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SOME COMMENTS ON STOCHASTIC QUANTUM MECHANICS*

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ABSTRACT: In this note we briefly comment on some aspects of stochastic quantum mechanics. First we present a formal but elementary derivation, based on classical principles, of Schrödinger's equation; this suggests once more the stochastic (Markovian) origin of quantum phenomena. We show that for stationary states without a net flux of matter, the particles belonging to the quantum ensemble distribute themselves in such a way as to guarantee a (relative) minimum value for the total energy of the ensemble. Finally, we show explicitly that the analogy between quantum mechanics and a classical fluid described by the Navier-Stokes equations lacks of any profound physical content.

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I. INTRODUCTION

The very nature of Schrödinger's equation as an equation of motion for quantum particles has led to several attempts of interpreting it in terms of particle trajectories. The first step in this direction is due to one of the founders of quantum mechanics, L. de Broglie[1]. Several theories have since been developed. On the one hand, Bohm[2] has shown that the electron may be treated as a classical particle subject to an additional potential ϕ_B, which is itself a function of the amplitude ψ; hence, ψ is assumed to represent a real field. On the other hand, Fényes[3] and Weizel[4] have attempted to demonstrate the possibility of understanding Schrödinger's theory by postulating the randomness of the electron's trajectories. (This point of view is supported by the work of Moyal[5] who has shown that, at least formally, quantum mechanics may be considered a special kind of Markov process in phase space). The electromagnetic field would thus necessarily contain a radiation component due to the stochastically moving charged particles. The existence of this random field may alternatively be used as a starting point, as done by Sokolov and Tumanov[6], who have been able to quantize the classical harmonic oscillator by adding to it a fluctuating force due to such an electromagnetic vacuum. More recently, Marshall[7] and Boyer[8] have shown, using exclusively classical arguments, that the mere existence of the zero-point energy implies the quantization of the electromagnetic field. Several authors[9-15] have since then continued these investigations, in an attempt to establish a more conclusive connection between quantum mechanics and the theory of Markov processes. (We shall not mention here a series of valuable works, connected to ours through the use of formally similar procedures, but being of no immediate interest for us. For references, the reader may resort to the cited literature).

From the work done up to now, it seems reasonable to conclude that quantum mechanics is indeed a stochastic problem or at least, may be derived from a classical theory of random motion. However, this extremely interesting question is still in its beginnings and a great deal of work will be necessary to obtain a definite answer. At this stage, it seems desirable to discuss the ideas and postulates of the stochastic theory of quantum mechanics in a systematic way, to which the present note intends to contribute by discussing a few elementary but interesting aspects of it.

As was mentioned before, Bohm[2] has shown that the knowledge of Schrödinger's equation allows us to derive from it a classical description, while several other authors[9-15] have shown that the inverse procedure works even better. In particular, Santos[15] has derived Schrödinger's equation from a stochastic theory, through the use of a classical variational principle. However, in his derivation the author resorts to some considerations out of the realm of mechanics; since it seems to us that his method offers a very simple way of obtaining Schrödinger's equation from a classical stochastic theory, we present in Sect. II a similar formal derivation, though more straightforward, simpler and devoid of conceptual difficulties. Incidentally, this derivation seems more convincing and complete than that given in the first paper of reference 13 and hence, may be considered an improved version of it.

Making use once more of variational methods, we demonstrate in Sect. III that in a stationary bounded system the particles distribute themselves in such a way as to guarantee a minimum value for the energy – a well-known result in ordinary quantum mechanics. The present proof, although more complex than the usual one found in textbooks, seems interesting because use is made only of the mathematical structure of the kinetic energy term associated to the stochastic motion.

Some aspects of the mathematical structure of quantum mechanics have led several authors[16] to establish an analogy with classical fluid mechanics, which in our opinion is only apparent. In Sect. IV we show that if such an analogy is constructed, the corresponding "stress tensor" of quantum mechanics has the wrong properties and hence is not a stress tensor.

II. DERIVATION OF SCHRODINGER'S EQUATION

Consider a classical particle acted on by a stochastic background. We postulate that the system may be approximately described in terms of a probability distribution in configuration space, satisfying the diffusion equation

$$\frac{\partial \rho}{\partial t} = - \nabla \cdot (\rho \mathbf{c}) + D\nabla^2 \rho \,; \tag{1}$$

here ρ may represent in general not only the inconditional distribution function, but also the transition frequency distribution of $\mathbf{x}(t)$, $\mathbf{x}$ being approximated by a Wiener process[17]. In Eq. (1), $\mathbf{c}$ represents the particle's velocity and D the diffusion coefficient, i.e., a measure of the dispersion of the particle's displacement due to the stochastic background. Although somewhat more restrictive than necessary, postulate (1) is sufficient for our purposes[14]. We may express it in the form of a continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\mathbf{v}\rho) = 0\,, \tag{2}$$

as long as

$$\mathbf{v} = \mathbf{c} - D\,\frac{\nabla\rho}{\rho} \tag{3}$$

holds. The term $D\rho^{-1}\nabla\rho$ represents the diffusion velocity[12], first introduced by Einstein[18] as the osmotic velocity in Brownian movement and is precisely the stochastic velocity $\boldsymbol{u}$ defined in our earlier work[14]; hence,

$$\mathbf{c} = \mathbf{v} + \boldsymbol{u}\,. \tag{4}$$

From Eq. (2) it follows that $\mathbf{v}$ represents a flow or current velocity; hence we name it the systematic velocity and interpret Eq. (4) by saying that the particle's velocity has systematic (current) and stochastic (diffusion) components.

Consider now a volume element d^3x containing a total mass $m\rho d^3x$, where ρ is the expected mean number of particles per unit volume. From Eq. (2) it follows that the kinetic energy associated to this volume element is $1/2\ m\rho\,\mathbf{v}^2\,d^3x$; however, this is not all the kinetic energy, since according to Eq. (4) there must be also a contribution from the stochastic velocity $\boldsymbol{u}$. In order to take this into account, we introduce a second postulate, proceeding in a similar, but less restrictive way as in reference (15): we *assume* that $\boldsymbol{u}$ contributes to the total energy with a term given by

$$U = \int \frac{1}{2} m\rho\,\boldsymbol{u}^2 d^3x\,. \tag{5}$$

In order to write this term in a more convenient form, we use the definition of $\boldsymbol{u}$ to obtain, for any finite system,

$$\begin{aligned}\int \rho\boldsymbol{u}^2 d^3x &= -D^2\int \rho\nabla\cdot(\nabla\rho/\rho)\,d^3x \\ &= -D\int(\nabla\cdot\boldsymbol{u})\rho\,d^3x\,,\end{aligned} \tag{6}$$

whence Eq. (5) transforms into

$$U = \int \rho\,\mathcal{U}\,d^3x \tag{7}$$

with

$$\mathcal{U} = -\frac{1}{2} m \left[\lambda u^2 + D(1+\lambda) \nabla \cdot \mathbf{u} \right], \tag{8}$$

since according to Eq. (6), the two terms containing the arbitrary parameter λ cancel out.

Note that the kinetic energy associated to the u-motion does not depend on the systematic velocity $\mathbf{v}$; hence, if $\mathbf{x}$ and $\mathbf{v}$ are considered as the independent variables in our lagrangian*, $\mathcal{U}$ becomes a function of the position coordinates only, and we may therefore treat it formally as a potential energy term. The stochastic problem has thus been reduced to an equivalent classical problem whose corresponding Hamilton-Jacobi equation is:

$$\frac{\partial S_0}{\partial t} + \frac{1}{2m}(\nabla S_0)^2 + V + \mathcal{U} = 0,$$

where

$$\nabla S_0 = m\mathbf{v}.$$

Writing down this equation is equivalent to introducing a classical Hamilton's principle, namely,

$$\int_{t_1}^{t_2} dt \int d^3x\, \mathcal{L}\rho = \text{extremum},$$

with ρ satisfying Eq. (2) and with $\mathcal{L}$ given by

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{U}$$

where $\rho\mathcal{L}_0$ stands for the classical lagrangian density. In this sense, S_0 is the Lagrange multiplier for the above variational problem conditioned by $\int \rho\, d^3x = 1$. It is convenient to introduce a dimensionless function S, defined by

$$S_0 = 2DmS$$

*Note that we are *not* assuming $\mathbf{v} = \dot{\mathbf{r}}$; this equality is valid only in the statistical sense.

in terms of which

$$2Dm \frac{\partial S}{\partial t} + 2D^2 m (\nabla S)^2 + V + \mathcal{U} = 0 \tag{9}$$

and

$$\mathbf{v} = 2D\nabla S \,. \tag{10}$$

From Eq. (9) we see that if $\mathcal{H}$ represents the equivalent hamiltonian

$$\mathcal{H} = \frac{(m\mathbf{v})^2}{2m} + \mathcal{U} + V, \tag{11}$$

then

$$\mathcal{H} = -2Dm \frac{\partial S}{\partial t}, \tag{12}$$

and the total energy is therefore

$$E = \int \mathcal{H}\rho d^3x = -2Dm\int \rho \frac{\partial S}{\partial t} d^3x \,.$$

It has been demonstrated in a number of ways that the system of Eqs. (2) and (9), together with the definitions of $\mathbf{u}$ and $\mathbf{v}$, yield Schrödinger's equation if $\lambda = 1$ (see for instance references 12 to 15). In the general case, for arbitrary λ, we have[14]

$$2iDm \frac{\partial \psi}{\partial t} = -2mD^2\nabla^2\psi + (V + V_u)\psi, \tag{13}$$

with

$$V_u = \frac{1}{2} m(1-\lambda)(\mathbf{u}^2 + D\nabla \cdot \mathbf{u}), \tag{14}$$

so that in fact, for $\lambda = 1$ we obtain Schrödinger's equation if $2Dm = \hbar$ (which is an experimental datum!). To obtain Eq. (13), we have combined Eqs. (2) and (9) and introduced

$$\psi = \rho^{1/2} \exp(iS) \tag{15}$$

and its complex conjugate.

Hence, we have shown that Schrödinger's equation can in fact be derived from a stochastic (Markovian) problem with the help of classical arguments, if the stochastic contribution to the kinetic energy is explicitly taken into account. In this alternative, formal derivation, two parameters appear whose value must be taken from experiment, namely, $D = \hbar/2m$ and $\lambda = 1$.

III. THE ENERGY AS AN EXTREMUM

We proceed to consider the simple, but important case in which the net velocity of flux $\mathbf{v}$ is zero. In this case, according to Eq. (2), ρ does not vary with time. From Eqs. (10) and (12) we conclude that $S = \epsilon t/\hbar$ and therefore Eq. (9) reduces to:

$$-\frac{1}{2} m\mathbf{u}^2 - mD\nabla \cdot \mathbf{u} + V = \epsilon , \tag{16}$$

where ϵ represents the local value of the total energy per particle and is constant due to the stationary character of the problem. The total energy is therefore

$$E = \int \epsilon \rho d^3x = \int \rho \left[-\frac{1}{2} m\mathbf{u}^2 - mD\nabla \cdot \mathbf{u} + V \right] d^3x$$

or, due to Eq. (6),

$$E = \int \rho \left[\frac{1}{2} m\mathbf{u}^2 + V \right] d^3x . \tag{17}$$

Since ρ has been normalized to unity, the energy of the system E and the energy per particle $\tilde{\epsilon}$ coincide. Eq. (17) together with the continuity equation in integral form

$$\int \rho d^3x = 1 \tag{18}$$

are equivalent to Eqs. (2) and (9) for the fluxless case.

At this point, let us study the consequences of assuming that the energy attains an extremum value. We must solve the following conditioned variational problem

$$\int \rho \left[\frac{1}{2} m \mathbf{u}^2 + V(x)\right] dx = \text{extremum}$$
$$\int \rho dx = \text{constant}, \tag{19}$$

(we work in one dimension for simplicity). It is convenient to introduce a new real function φ defined by

$$\varphi = \rho^{1/2}, \tag{20}$$

in terms of which we obtain, for any finite system,

$$\frac{1}{2} m\rho \mathbf{u}^2 = \frac{1}{2} mD^2 \rho^{-1} \left(\frac{d\rho}{dx}\right)^2 = 2mD^2 \left(\frac{d\varphi}{dx}\right)^2$$

and hence,

$$\int \left[\frac{\hbar^2}{2m}\left(\frac{d\varphi}{dx}\right)^2 + V\varphi^2\right] dx = E = \text{extremum}$$
$$\int \varphi^2 dx = 1 \tag{21}$$

We shall use a well-known result of the theory of the Sturm-Liouville equation[19], namely, that the Euler-Lagrange equation for the conditioned variational problem

$$\int_a^b \left[f(x)\left(\frac{d\varphi}{dx}\right)^2 - g(x)\varphi^2\right] dx = \text{extremum}$$
$$\int_a^b \varphi^2(x)\, h(x)\, dx = \text{constant} \tag{22}$$

is the Sturm-Liouville equation

$$\frac{d}{dx}\left[f(x)\frac{d\varphi}{dx}\right] + \left[g(x) + \lambda h(x)\right]\varphi = 0. \tag{23}$$

The system of Eqs. (21) is the same as that of Eqs. (22) for

$$f(x) = \frac{\hbar^2}{2m}\,, \quad h(x) = 1 \text{ and } g(x) = -V(x)\,.$$

Substituting these values into Eq. (23) we arrive at the stationary Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2\varphi}{dx^2} + V(x)\,\varphi = -\lambda\varphi\,, \tag{24}$$

the Lagrange multiplier being determined by the first of Eqs. (21) as the total energy

$$E = \int\varphi\left[-\frac{\hbar^2}{2m}\frac{d^2\varphi}{dx^2} + V\varphi\right]dx = \int\varphi(-\lambda)\,\varphi\,dx = -\lambda\,.$$

This result allows us to state that the particles of any finite quantum-mechanical ensemble without net flux of matter are distributed so as to guarantee an extremal value (actually a minimum from stability considerations) for the total energy of the system. Hence, in conclusion, we may say that the stationary Schrödinger equation is the Euler-Lagrange equation of the variational problem associated to a stationary stochastic (Markovian) ensemble.

IV. COMMENTS ON THE HYDRODYNAMICAL ANALOGY

As stated in the introduction, the formal similarity between quantum mechanics and classical fluid mechanics has led several authors to interpret the equations studied in Section II as describing an analog classical fluid[16]. In order to analyze this interpretation, we procced to derive the tentative Navier-Stokes analog along lines similar to those of reference (16). Since we are particularly interested in the quantum-mechanical case ($\lambda = 1$) we define

$$\phi_B = \mathcal{U}(\lambda = 1) = -\frac{1}{2}\,m\,(\boldsymbol{u}^2 + 2D\nabla\cdot\boldsymbol{u})\,. \tag{25}$$

ϕ_B is the potential introduced by Bohm in his causal interpretation of quantum mechanics[2] and afterwards interpreted by several authors[13,16] as a term of kinetic origin; indeed, we have seen before that ϕ_B measures the contribution of the u-motion to the kinetic energy. Using Eq. (25), we may rewrite Eq. (2) and the gradient of Eq. (9) as follows (we use tensorial notation for simplicity):

$$m\,\frac{\partial v_i}{\partial t} + m\,(v_k\,\partial_k)\,v_i = -\,\partial_i\,[\phi_B + V]$$

$$\frac{\partial \rho}{\partial t} + \partial_k(\rho v_k) = 0\,. \tag{26}$$

These equations may be combined to yield

$$m\,\frac{d}{dt}(\rho v_i) = -\,\partial_k \pi_{ik} - \rho\partial_i V \tag{27}$$

with the components of the tensor π given by

$$\pi_{ij} = m\rho v_i v_j - \frac{1}{2}\hbar\rho\partial_i u_j \tag{28}$$

or equivalently,

$$\pi_{ij} = m\rho v_i v_j - \sigma_{ij}$$

$$\sigma_{ij} = \frac{1}{4}\hbar\rho(\partial_i u_j + \partial_j u_i) \tag{29}$$

since u is irrotational. π has been formally identified with a flux density tensor, and σ with the corresponding stress tensor:

$$\sigma_{ij} = \eta\,(\partial_i u_j + \partial_j u_i)\,, \tag{30}$$

the dynamical viscosity η being

$$\eta = \frac{1}{4}\hbar\rho\,.$$

Since $\boldsymbol{u}$ is a velocity, one is indeed tempted to identify σ as a stress tensor, as defined in connection with the motion of a fluid with flux velocity $\mathbf{v}$[20]:

$$\sigma_{ij} = \eta(\partial_i v_j + \partial_j v_i) . \tag{31}$$

Eq. (27) would then be the corresponding momentum-flow equation

$$m\rho\left(\frac{\partial}{\partial t} + v_k \partial_k\right) v_i = - \partial_k \pi_{ik} - \rho \partial_i V \tag{32}$$

However, several objections may be raised against this interpretation. Firstly, there appears a contradiction related to the sign of the viscosity η, which is in general determined by the sign of the parameter λ and therefore varies according to the particular problem. For the (frictionless) case of quantum mechanics, $\lambda = 1$ and there would indeed be a viscosity, while for the Smoluchowsky approximation to Brownian motion, $\lambda = -1$ (as has been shown elsewhere[14]) and consequently there would be no viscosity in the dissipative Brownian movement. These considerations suggest introducing a minus sign into the definition of both η and σ_{ij} in Eq. (30).

In the second place, since the kinetic energy associated to the transfer of mass is given by

$$T = \int \frac{1}{2} m\rho \mathbf{v}^2 d^3x , \tag{33}$$

we obtain from Eq. (27) and the continuity equation

$$\begin{aligned} \frac{\partial T}{\partial t} &= - \frac{m}{2}\int \sigma_{ij} (\partial_i v_j + \partial_j v_i)\, d^3x \\ &= - \frac{m\hbar}{2}\int \rho(\partial_i u_j + \partial_j u_i)(\partial_i v_j + \partial_j v_i)\, d^3x ; \end{aligned} \tag{34}$$

therefore, we conclude that the sign of $\partial T/\partial t$ is not definite and hence σ_{ij} does not involve in general a decrease in mechanical energy, i.e., it is not a stress tensor associated to a viscosity. This is a consequence of the dependence of σ_{ij} on the derivatives of the stochastic velocity $\boldsymbol{u}$ instead of those of the flux velocity $\mathbf{v}$ (cf. Eqs. 30 and 31).

In concluding, we may say that the objections raised against the stochastic interpretation from the point of view of the hydrodynamical analogy,[16] do not apply to the kind of theory discussed in the present paper in which energy and momentum are in the average conserved. More precisely, we may say that the stochastic theory *explains* in a natural way the origin of terms such as Bohm's potential.

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RESUMEN

La presente nota contiene breves comentarios sobre algunos aspectos de la reinterpretación estocástica de la mecánica cuántica. En la primera parte se presenta una deducción formal y elemental, apoyada en principios clásicos, de la ecuación de Schrödinger; esta deducción sugiere una vez más la interpretación estocástica (Markoviana) de los fenómenos cuánticos. A continuación se hace ver que, para estados estacionarios sin flujo neto de materia, las partículas del ensemble cuántico se distribuyen necesariamente de tal manera que la energía total del mismo es mínima. Por último, se demuestra explícitamente que la analogía que ha sido propuesta entre la mecánica cuántica y un fluido clásico descrito por las ecuaciones de Navier-Stokes, carece de contenido físico profundo.