significant value. Eq. (14) indicates also that the localization of a relativistic particle cannot be better than the Compton wavelength $\lambda_c = \hbar/mc$ associated to the particle. These results are in complete agreement with the points of view usually adopted in connection with relativistic equations in quantum mechanics[6], although the arguments leading to them are quite different. From Eq. (14) we conclude, in particular, that the theory cannot be applied to electrons for time intervals $\Delta t \lesssim t \sim 10^{-20}$ sec and that for nucleons, this limit reduces to 10^{-23} sec. The same estimate has been given before from similar, but non-relativistic considerations[2].

In what follows we shall use the value of $D_{\mu\nu}$ given by Eq. (13) and thus write

$$\mathcal{D} = c_\mu \partial_\mu + D \partial_\mu \partial_\mu + \dots . \tag{15}$$

It appears convenient to write the equations in terms of four-momenta instead of four-velocities; hence, if m is the particle's rest mass, we define the systematic and stochastic four-momenta p_μ and q_μ as

$$p_\mu = m v_\mu$$
$$q_\mu = m u_\mu \tag{16}$$

so that according to Eq. (11),

$$m\mathcal{D}_c = p_\mu \partial_\mu + \dots$$
$$m\mathcal{D}_s = q_\mu \partial_\mu + mD \partial_\mu \partial_\mu + \dots \tag{17}$$

where we have used the property[1] $D = \tilde{D}$. Now let f_μ represent the total four-force acting on the particle. In the general case[7], f_μ may be decomposed into the sum of two parts having different behaviour under $\hat{T}$, namely,

$$f_\mu = f_\mu^{(+)} + f_\mu^{(-)} ,$$
$$\tilde{f}_\mu^{(\pm)} = \pm g_{\mu\nu} f_\nu^{(\pm)} . \tag{18a}$$

In order to establish the relationship between forces and accelerations, we introduce two postulates corresponding to those used in the non-relativistic theory. First, let us assume that the total force f_μ is related to the total acceleration $\mathcal{D}c_\mu$ as follows:

$$f_\mu = m\mathcal{D}c_\mu , \tag{19}$$

or, more explicitly,

$$f_\mu^{(+)} + f_\mu^{(-)} = \mathcal{D}_c p_\mu + \mathcal{D}_s q_\mu + \mathcal{D}_c q_\mu + \mathcal{D}_s p_\mu . \tag{20a}$$

The $\hat{T}$-transform of this equation is

$$f_\mu^{(+)} - f_\mu^{(-)} = \mathcal{D}_c p_\mu + \mathcal{D}_s q_\mu - \mathcal{D}_c q_\mu - \mathcal{D}_s p_\mu . \tag{20b}$$

This system of equations may be written in the simpler form:

$$\mathcal{D}_c p_\mu + \mathcal{D}_s q_\mu = f_\mu^{(+)}$$
$$\mathcal{D}_c q_\mu + \mathcal{D}_s p_\mu = f_\mu^{(-)} . \tag{21}$$

We now introduce a second dynamical postulate, which establishes that the total force f_μ differs from the external four-force $f_{0\mu}$ by a term of stochastic origin:

$$f_\mu = f_{0\mu} + (1+\lambda)\,\mathcal{D}_s q_\mu \,. \tag{22}$$

Using the same arguments as in the non-relativistic case, we can see that the relation expressed by Eq. (22) is the most general one compatible with the theory and having the correct non-stochastic limit. As in the non-relativistic case, λ is a free parameter.

From Eq. (22) and its $\hat{T}$- transform we readily obtain:

$$\begin{aligned} f_\mu^{(+)} &= f_{0\mu}^{(+)} + (1+\lambda)\,\mathcal{D}_s q_\mu \\ f_\mu^{(-)} &= f_{0\mu}^{(-)} \,. \end{aligned} \tag{23}$$

Introducing these results into Eqs. (21) we arrive at the fundamental equations of the theory:

$$\begin{aligned} \mathcal{D}_c p_\mu - \lambda\,\mathcal{D}_s q_\mu &= f_{0\mu}^{(+)} \\ \mathcal{D}_c q_\mu + \mathcal{D}_s p_\mu &= f_{0\mu}^{(-)} \,. \end{aligned} \tag{24}$$

There exists a simple alternative form of writing Eqs. (24) in the particular case of interest for quantum mechanics[1], i.e., $\lambda = 1$. To show this, let us introduce the following complex quantities

$$\begin{aligned} \mathfrak{p}_\mu &= p_\mu - i q_\mu \\ f_\mu^{q} &= f_{0\mu}^{(+)} - i f_{0\mu}^{(-)} \end{aligned} \tag{25}$$

and write Eqs. (24) in terms of them and their complex conjugates (c.c.):

$$\frac{\lambda+3}{4}\mathcal{D}_q\mathcal{P}_\mu+\frac{\lambda-1}{4}\mathcal{D}_q^*\mathcal{P}_\mu^*-\frac{\lambda-1}{4}(\mathcal{D}_q^*\mathcal{P}_\mu+\mathcal{D}_q\mathcal{P}_\mu^*)=f_\mu^q$$

$$\frac{\lambda+3}{4}\mathcal{D}_q^*\mathcal{P}_\mu^*+\frac{\lambda-1}{4}\mathcal{D}_q\mathcal{P}_\mu-\frac{\lambda-1}{4}(\mathcal{D}_q^*\mathcal{P}_\mu+\mathcal{D}_q\mathcal{P}_\mu^*)=f_\mu^{q^*}\ .$$

These equations are considerably simplified when λ assumes the quantum-mechanical value $\lambda = 1$:

$$\mathcal{D}_q\mathcal{P}_\mu=f_\mu^q$$
$$\mathcal{D}_q^*\mathcal{P}_\mu^*=f_\mu^{q^*}\ . \tag{26}$$

In what follows we shall restrict ourselves to this particular case. Since (26) is a set of c.c. equations, we shall use only one of them, say the first one, and consider it as the fundamental equation.

Let us now go over to the particular case of greatest interest, namely, when $f_{0\mu}$ corresponds to an external electromagnetic field. According to previous results[7] the corresponding forces must be written as:

$$f_{0\mu}^{(+)}=\frac{e}{mc}F_{\mu\nu}p_\nu-\frac{e}{c}\mathcal{D}_c A_\mu+\frac{e}{mc}p_\nu\partial_\nu A_\mu$$
$$f_{0\mu}^{(-)}=\frac{e}{mc}F_{\mu\nu}q_\nu-\frac{e}{c}\mathcal{D}_s A_\mu+\frac{e}{mc}q_\nu\partial_\nu A_\mu\ . \tag{27}$$

Here $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ stands for the electromagnetic tensor and the Lorentz gauge $\partial_\mu A_\mu = 0$ is imployed; Eqs. (27) represent the covariant form of the three-forces previously obtained and have been derived by an analogous procedure. Introducing them into the first one of Eqs. (26), we arrive at

$$\mathcal{D}_q\left(\mathcal{P}_\mu+\frac{e}{c}A_\mu\right)=\frac{e}{mc}\mathcal{P}_\nu\partial_\mu A_\nu\ . \tag{28}$$

III. THE KLEIN-GORDON AND CONTINUITY EQUATIONS.

The integration of Eq. (28) can be performed by translating into covariant language the arguments used in previous works[7]. We shall therefore omit most of the intermediate steps in the derivation of the corresponding "wave equation".

When D and all higher order coefficients are constant, the commutator of $\mathcal{D}_q$ and ∂_λ is

$$[\partial_\lambda, m\mathcal{D}_q] = (\partial_\lambda \mathcal{P}_\mu)\,\partial_\mu . \tag{29}$$

Hence, if we introduce a complex, dimensionless function w such that

$$\mathcal{P}_\mu + \frac{e}{c} A_\mu = \hbar \partial_\mu w , \tag{30}$$

we obtain with the help of Eq. (29) the following mathematical identity:

$$\mathcal{D}_q\left(\mathcal{P}_\mu + \frac{e}{c} A_\mu\right) = \frac{e}{mc}\, \mathcal{P}_\nu\, \partial_\mu A_\nu + \frac{1}{m}\, \partial_\mu W ,$$

where

$$W = m\hbar \mathcal{D}_q w - \frac{\hbar^2}{2}\, \partial_\nu w\, \partial_\nu w + \frac{e^2}{2c^2} A_\nu A_\nu .$$

Upon comparison with Eq. (28), we see that necessarily $\partial_\mu W = 0$, and hence W is a constant, so that

$$2\hbar m \mathcal{D}_q w - \hbar^2\, \partial_\mu w\, \partial_\mu w + \frac{e^2}{c^2} A_\mu A_\mu = - m^2 c^2 \tag{31}$$

is a first integral of Eq. (28). The value of the integration constant has been fixed by demanding that Eq. (31) give the correct result in the non-stochastic limit. This can be seen from the expansion of Eq. (31) to second order:

$$\mathbb{p}_\mu \mathbb{p}_\mu - i\hbar\, \partial_\mu \mathbb{p}_\mu = - m^2 c^2$$

which in the non-stochastic limit $\hbar = 2mD \rightarrow 0$ and $\mathbb{p}_\mu \rightarrow p_\mu$ yields correctly $p_\mu p_\mu = - m^2 c^2$.

When an approximation to second order is sufficient, Eq. (31) may be given a simpler form if written in terms of another function ψ, related to w by

$$w = - i \ln\psi , \tag{32}$$

the result being

$$\left(- i\hbar \partial_\mu - \frac{e}{c} A_\mu\right)^2 \psi + m^2 c^2 \psi = 0 \tag{33}$$

i.e., the Klein-Gordon equation. Note that only if condition (12) is fulfilled, does our system lead to the Klein-Gordon term $\partial_\mu \partial_\mu \psi$. Furthermore since in writing down Eq. (33) we have equated all higher-order coefficients to zero, we may say that, from the point of view of this theory, the Klein-Gordon equation corresponds to a relativistic description of a Markov process under local equilibrium.

We know from the non-relativistic theory[1] that Eq. (31), i.e., the first integral of the fundamental equations, contains the energy and continuity equations. Let us briefly review some of these questions in the relativistic case. In the Markovian approximation, a separation of the real and imaginary parts of Eq. (31) yields

$$2 p_\mu q_\mu + \hbar \partial_\mu p_\mu = 0 , \tag{34a}$$

$$p_\mu p_\mu - q_\mu q_\mu - \hbar \partial_\mu q_\mu = - m^2 c^2 . \tag{34b}$$

In the non-relativistic theory, the stochastic velocity $\mathbf{u}$ is related to the density of particles by $\mathbf{u}\rho = D\nabla\rho$; hence, we write in covariant form:

$$q_\mu \rho = \frac{\hbar}{2} \partial_\mu \rho \tag{35}$$

which substituted into Eq. (34a) gives:

$$\partial_\mu (\rho p_\mu) = 0 , \tag{36}$$

i.e., a continuity equation for the four-current

$$j_\mu = \rho v_\mu . \tag{37}$$

The scalar ρ may be interpreted as the density of particles in the frame in which the volume element concerned is at rest. The relation between ρ and ψ is readily established from the definitions given previously; in fact, we have from Eqs. (25), (30), (32) and (35) that

$$q_\mu = - Im\, \mathcal{P}_\mu = - \hbar \partial_\mu (Im\, w) = \frac{\hbar}{2} \partial_\mu (ln\, \psi^* \psi) = \frac{\hbar}{2} \partial_\mu\, ln\, \rho ,$$

so that

$$\rho = \psi^* \psi , \tag{38}$$

which is exactly the non-relativistic relation; however, it must be stressed that, according to Eq. (36), the density associated to a conserved quantity is not ρ but $j_0 = \rho v_0$. Eq. (36) is the usual continuity equation derived from the Klein-Gordon equation; in fact, we have

$$j_\mu = iD \left[\psi \partial_\mu \psi^* - \psi^* \partial_\mu \psi \right] - \frac{e}{mc} \psi^* A_\mu \psi , \tag{39}$$

as may be easily established. But this continuity equation is also a relativistic version of the diffusion equation, as can be seen by using $p_\mu = mc_\mu - q_\mu$ and Eq. (35) to rewrite Eq. (36) as follows:

$$\partial_\mu (\rho c_\mu) - D \partial_\mu \partial_\mu \rho = 0 . \tag{40}$$

This result is a particular case of the continuity equation for a relativistic Markov process under local equilibrium, previously derived by Hakim[3]. Its mathematical structure (in particular, its hyperbolic form) reflects its approximate character; in fact, it appears convenient to consider it a degenerate asymptotic form of the relativistic Fokker-Planck equation in μ-space, the approximation being valid only under local equilibrium.

Note that the assumption of local equilibrium already introduces into the theory a time arrow, since $\tau > 0$; this practically compensates for the loss of irreversibility implied by the use of a hyperbolic differential equation instead of a parabolic equation, i.e., the non-relativistic diffusion equation.

The analysis of Eq. (34b), the introduction of operators associated to the dynamical variables, etc. may be performed in analogy with the non-relativistic case.

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RESUMEN

En este trabajo se desarrolla una formulación relativista aproximada de la teoría estocástica de la mecánica cuántica para partículas sin espín. Se emplea un método similar al utilizado para construir la teoría no relativista, lo que implica trabajar en la aproximación de equilibrio local. Se demuestra que esta restricción implica la validez de la teoría sólo para intervalos de tiempo mayores que $\hbar/mc^2$.