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TIME DEPENDENT BEHAVIOUR OF A CLASSICAL MODEL FOR REACTIONS INVOLVING ISOBARIC ANALOGUE STATES*

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ABSTRACT: In the present paper we study the time dependent behaviour of a classical model for two-channel reactions involving isobaric analogue states. It is first shown that the poles of the S-matrix for the classical system are excluded from the upper half of the first Riemann sheet of the wave-number plane. The influence of the S-matrix poles in a time-dependent description of the problem is investigated. It is found that the poles in the first Riemann sheet give the usual exponentially decaying response in time (resonance effect), plus a "diffraction in time" effect, consisting of terms that also go to zero as $t \to \infty$, but as an inverse power of t. On the other hand, the poles in the second Riemann sheet give rise only to diffraction in time effects. It is made plausible, although it is not proved, that in the corresponding quantum-mechanical problem, only the poles in the lower half of the first sheet, that are in the fourth quadrant above the bisector at $7\pi/4$, will behave as resonance states.

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I. INTRODUCTION

In a recent paper[1] the authors discussed a classical model for reactions involving isobaric analogue states. This model had all the properties associated with these reactions and it allowed us to understand in a simple fashion a number of features of the physical problem, among them the Robson enhancement factor[2] due to the isobaric analogue state.

We obtained in reference 1 the **S**-matrix of the nuclear reaction problem. In this paper we shall discuss first the poles of the **S**-matrix showing that they are excluded from the upper half of the first Riemann sheet of the wave-number plane. We intend then to understand the physical significance of the poles of the **S**-matrix on those parts of the Riemann surface where they are allowed. This we achieve by discussing the time-dependent behaviour of the model which indicates that only the poles on the lower part of the first sheet gives states decaying with time. At the concluding section of this paper we shall discuss the significance of this result for the nuclear reaction problem.

We start our discussion with the presentation of the classical model, referring the reader to the previous paper[1] for all details.

The model (see Fig. 1) consists of a string capable of vibrating in any direction perpendicular to its length. The string is embedded in a rubber band and the displacements perpendicular and parallel to the rubber band are denoted by

$$u_1(x,t), \qquad u_2(x,t), \tag{1.1}$$

x being the length along the string

$$0 \leqslant x < \infty . \tag{1.2}$$

The string terminates at $x = 0$ on a mass connected to two springs perpendicular to each other, the direction of the first of which forms an angle α with $u_1(x,t)$. The displacements of the mass along the directions of the springs are denoted by

$$w_1(t) \text{ and } w_2(t) . \tag{1.3}$$

The equations of motion of our model are

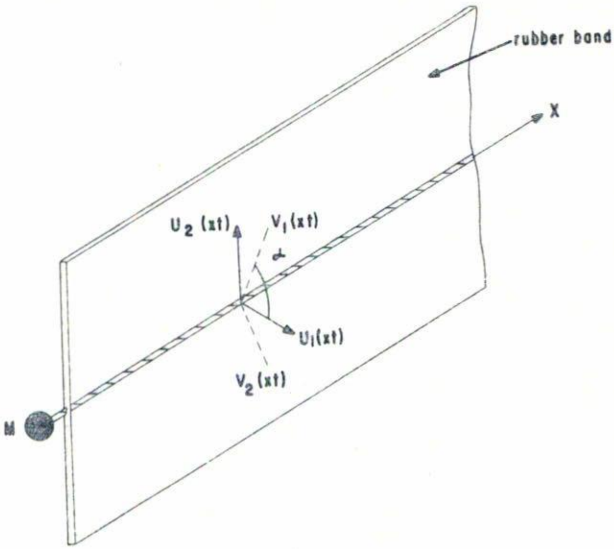

Fig. 1a Classical model for the external region of the two-channel nuclear problem, when the two fragments are separated by a variable distance x. The model consists of a string embedded in a massless elastic rubber band in the plane of the figure and connected to a mass M at $x = 0$. The string can vibrate in both directions perpendicular to its length.

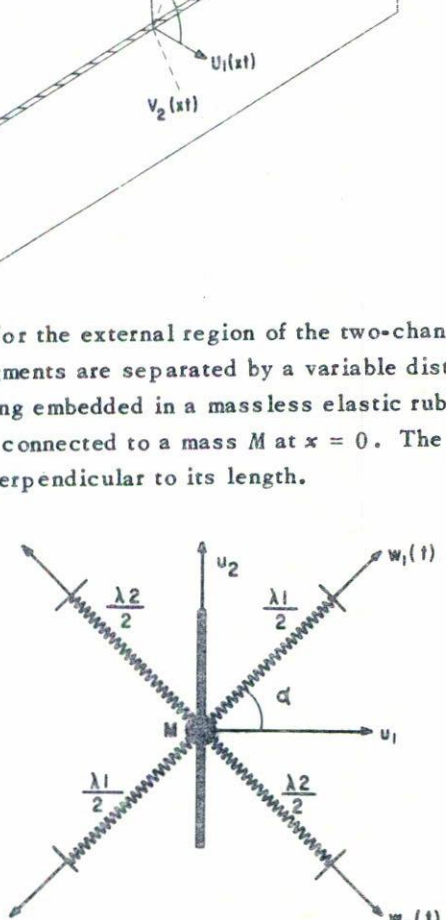

Fig. 1b Classical model for the internal region of the two-channel nuclear problem. Spring attachments of the mass M at the end of the string are shown, as well as a cross section of the rubber band.

$$\rho \frac{\partial^2 u_1}{\partial t^2} = \tau \frac{\partial^2 u_1}{\partial x^2} \qquad \rho \frac{\partial^2 u_2}{\partial t^2} = \tau \frac{\partial^2 u_2}{\partial x^2} - \lambda u_2 \tag{1.4a}$$

$$M \frac{d^2 w_1}{dt^2} = -\lambda_1 w_1 + \tau \left(\frac{\partial v_1}{\partial x} \right)_{x=0} \qquad M \frac{d^2 w_2}{dt^2} = -\lambda_2 w_2 + \tau \left(\frac{\partial v_2}{\partial x} \right)_{x=0}, \tag{1.4b}$$

where $v_1(x, t)$, $v_2(x, t)$ are the displacements of the string along the directions of the springs and they are related to $u_1(x, t)$, $u_2(x, t)$ by

$$\begin{bmatrix} v_1(x,t) \\ v_2(x,t) \end{bmatrix} = \mathbf{O} \begin{bmatrix} u_1(x,t) \\ u_2(x,t) \end{bmatrix}, \quad \mathbf{O} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{bmatrix}. \tag{1.4c}$$

In (1.4) we indicate by ρ and τ the density and tension of the string, by M the mass of the particle, by λ, λ_1 and λ_2 the spring constants of the rubber band and of the two springs to which the mass is connected. As the displacement of the mass and the string at $x = 0$ must be identical we have furthermore that

$$w_1(t) = v_1(0, t), \qquad w_2(t) = v_2(0, t). \tag{1.4d}$$

The $\boldsymbol{R}$-matrix for the problem (1.4) defined by

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_{x=0} = \boldsymbol{R}(\omega^2) \begin{bmatrix} \partial u_1/\partial x \\ \partial u_2/\partial x \end{bmatrix}_{x=0}, \quad u_c(x,t) = u_c(x)\, e^{-i\omega t}, \; c = 1,2 \tag{1.5a}$$

was shown to have the form[1]

$$\boldsymbol{R}(\omega^2) = \boldsymbol{ODO}, \tag{1.5b}$$

where $\boldsymbol{D}$ is a diagonal matrix

$$\mathbf{D} = \begin{bmatrix} d_1(\omega^2) & 0 \\ 0 & d_2(\omega^2) \end{bmatrix}, \tag{1.5c}$$

with

$$d_c(\omega^2) = \frac{\tau/M}{(\lambda_c/M) - \omega^2} .$$

We showed in reference 1 that the problem could be generalized by connecting the mass M to a system of masses interconnected with springs. This generalization allowed us to discuss reactions involving isobaric analogue states in which we still have only one $T_>$-state but several $T_<$-states. The **R**-matrix in this case retains the form (1.5) with the same $d_1(\omega^2)$, but with $d_2(\omega^2)$ taking the form

$$d_2(\omega^2) = \sum_\mu \frac{\gamma_\mu^2}{E_\mu - \omega^2} , \tag{1.5d}$$

where E_μ corresponds to the poles and γ_μ^2 to the reduced widths for the *R*-function of the $T_<$-states.

The **S**-matrix for the problem (denoted by $\hat{\mathbf{S}}$) was also discussed in reference 1 and it is related to the **R**-matrix in the usual fashion by

$$\hat{\mathbf{S}} = (1 - i\mathbf{RK})^{-1}(1 + i\mathbf{RK}) , \tag{1.6}$$

where $\mathbf{K}$ is the 2 x 2 matrix of the momenta in the two channels

$$\mathbf{K} = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}, \tag{1.7a}$$

with

$$k_1 = \frac{\omega}{c} , \quad k_2 = \sqrt{\left(\frac{\omega}{c}\right)^2 - \frac{\lambda}{\tau}} \tag{1.7b, c}$$

The matrix (1.6) is not unitary but it is easily seen that

$$S = K^{\frac{1}{2}} \hat{S} K^{-\frac{1}{2}} \tag{1.8}$$

is unitary, and, in fact S is defined as the coefficient of the outgoing wave when both the incoming and outgoing waves are normalized to unit flux in the corresponding quantum mechanical problem.

Before proceeding to make use of the S-matrix (1.6) to discuss the time-dependent behaviour of the scattering problem, we first show that the poles of the S-matrix are excluded from the upper part of the first sheet of the wave number plane.

II. DISTRIBUTION OF POLES OF THE S-MATRIX

From the expression (1.6) for the S-matrix we see that its poles are given by the equation

$$\det \| iK^{-1} + R \| = 0 . \tag{2.1}$$

We shall investigate the distribution of these poles in the k_1 complex plane rather than in the frequency plane $\omega = k_1 c$ $(c = \sqrt{\tau/\rho})$. From (1.7b, c) we see that

$$k_2 = \sqrt{k_1^2 - (\lambda/\tau)} \tag{2.2}$$

and, as the determinant (2.1) is also a function of k_2, the complex k_1 plane of Fig. 2 has two sheets connected by a cut from $-\sqrt{\lambda/\tau}$ to $\sqrt{\lambda/\tau}$. The first sheet is characterized by the fact that for real $k_1 > \sqrt{\lambda/\tau}$, k_2 defined by (2.2) is positive.

As ω^2 appears in our R-matrix (see Eq. (1.5)) we shall in this section use for it the notation

$$\omega^2 = k_1^2 c^2 \equiv E . \tag{2.3}$$

We proceed now to show that the S-matrix has no poles in the upper part of the first sheet of the Riemann surface of the k_1 -plane[3].

From (2.2, 2.3) we see that

$$2k_{2x}k_{2y} = 2k_{1x}k_{1y} = (E_y/c^2)\ , \qquad (2.4)$$

where k_{cx} , E_x are the real and k_{cy} , E_y the imaginary parts of k_c , $E, (c=1,2)$. The real and imaginary parts of k_2 cannot then change sign within each quadrant, and in the upper half of the first sheet of the k_1 Riemann surface, they have the signs indicated in Fig. 2 .

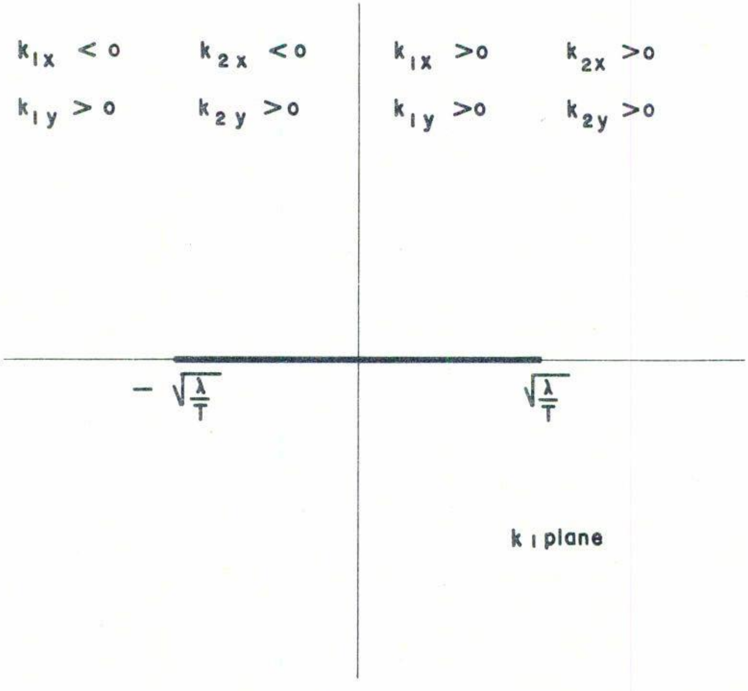

Fig. 2 The complex wave number plane.

We shall now assume that equation (2.1) has a root in the upper half of the first sheet of the k_1 plane, outside the imaginary axis, and show that this leads to a contradiction. If the root exists, then for that value of k_1 the set of linear equations

$$(iK^{-1} + R)\ \mathbf{x} = 0 \qquad (2.5)$$

has a solution where $\mathbf{x}$ is, in our problem, a two dimensional vector. This implies that

$$\mathbf{x}^\dagger \mathbf{R}\mathbf{x} = -i\mathbf{x}^\dagger \mathbf{K}^{-1}\mathbf{x} = -i \sum_{c=1}^{2} |x_c|^2 k_c^{-1} =$$

$$= \sum_c |x_c|^2 |k_c|^{-2} (-ik_{cx} + k_{cy}) \tag{2.6}$$

From (2.4) and (2.6) we conclude that

$$E_y \operatorname{Im}(\mathbf{x}^\dagger \mathbf{R}\mathbf{x}) = -2 \frac{T}{\rho} \sum_c |x_c|^2 |k_c|^{-2} k_{cx}^2 k_{cy} , \tag{2.7}$$

where the right hand side is clearly negative in the upper half of the k_1 plane, excluding the imaginary axis.

If we have though for the R matrix the general Wigner form[4]

$$R_{cc'}(E) = \sum_\mu \frac{\gamma_{\mu c}\gamma_{\mu c'}}{E_\mu - E} \tag{2.8}$$

then

$$E_y \operatorname{Im}(\mathbf{x}^\dagger \mathbf{R}\mathbf{x}) = E_y \operatorname{Im} \sum_\mu \frac{|\sum_c \gamma_{c\mu} x_c|^2}{E_\mu - E} = \sum_\mu \frac{|\sum_c \gamma_{c\mu} x_c|^2 E_y^2}{(E_\mu - E_x)^2 + E_y^2} , \tag{2.9}$$

and so the left side of (2.7) is positive. Thus we have proved our theorem that there are no poles of $\mathbf{S}(k_1)$ in the upper part of the first sheet of the k_1 complex plane, except possibly for the imaginary axis.

The theorem as it stands is also true for the nuclear case. For the mechanical model the theorem can be made stronger by using a causality argument to eliminate poles in the upper imaginary axis also. If such poles $k_1 = i\kappa$ with $\kappa > 0$ real exist, then the problem admits a solution of the type

$$\exp[\kappa(ct - x)] \tag{2.10}$$

which becomes ∞ when $t \to \infty$, thus violating the possibility of a causal description of the type to be discussed in the next section.

III. TIME-DEPENDENT BEHAVIOUR OF THE PROBLEM

As we indicated in the Introduction we intend to analyze the time-dependent behaviour of our model so as to get some insight on the physical significance of the poles of the **S**-matrix in the different regions of the k_1-plane. For this purpose it suffices to discuss the time-dependent behaviour when we assume that for $t \leqslant 0$ we have only incoming waves on an infinite string, and that at $t = 0$ we connect, at $x = 0$, the resonating system which has a finite number of degrees of freedom. Denoting, as in reference 1, by $u_{cc'}(x, t)$ the displacement of the string in the $c\,(= 1, 2)$ direction when the incoming vibration has the direction $c'\,(= 1, 2)$, we have

$$u_{cc'}(x, t \leqslant 0) = \exp\left[-ik_{c'}x\right] \exp\left[-i\omega t\right] \delta_{cc'} \ .$$

The initial conditions at $t = 0$ for the classical problem described by the equations of motion (1.4) are then

$$u_{cc'}(x, t = 0) = \exp\left[-ik_{c'}x\right] \delta_{cc'} \qquad (3.2\text{a})$$

$$\left(\frac{\partial u_{cc'}}{\partial t}\right)_{t=0} = -i\omega \exp\left[-ik_{c'}x\right] \delta_{cc'} \qquad (3.2\text{b})$$

$$w_{cc'}(t = 0) = 0, \quad \left(\frac{\partial w_{cc'}}{\partial t}\right)_{t=0} = 0, \quad w_{cc'}(t) = v_{cc'}(0, t)\,,$$

and the

$$v_{cc'}(x, t = 0)\,, \ (\partial v_{cc'}/\partial t)_{t=0}\,,$$

being related to the

$$u_{cc'}(x, t = 0)\,, \ (\partial u_{cc'}(x, t)/\partial t)_{t=0}$$

of (3.2 a, b) by the transformation (1.4c).

To solve the classical problem with the initial conditions (3.2), we first take the Laplace transform of the equations of motion (1.4). Defining

$$\bar{u}_{cc'}(x,s)=\int_0^\infty u_{cc'}(x,t)\,e^{-st}\,dt,\quad \bar{w}_{cc'}(s)=\int_0^\infty w_{cc'}(t)\,e^{-st}\,dt, \tag{3.3 a$'$, b$'$}$$

and similarly for $\bar{v}_{cc'}(x,s)$, the Laplace transformed equations become

$$\frac{d^2\bar{u}_{cc'}}{dx^2}-\kappa_c^2\,\bar{u}_{cc'}=-\frac{\rho}{\tau}(s-i\omega)\,e^{-ik_{c'}x}\,\delta_{cc'}\,, \tag{3.3 a}$$

with

$$\kappa_1=\frac{s}{c}\,,\quad \kappa_2=\left[\left(\frac{s}{c}\right)^2+\frac{\lambda}{\tau}\right]^{\frac{1}{2}}=\left[\kappa_1^2+\frac{\lambda}{\tau}\right]^{\frac{1}{2}},\quad c=\sqrt{\frac{\tau}{\rho}} \tag{3.3b}$$

and

$$(Ms^2+\lambda_c)\,\bar{w}_{cc'}=\tau\left(\frac{\partial\bar{v}_{cc'}}{\partial x}\right)_0, \tag{3.3 c}$$

$$\bar{v}_{cc'}(0,s)=\bar{w}_{cc'}(s)\,. \tag{3.3 d}$$

From the last two equations we have then, for the $\boldsymbol{D}$-matrix (1.5c),

$$\begin{bmatrix}\bar{v}_{1c'}\\ \\ \bar{v}_{2c'}\end{bmatrix}_{x=0}=\begin{bmatrix}\dfrac{(\tau/M)}{s^2+(\lambda_1/M)} & 0\\ \\ 0 & \dfrac{(\tau/M)}{s^2+(\lambda_2/M)}\end{bmatrix}\begin{bmatrix}\dfrac{\partial\bar{v}_{1c'}}{\partial x}\\ \\ \dfrac{\partial\bar{v}_{2c'}}{\partial x}\end{bmatrix}_{x=0} \tag{3.4}$$

so that from (1.5) we can write equations (3.3) in the matrix form

$$\frac{d^2\overline{U}}{dx^2} - \overline{K}^2\overline{U} = -\frac{1}{c^2}(s - i\omega)\begin{bmatrix} e^{-ik_1x} & 0 \\ 0 & e^{-ik_2x} \end{bmatrix}, \quad c = \sqrt{\frac{T}{\rho}} \tag{3.5a}$$

$$\overline{U}(x=0) = R(-s^2)\left(\frac{\partial \overline{U}}{\partial x}\right)_{x=0} \tag{3.5b}$$

where

$$\overline{U}(x,s) = \left\| \overline{u}_{cc'}(x,s) \right\|, \quad \overline{K} = \begin{bmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{bmatrix} \tag{3.5c,d}$$

The equation (3.5a) admits the particular solution

$$\frac{1}{s+i\omega}\begin{bmatrix} e^{-ik_1x} & 0 \\ 0 & e^{-ik_2x} \end{bmatrix} \tag{3.6}$$

Furthermore, the most general solution of the homogeneous part of the equation (3.5a), which would be bounded when $x \to \infty$, could be written in the form

$$\frac{1}{s+i\omega}\begin{bmatrix} e^{-\varkappa_1x} & 0 \\ 0 & e^{-\varkappa_2x} \end{bmatrix}\overline{S} \tag{3.7}$$

where $\overline{S}$ is a, so far, undetermined matrix. Writing then

$$\overline{U} = \frac{1}{s+i\omega}\left\{\begin{bmatrix} e^{-ik_1x} & 0 \\ 0 & e^{-ik_2x} \end{bmatrix} + \begin{bmatrix} e^{-\varkappa_1x} & 0 \\ 0 & e^{-\varkappa_2x} \end{bmatrix}\overline{S}\right\} \tag{3.8}$$

and introducing it in (3.5b), we obtain

$$\overline{S} = -[I + R(-s^2)\overline{K}]^{-1}[I + iR(-s^2)K], \tag{3.9}$$

with $\bar{K}$, K given by (3.5d) and (1.7) respectively.

The expression (3.9) for S holds not only for the problem (1.4), but also for the more general case when $d_2(\omega)$ has the form (1.5d), in which case $R(-s^2)$ is again given by (1.5b), but with D of tne general form (1.5c,d).

The time-dependent matrix

$$U(x,t) = \| u_{cc'}(x,t) \| , \tag{3.10}$$

is then given by the integral

$$U(x,t) = \frac{1}{2\pi i} \int_B \bar{U}(x,s)\, e^{st}\, ds , \tag{3.11}$$

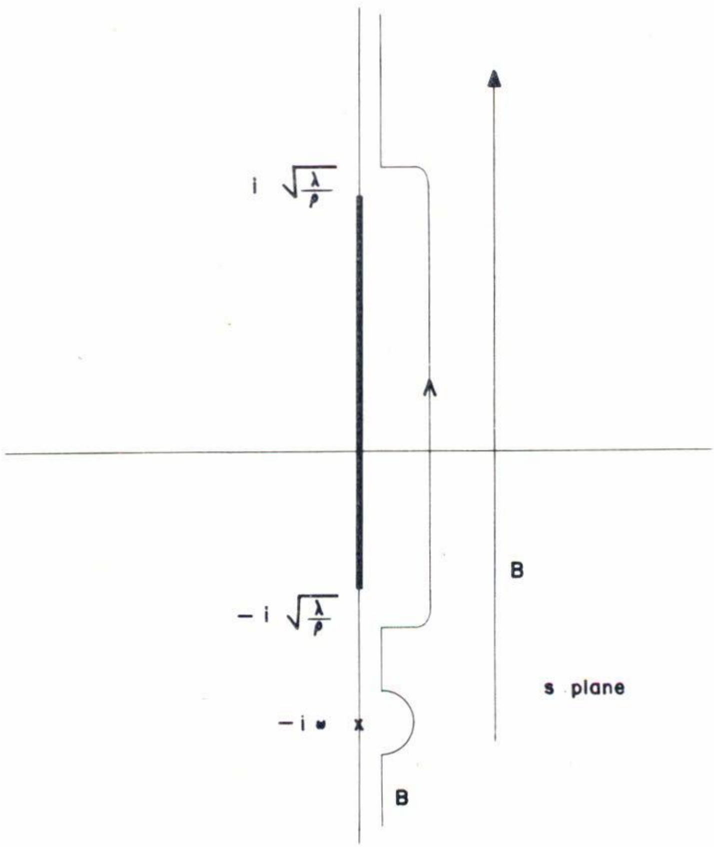

Fig. 3 The complex s-plane and the Bromwich integration contour.

over the Bromwich contour in the s plane indicated in Fig. 3. That contour must be to the right of all singularities of the function $U(x, s)$.

As κ_2 of (3.3b) appears in the $\bar{S}$ of (3.8), we have that $U(x, s)$ has branch points at

$$s = \pm ic\sqrt{\frac{\lambda}{\tau}} = \pm i\sqrt{\frac{\lambda}{\rho}}\,, \tag{3.12}$$

and so we must introduce a cut between them, as marked in Fig. 3. Besides, we have the pole $s = -i\omega$ also marked in the figure, as well as the poles of the S matrix given by the equation

$$\det\left|\bar{K}^{-1} + R(-s^2)\right| = 0\,. \tag{3.13}$$

Clearly, equation (3.13) becomes identical to (2.1) if we put $s = -i\omega$, and as $\omega = ck_1$, we see that, except for a change of scale, the planes k_1 and s can be transformed into each other if we rotate the former by $-\frac{\pi}{2}$. We saw though that equation (2.1) has no roots in the upper k_1 plane, which implies that (3.13) has no roots in the right hand side of the s plane and so there are no poles of $U(x, s)$ there. The Bromwich contour can then be changed to the one bordering the imaginary axis, but bypassing the cut and the pole $s = -i\omega$, as shown also in Fig. 3.

We wish now to evaluate explicitly $U(x, t)$ of (3.11). To do this it proves convenient to introduce a new complex variable defined by[5]

$$\zeta \equiv (\tau/\lambda)^{\frac{1}{2}}(\kappa_2 + \kappa_1) = (\tau/\lambda)^{\frac{1}{2}}\left\{\left[\left(\frac{s}{c}\right)^2 + \left(\frac{\lambda}{\tau}\right)\right]^{\frac{1}{2}} + \frac{s}{c}\right\} \tag{3.14a}$$

which implies that

$$\zeta^{-1} = (\tau/\lambda)^{\frac{1}{2}}(\kappa_2 - \kappa_1)\,, \tag{3.14b}$$

so that we get

$$\kappa_2 = \frac{1}{2}(\lambda/\tau)^{\frac{1}{2}}(\zeta + \zeta^{-1})\,,\quad \kappa_1 = \frac{1}{2}(\lambda/\tau)^{\frac{1}{2}}(\zeta - \zeta^{-1}) \tag{3.15a, b}$$

The two sheets of the Riemann surface of s are then mapped on a single plane ζ, with the cut corresponding to the circle of radius unity in the ζ plane. The first sheet of the plane is mapped on the outside, while the second is mapped on the inside of the unit circle in the ζ plane.

From (3.15) we note that $\mathbf{S}$ becomes now a rational function of ζ. Furthermore we have that

$$\frac{ds}{s + i\omega} = \frac{(1 + \zeta^2)\, d\zeta}{\zeta(\zeta - z)(\zeta + z^{-1})} \tag{3.16a}$$

where

$$z \equiv -i\left(\frac{\tau}{\lambda}\right)^{\frac{1}{2}} \left\{ \frac{\omega}{c} + \left[\left(\frac{\omega}{c}\right)^2 - \frac{\lambda}{\tau}\right]^{\frac{1}{2}} \right\} \tag{3.16b}$$

For positive real

$$\frac{\omega}{c} > \left(\frac{\lambda}{\tau}\right)^{\frac{1}{2}},$$

z is negative imaginary with $|z| > 1$.

The matrix $\boldsymbol{U}(x, t)$ of (3.11) takes then the form

$$\boldsymbol{U}(x,t) = \begin{bmatrix} e^{-ik_1 x} & 0 \\ 0 & e^{-ik_2 x} \end{bmatrix} e^{-i\omega t}$$

$$+ \frac{1}{2\pi i} \int_B \frac{(1 + \zeta^2)}{\zeta(\zeta - z)(\zeta + z^{-1})} \begin{bmatrix} e^{-\varkappa_1 x} & 0 \\ 0 & e^{-\varkappa_2 x} \end{bmatrix}$$

$$\overline{\mathbf{S}}(\zeta) \exp\left[\frac{1}{2}(\lambda/\rho)^{\frac{1}{2}}(\zeta - \zeta^{-1})\, t\right] d\zeta \tag{3.17}$$

where the first term on the right hand side is obtained by direct integration when introducing (3.8) into (3.11). The Bromwich contour is marked ir Fig. 4 and κ_1, κ_2 are given in terms of ζ by (3.15). It is clear that if

$ct < x$, we could close the Bromwich contour from the right and get a null value for the integral. We proceed now to evaluate the integral when $ct > x$, in which case we must close from the left.

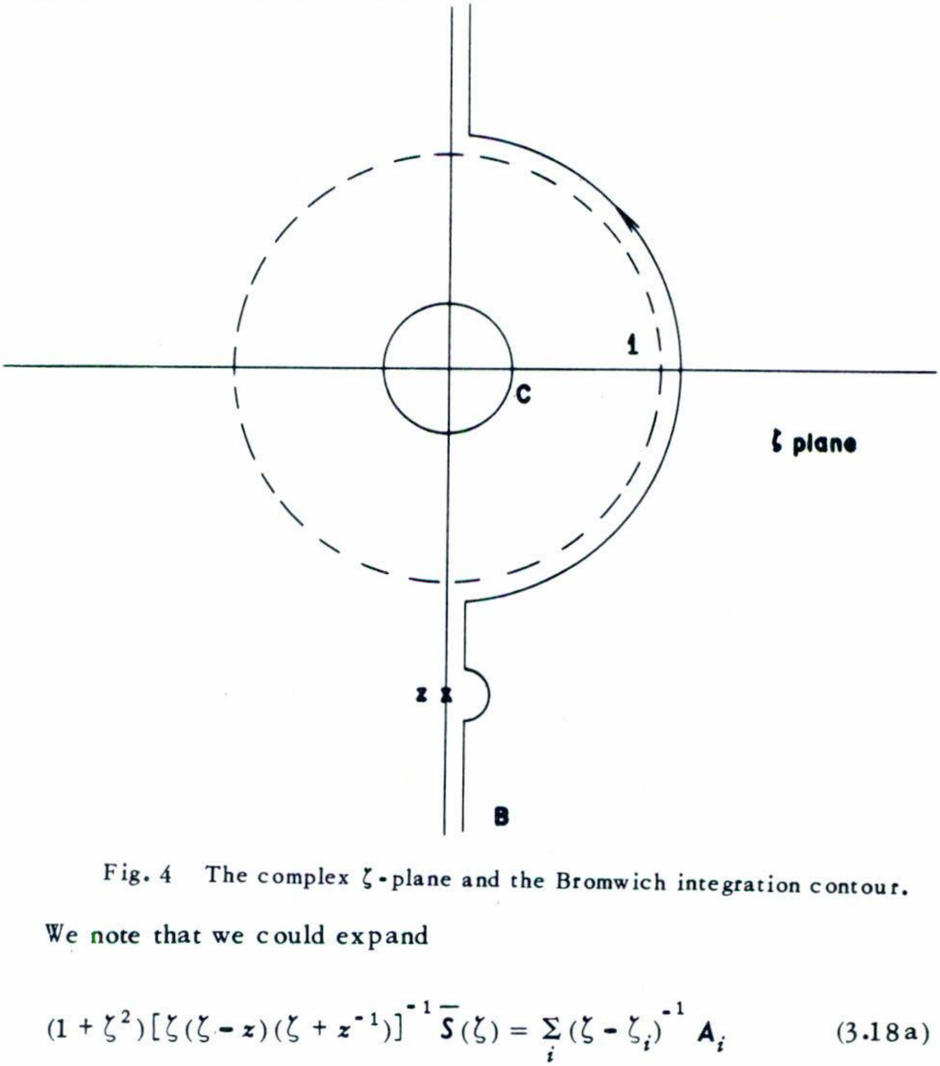

Fig. 4 The complex ζ-plane and the Bromwich integration contour.

We note that we could expand

$$(1+\zeta^2)[\zeta(\zeta-z)(\zeta+z^{-1})]^{-1}\,\overline{\mathbf{S}}(\zeta) = \sum_i (\zeta-\zeta_i)^{-1}\,\mathbf{A}_i \qquad (3.18a)$$

where

$$\mathbf{A}_i \equiv \begin{bmatrix} a^i_{11} & a^i_{12} \\ a^i_{21} & a^i_{22} \end{bmatrix}, \qquad (3.18b)$$

and ζ_i are the poles of the expression on the left hand side which include

$$\{\zeta_i\} = 0, z, -z^{-1}, \tag{3.19}$$

plus poles of the matrix $\bar{S}(\zeta)$.

The latter poles are the roots of the equation (3.13), and are all to the left of the Bromwich contour in Fig. 4.

We define now the basic time dependent solutions

$$u_c(x, t, \zeta_i) \equiv \frac{1}{2\pi i}\int_{B_c} \frac{e^{\frac{1}{2}\eta_c(\zeta-\zeta^{-1})}}{\zeta-\zeta_{ci}}\, d\zeta, \quad c = 1,2 \tag{3.20a}$$

where

$$\eta_1 = (\lambda/T)^{\frac{1}{2}}(ct-x), \quad \eta_2 = (\lambda/T)^{\frac{1}{2}}(c^2t^2-x^2)^{\frac{1}{2}},$$

$$\zeta_{1i} = \zeta_i, \quad \zeta_{2i} = \left[\frac{ct-x}{ct+x}\right]^{\frac{1}{2}}\zeta_i \quad c = \sqrt{T/\rho}. \tag{3.20b, c,d,e}$$

The Bromwich contour B_1 is the contour B of Fig. 4, while B_2 is B multiplied by the scale factor

$$[(ct-x)/(ct+x)]^{\frac{1}{2}}.$$

From (3.17) and (3.18), and introducing at the appropriate point the change of variable

$$\zeta' = \left[\frac{ct-x}{ct+x}\right]^{\frac{1}{2}}\zeta,$$

we immediately see that the components $u_{cc'}(x, t)$ of $U(x, t)$ can be written in the form

$$u_{cc'}(x, t) = \sum_i a^i_{cc'}\, u_c(x, t, \zeta_i) + \exp[-i(k_c x + \omega t)]\,\delta_{cc'} \tag{3.21}$$

It remains therefore only to evaluate explicitly the functions (3.20). For this purpose we note that for $\zeta_i \neq 0$, the Bromwich contours B_α can be deformed into two small circles, one denoted by C of radius smaller than $|\zeta_{ci}|$, surrounding the essential singularity at $\zeta = 0$, and the other surrounding the pole ζ_{ci}. The latter can be evaluated immediately and so we can write for $u_c(x, t, \zeta_i)$ the expression

$$u_c(x, t, \zeta_i) = \exp[\eta_c(\zeta_{ci} - \zeta_{ci}^{-1})/2]$$

$$+ \frac{1}{2\pi i} \oint_c \exp[\eta_c(\zeta_{ci} - \zeta_{ci}^{-1})/2]\, d\zeta \qquad (3.22)$$

To evaluate the last integral in (3.22), we consider the following expansions[6]

$$\exp[\eta_c(\zeta - \zeta^{-1})/2] = \sum_{n=0}^{\infty} \zeta^n J_n(\eta_c) + \sum_{n=1}^{\infty} (-1)^n \zeta^{-n} J_n(\eta_c), \qquad (3.23a)$$

$$(\zeta - \zeta_{ci})^{-1} = -\zeta_{ci}^{-1}\left(1 - \frac{\zeta}{\zeta_{ci}}\right)^{-1} = -\zeta_{ci}^{-1} \sum_{m=0}^{\infty} \left(\frac{\zeta}{\zeta_{ci}}\right)^m, \qquad (3.23b)$$

where $J_n(\eta_c)$ are Bessel functions of η_c. Taking the product of the two functions (3.23) and carrying out the integration over the circle c, which we may make as small as we wish, we see that only terms containing ζ^{-1} survive[6], so we get

$$u_c(x, t, \zeta_i) = \exp[\eta_c(\zeta - \zeta^{-1})/2] - \sum_{n=1}^{\infty} (-\zeta_{ci})^n J_n(\eta_c) \qquad (3.24)$$

We have then a closed expression for the basic functions albeit in the form of an infinite series. We can express $u_c(x, t, \zeta_i)$ in terms of known functions if we recall the definition of the Lomel functions of two variables[7]

$$L_p(t, \eta) \equiv \sum_{m=0}^{\infty} (-1)^m \left(\frac{t}{\eta}\right)^{p+2m} J_{p+2m}(\eta)\,. \qquad (3.25)$$

The basic functions can then be written as

$$u_c(x, t, \zeta_i) = \exp[\eta_c(\zeta - \zeta^{-1})/2] - L_0\left(\frac{i\eta_c}{\zeta_{ci}}, \eta_c\right)$$

$$- iL_1\left(\frac{i\eta_c}{\zeta_{ci}}, \eta_c\right) + J_0(\eta_c) \tag{3.26a}$$

The expression (3.26) gives the basic function for all $\zeta_i \neq 0$. For $\zeta_i = 0$ we see immediately from (3.23) that the integral in (3.20) reduces to

$$u_c(x, t, 0) = J_0(\eta_c) \ . \tag{3.26b}$$

If we introduce (3.26) into (3.21), we get the general time-dependent solution of our mechanical problem. It now remains to understand this solution from a more physical point of view.

Let us consider separately the terms in the summation (3.21), starting with the term associated with the pole $\zeta_i = z$. The residue of the expression (3.18) at $\zeta = z$ is then just $\mathbf{S}(z)$, which is the negative of the ordinary $\hat{\mathbf{S}}$-matrix (1.6). The contribution of this pole to the matrix $\mathbf{U}(x, t)$ is then given by

$$\begin{bmatrix} e^{i(k_1 x - \omega t)} & 0 \\ 0 & e^{i(k_2 x - \omega t)} \end{bmatrix} \hat{\mathbf{S}} + \begin{bmatrix} v_1(x, t, z) & 0 \\ 0 & v_2(x, t, z) \end{bmatrix} \hat{\mathbf{S}} \tag{3.27}$$

where, from (3.24),

$$v_1(x, t, z) \equiv - \sum_{n=1}^{\infty} (-z)^{-n} J_n\left[(\lambda/\tau)^{\frac{1}{2}}(ct - x)\right] \ ,$$

$$v_2(x,t,z) \equiv -\sum_{n=1}^{\infty}\left[-z\left(\frac{ct-x}{ct+x}\right)^{\frac{1}{2}}\right]^{-n} J_n\left[(\lambda/T)^{\frac{1}{2}}(c^2t^2-x^2)^{\frac{1}{2}}\right].$$

(3.28a,b)

The first term in (3.27) is then just the stationary outgoing wave with the appropriate matrix amplitude. The second term, is easily seen from (3.28) to vanish for $t \to \infty$, as $|z| > 1$. It therefore represents the transient effect that connects the initial condition at $t = 0$, in which we have no outgoing wave,(in fact, we have no outgoing wave until $t > \frac{x}{c}$), to the stationary state in which the outgoing wave is given by the first term in (3.27). These transient effects were given the name of diffraction in time and are closely connected with the operation of the time-energy uncertainty relation[8]

The situation for the other poles can be discussed in a similar fashion. Let us consider first all poles for which $|\zeta_i| > 1$. Then the contribution of these poles to the summation (3.21) can be put in the form

$$\begin{bmatrix} e^{i(k_{1i}x-\omega_i t)} & 0 \\ 0 & e^{i(k_{2i}-\omega_i t)} \end{bmatrix} \mathbf{A}_i + \begin{bmatrix} v_1(x,t,\zeta_i) & 0 \\ 0 & v_2(x,t,\zeta_i) \end{bmatrix} \mathbf{A}_i,$$

(3.29)

where $\mathbf{A}_i$ is given by (3.18) and $v_\alpha(x,t,\zeta_i)$ have the form (3.28a,b) with z replaced by ζ_i, and

$$k_{1i} = \frac{\omega_i}{c} = \frac{i}{2}\left(\frac{\lambda}{T}\right)^{\frac{1}{2}}(\zeta_i - \zeta_i^{-1}), \quad k_{2i} = i\left(\frac{\lambda}{T}\right)^{\frac{1}{2}}(\zeta_i + \zeta_i^{-1})$$

The first term in (3.29) represents decaying states in time, as ω_i has a negative imaginary part, in view of the fact that ζ_i with $|\zeta_i| > 1$ is on the left-hand side of the ζ plane outside of the unit circle. Again, because $|\zeta_i| > 1$, the terms $v_\alpha(x,t,\zeta_i)$ tend to zero when $t \to \infty$, and they represent diffraction in time effects in these decaying states.

For the poles $|\zeta_i| < 1$, there is an expansion similar to (3.29), but the $v_c(x,t,\zeta_i)$ do not tend to zero when $t \to \infty$ precisely because $|\zeta_i| < 1$. We can though expand the plane waves in the first part of (3.29), in terms of Bessel functions using (3.30) and (3.23a). We then get an expansion in terms of ζ_i^{-1} instead of ζ_i, and as $|\zeta_i^{-1}| > 1$, this expansion will go to zero when $t \to \infty$. Thus for poles within the unit circle, and this includes $\zeta_i = -z^{-1}$ and 0, we have just the behaviour associated with diffraction-in-time effects.

As the left hand side of the ζ-plane outside of the unit circle corresponds to the lower part of the first sheet of the Riemann surface for the k_1 - or ω-plane, we conclude that only the poles of the S-matrix in the lower part of this first sheet give rise to the states decaying in time, that we usually associate with the resonances in nuclear reactions.

V. CONCLUSIONS

For the classical model described in the Introduction, we have first shown that the poles of the S-matrix are excluded from the upper half of the first Riemann sheet of the wave number k_1 and they have no restrictions in the second sheet. The influence of these poles in a time-dependent description of the problem has been investigated. It was found that the poles in the first Riemann sheet give the usual exponentially decaying response in time (resonance effect), plus a diffraction in time effect, consisting of terms that also go to zero as $t \to \infty$, but as an inverse power of t. On the other hand, the poles in the second Riemann sheet give rise only to diffraction in time effects.

The extension of the previous analysis to the quantum mechanical case with two channels is not obvious. For this purpose one might be guided by comparing the classical and quantum results for the one-channel case. In this case we have only one Riemann surface for the variable k. In the classical model every pole in the allowed lower region contributes a term that decays exponentially with time. In the quantum mechanical case[5] the only poles that have a resonance behaviour are those located in the 4th quadrant above the bisector at $7\pi/4$. We could conclude by analogy, that in the two-channel case discussed here, when we go to the quantum mechanical picture, (i.e., $E = k^2/2$ rather than $\omega = ck$) only the poles in the lower half of the first sheet that are in the fourth quadrant and above the bisector at $7\pi/4$, will behave as resonance states.

REFERENCES

1. P.A. Mello and M. Moshinsky, Am. J. Phys. (submitted for publication).
2. D. Robson, Phys. Rev. 137 (1965) B535.
3. M. Moshinsky, Phys. Rev. 91 (1953) 984.
4. A.M. Lane and R.G. Thomas, Rev. Mod. Phys. 30 (1958) 257.
5. M. Moshinsky, Phys. Rev. 84 (1951) 525. Proceedings of the Conference on Novas Tecnicas de Pesquisa en Física (Río de Janeiro 1952) p. 285.
6. E.T. Whittaker and G.N. Watson, "Modern Analysis" (MacMillan Co., New York 1943).
7. G.N. Watson, "The Theory of Bessel Functions", 2nd Ed. (Cambridge University Press 1944) p. 537.
8. M. Moshinsky, Phys. Rev. 88 (1952) 625; Rev. Mex. Fís. 1 (1952) 28.

RESUMEN

En el presente artículo se estudia el comportamiento temporal de un modelo clásico que simula reacciones con dos canales, en las que intervienen estados isobáricos análogos. Primero que nada se muestra que los polos de la matriz S para el sistema clásico están excluidos del semiplano superior de la primera hoja de Riemann del número de onda. Enseguida se estudia la influencia de los polos de S en una descripción temporal del sistema. Se encuentra que los polos en la primera hoja de Riemann dan lugar a un transitorio que decae exponencialmente con el tiempo (que es la respuesta usual de una resonancia), más un efecto de "difracción en el tiempo", que consiste de términos que también tienden a cero cuando $t \to \infty$, pero como una potencia inversa de t. Por otro lado, los polos en la segunda hoja de Riemann sólo dan lugar a efectos de difracción en el tiempo. No se demuestra, pero se hace plausible, que en el correspondiente problema cuántico sólo los polos que están en el cuarto cuadrante de la primera hoja y arriba de la bisectriz a $7\pi/4$, se comportan como estados resonantes.