TRANSFORMATION MATRICES BETWEEN THE $U_3 \supset U_2$ AND $U_3 \supset O_3$ CHAINS*

Kurt Bernardo Wolf†

Department of Physics and Astronomy,
Tel-Aviv University
Ramat Aviv, Israel

(Recibido: febrero 20, 1969)

ABSTRACT: The transformation matrices between the bases for irreducible representations of the U_3 group classified by the $U_3 \supset U_2 \supset U_1$ (canonical) and the $U_3 \supset O_3 \supset O_2$ (orbital) chain decompositions have been computed directly. The results, for all representations up to order six are presented in a table as (square roots of) rational numbers.

*Research sponsored by the Air Force Office of Scientific Research, Office of Aerospace Research, United States Air Force under AFOSR grant number EOOAR-68-0010, through the European Office of Aerospace Research.

†Under scholarship from the Comisión Nacional de Energía Nuclear, México D.F., México. Present address: Chalmers Tekniska Hogskola, Goteborg.

When a set of wave functions describing a system is separable in two different coordinate systems[1], as eigenfunctions of two group chains, the system possesses a higher symmetry group containing both. The separability of the Coulomb potential in spherical and paraboloidal coordinates[2] classifies them by the irreducible representation (IR) labels of the $O_4 \supset O_3 \supset O_2$ and $O_4 \cong O_3 \times O_3 \supset O_2 \times O_2$ group chains respectively. The transformation matrices between these two bases is an O_3 Clebsch-Gordan coefficient. The importance of the $U_3 \supset U_2 \supset U_1$ (canonical) and the $U_3 \supset O_3 \supset O_2$ (orbital) chains has been seen in nuclear physics[3,4,5]. Elementary particle theory has used mainly the first chain (where U_2 is the isospin subgroup); however, some quark-model calculations[6] related to harmonic oscillators require the transformation brackets between the two. It is therefore worthwhile to have a compact table of the transformation matrices between the two bases.

Although closed (but rather cumbersome) expressions exist[7] for these matrices, it seems that no table has yet been compiled. The present values have been computed using a Fortran program[8] which handles bases for group representations of unitary or linear groups. Known techniques[9] were used in order to generate the complete basis of U_3 in the canonical chain. In the orbital chain the operators which lower the O_3 IR contained in U_3 (keeping it of highest weight in O_3)[5] and of O_2 contained in O_3, were programmed and applied to the highest weight state in order to obtain the complete basis in this chain. It is known that there exist ambiguities concerning the operator which distinguishes between the basis vectors when there is more than one O_3 IR contained in the U_3 one. We have made no attempt in using the known discriminating operators, but in the only case when such a degeneracy occurred (for the $[4,2,0]$ IR), orthogonal states were obtained in the lowering process.

We present in Table I, for convenience, the relations between the generators of both chains as well as the eigenvalues of the diagonal ones between Gel'fand kets in the canonical chain, and the definition of the states classified by the orbital chain. The quantum numbers used in particle physics[10] are also included. In Table II, we give the O_3 IR's[3] contained in the tabulated U_3 representations. The transformation matrices appearing in Table III were computed directly through the scalar product between the states obtained by lowering along the two chains.

The transformation matrix element

$$\left\langle \begin{matrix} h_1 & h_2 & h_3 \\ & L\,\gamma\,M & \end{matrix} \middle| \begin{matrix} h_1 & h_2 & h_3 \\ q_1 & & q_2 \\ & r & \end{matrix} \right\rangle$$

is to be found under the heading $[h_1, h_2]$, the row- and column-indices being $L\gamma M$ and q_1, q_2, r, respectively. The label γ is absent when the value L appears only once. The entry $\pm\, n/m$ is to be read as $\pm\sqrt{n/m}$.

The numerical calculation was performed at the Tel-Aviv Computation Centre.

TABLE I.

U_3 generators: C_j^k, $j, k = 1,2,3$

U_2 generators: C_j^k, $j, k = 1,2$

U_1 generator : C_1^1 .

Commutation relations

$$\left[C_j^k, C_m^n\right] = \delta_m^k\, C_j^n - \delta_j^n\, C_m^k \ .$$

To the partition $[h_1, h_2, h_3]$ corresponds the Young diagram

$$\begin{matrix} & & h_1 \\ & h_2 & \\ h_3 & & \end{matrix}$$

The Gel'fand state

$$\left| \begin{matrix} h_1 & & h_2 & & h_3 \\ & q_1 & & q_2 & \\ & & r & & \end{matrix} \right\rangle$$

belongs to the IR

TABLE I

(continued)

$[h_1, h_2, h_3]$ of U_3

$[q_1, q_2]$ of U_2

and $[r]$ of U_1 .

This ket is eigenstate of the weight generators C_j^j with eigenvalues

$$w_1 = r$$

$$w_2 = q_1 + q_2 - r$$

$$w_3 = h_1 + h_2 + h_3 - q_1 - q_2$$

Restriction to $h_3 = 0$ yields IR's of SU_3 .

The harmonic oscillator state

$$\left| \begin{matrix} n & & 0 & & 0 \\ & n_{12} & & 0 & \\ & & n_1 & & \end{matrix} \right\rangle$$

has n energy quanta, eigenvalue of

$$C_1^1 + C_2^2 + C_3^3 \ ,$$

n_{12} quanta in the 1-2 plane and n_1 quanta along the 1-axis.
The state belonging to the (m, n) SU_3 multiplet, isospin I, z-component I_z and hypercharge Y, is denoted, in particle physics, by

TABLE I
(continued)

$$\left| \begin{matrix} m & & n & & 0 \\ & I + \frac{Y}{2} + \frac{m+n}{3} & & -I + \frac{Y}{2} + \frac{m+n}{3} & \\ & & I_z + \frac{Y}{2} + \frac{m+n}{3} & & \end{matrix} \right\rangle$$

O_3 generators

$$L_0 = C_1^1 - C_3^3$$

$$L_+ = - C_1^2 - C_2^3$$

$$L_- = C_2^1 + C_3^2 \ .$$

Commutation relations

$$[L_0, L_\pm] = \pm L_\pm \, , \quad [L_+ , L_-] = - L_0 \ .$$

The orbital state

$$\left| \begin{matrix} h_1 & h_2 & h_3 \\ & L \gamma M & \end{matrix} \right\rangle$$

belongs to the IR

$$\begin{matrix} [h_1 , h_2 , h_3] & \text{of } U_3 \\ L & \text{of } O_3 \\ M & \text{of } O_2 \end{matrix}$$

The extra label γ distinguishes between states in which a given L appears more than once in an U_3 multiplet.

TABLE II

O_3 irreducible representations (L) contained
in $[h_1, h_2, 0]$ of U_3

h_1, h_2	L	h_1, h_2	L	h_1, h_2	L
1 0	1	4 0	4,2,0	3 2	3,2,1
2 0	2,0	3 1	3,2,1	6 0	6,4,2,0
1 1	1	2 2	2,0	5 1	5,4,3,2,1
3 0	3,1	5 0	5,3,1	4 2	4,3,2,2′,0
2 1	2,1	4 1	4,3,2,1	3 3	3,1

TABLE III

[10]

	10,1
11	1

	10, 0
10	1

	00, 0
1 -1	1

[20]

	20,2
22	1

	20,1
21	1

	20, 0	10,1
20	2/3	1/3
00	-1/3	2/3

	10, 0
2 -1	1

	00, 0
2 -2	1

[11]

	11,1
11	1

	10,1
10	1

	10, 0
1-1	1

[30]

	30,3
33	1

	30,2
32	1

[30]

	30,1	20,2
31	4/5	1/5
11	-1/5	4/5

	30, 0	20,1
30	2/5	3/5
10	-3/5	2/5

	20, 0	10,1
3-1	4/5	1/5
1-1	-1/5	4/5

	10, 0
3-2	1

	00, 0
3-3	1

[21]

	21,2
22	1

	21,1	20,2
21	1/2	1/2
11	-1/2	1/2

	20,1	11,1
20	3/4	1/4
10	1/4	-3/4

	20,0	10,1
2-1	1/2	1/2
1-1	1/2	-1/2

	10,0
2-2	1

[40]

	40,4
44	1

	40,3
43	1

	40,2	30,3
42	6/7	1/7
22	-1/7	6/7

	40,1	30,2
41	4/7	3/7
21	-3/7	4/7

	40,0	30,1	20,2
40	8/35	24/35	3/35
20	-4/7	1/21	8/21
00	1/5	-4/15	8/15

[40] (continued)

	30,0	20,1
4-1	4/7	3/7
2-1	-3/7	4/7

	20,0	10,1
4-2	6/7	1/7
2-2	-1/7	6/7

	10,0
4-3	1

	00,0
4-4	1

[31]

	31,3
33	1

	31,2	30,3
32	2/3	1/3
22	-1/3	2/3

	31,1	30,2	21,2
31	4/15	5/9	8/45
21	-1/3	4/9	-2/9
11	2/5	0	-3/5

[31] (continued)

	30,1	21,1	20,2
30	8/15	4/15	1/5
20	1/3	-2/3	0
10	2/15	1/15	-4/5

	30,0	20,1	11,1
3-1	4/15	3/5	2/15
2-1	1/3	0	-2/3
1-1	2/5	-2/5	1/5

	20,0	10,1
3-2	2/3	1/3
2-2	1/3	-2/3

	10,0
3-3	1

[22]

	22,2
22	1

	21,2
21	1

	21,1	20,2
20	1/3	2/3
00	-2/3	1/3

	20,1
2-1	1

	20,0
2-2	1

[50]

	50,5
55	1

	50,4
54	1

	50,3	40,4
53	8/9	1/9
33	-1/9	8/9

	50,2	40,3
52	2/3	1/3
32	-1/3	2/3

	50,1	40,2	30,3
51	8/21	4/7	1/21
31	-8/15	1/5	4/15
11	3/35	-8/35	24/35

	50,0	40,1	30,2
50	8/63	40/63	5/21
30	-4/9	-1/45	8/15
10	3/7	-12/35	8/35

	40,0	30,1	20,2
5-1	8/21	4/7	1/21
3-1	-8/15	1/5	4/15
1-1	3/35	-8/35	24/35

	30,0	20,1
5-2	2/3	1/3
3-2	-1/3	2/3

	20,0	10,1
5-3	8/9	1/9
3-3	-1/9	8/9

	10,0
5-4	1

	00,0
5-5	1

[41]

	41,4
44	1

	41,3	40,4
43	3/4	1/4
33	-1/4	3/4

	41,2	40,3	31,3
42	3/7	7/16	15/112
32	-1/3	9/16	-5/48
22	5/21	0	-16/21

	41,1	40,2	31,2	30,3
41	1/7	27/56	15/56	3/28
31	-1/5	49/120	-3/8	1/60
21	5/14	5/84	-3/28	-10/21
11	-3/10	1/20	1/4	-2/5

	40,1	31,1	30,2	21,2
40	5/14	3/14	5/14	1/14
30	3/10	-1/2	1/30	-1/6
21	15/56	9/56	-10/21	-2/21
10	3/40	-1/8	-2/15	2/3

	40,0	30,1	21,1	20,2
4-1	1/7	4/7	5/28	3/28
3-1	1/5	4/45	-25/36	-1/60
2-1	5/14	-5/126	8/63	-10/21
1-1	3/10	-3/10	0	2/5

	30,0	20,1	11,1
4-2	3/7	27/56	5/56
3-2	1/3	-1/24	-15/24
2-2	5/21	-10/21	2/7

	20,0	10,1
4-3	3/4	1/4
3-3	1/4	-3/4

	10,0
4-4	1

[32]

	32,3
33	1

	32,2	31,3
32	1/3	2/3
22	-2/3	1/3

	31,2	30,3	22,2
31	3/5	4/15	2/15
21	0	1/3	-2/3
11	-2/5	2/5	1/5

	31,1	30,2	21,2
30	1/5	8/15	4/15
20	0	1/3	-2/3
10	-4/5	2/15	1/15

	30,1	21,1	20,2
3-1	5/9	8/45	4/15
2-1	4/9	-2/9	-1/3
1-1	0	-3/5	2/15

	30,0	20,2
3-2	1/3	2/3
2-2	2/3	-1/3

	20,0
3-3	1

[60]

	60, 6
66	1

	60, 5
65	1

	60, 4	50, 5
64	10/11	1/11
44	-1/11	10/11

	60, 3	50, 4
63	8/11	3/11
43	-3/11	8/11

	60, 2	50, 3	40, 4
62	16/33	16/33	1/33
42	-36/77	25/77	16/77
22	1/21	-4/21	16/21

	60, 1	50, 2	40, 3
61	8/33	20/33	5/33
41	-40/77	1/77	36/77
21	5/21	-8/21	8/21

	60, 0	50, 1	40, 2	30, 3
60	16/231	40/77	30/77	5/231
40	-24/77	-64/385	21/55	54/385
20	10/21	-1/7	0	8/21
00	-1/7	6/35	-8/35	16/35

	50, 0	40, 1	30, 2
6-1	8/33	20/33	5/33
4-1	-40/77	1/77	36/77
2-1	5/21	-8/21	8/21

[60] (continued)

	40, 0	30, 1	20, 2
6-2	16/33	16/33	1/33
4-2	-36/77	25/77	16/77
2-2	1/21	-4/21	16/21

	30, 0	20, 1
6-3	8/11	3/11
4-3	-3/11	8/11

	20, 0	10, 1
6-4	10/11	1/11
4-4	-1/11	10/11

	10, 0
6-5	1

	00, 0
6-6	1

[51]

	51, 5
55	1

	51, 4	50, 5
54	4/5	1/5
44	-1/5	4/5

	51, 3	50, 4	41, 4
53	8/15	9/25	8/75
43	-3/10	16/25	-3/50
33	1/6	0	-5/6

	51, 2	50, 3	41, 3	40, 4
52	4/15	32/75	6/25	1/15
42	-9/35	169/350	-81/350	1/35
32	1/3	1/30	-3/10	-1/3
22	-1/7	2/35	8/35	-4/7

[51] (continued)

	51,1	50,2	41,2	40,3	31,3
51	8/105	28/75	48/175	7/30	3/70
41	-4/35	121/350	-72/175	9/140	-9/140
31	4/15	1/6	0	-5/12	-3/20
21	-2/7	4/35	1/35	-2/7	2/7
11	9/35	0	-2/7	0	16/35

	40,0	30,1	21,1	20,2
5-2	4/15	8/15	2/15	1/15
4-2	9/35	1/70	-7/10	-1/35
3-2	1/3	-1/6	1/6	-1/3
2-2	1/7	-2/7	0	4/7

	50,1	41,1	40,2	31,2	30,3
50	8/35	16/105	3/7	1/7	1/21
40	8/35	-12/35	3/28	-9/28	0
30	8/25	16/75	-3/20	-1/20	-4/15
20	6/35	-9/35	-1/7	3/7	0
10	9/175	6/175	-6/35	-2/35	24/35

	30,0	20,1	11,1
5-3	8/15	6/15	1/15
4-3	3/10	-1/10	-9/15
3-3	1/6	-1/2	1/3

	20,0	10,1
5-4	4/5	1/5
4-4	1/5	-4/5

	50,0	40,1	31,1	30,2	21,2
5-1	8/105	10/21	6/35	5/21	4/105
4-1	4/35	5/28	-81/140	0	-9/70
3-1	4/15	1/60	3/20	-8/15	-1/30
2-1	2/7	-1/14	-1/14	0	4/7
1-1	9/35	-9/35	1/35	8/35	-8/35

	10,0
5-5	1

[42]

	43,4
44	1

	42,3	41,4
43	1/2	1/2
33	-1/2	1/2

	41,1	40,2	31,2	30,3	22,2
40	4/35	169/420	9/28	4/35	1/21
30	0	5/12	-1/4	0	-1/3
$2^2$0	-1/2	0	0	1/2	0
$2^1$0	-5/42	10/63	-2/21	-5/42	32/63
00	4/15	-1/45	-1/3	4/15	1/9

	42,2	41,3	40,4	32,3
42	1/7	25/42	1/7	5/42
32	-1/3	1/18	1/3	-5/18
$2^2$2	-1/6	-1/9	1/6	5/9
$2^1$2	5/14	-5/21	5/14	-1/21

	40,1	31,1	30,2	21,2
4-1	9/28	5/28	8/21	5/42
3-1	5/12	-1/12	0	-1/2
2^2-1	-1/12	-5/12	1/2	0
2^1-1	5/28	-9/28	-5/42	8/21

	41,2	40,3	32,2	31,3
41	8/21	9/28	5/42	5/28
31	0	5/12	-1/2	-1/12
$2^2$1	-1/2	1/12	0	5/12
$2^1$1	-5/42	5/28	8/21	-9/28

[42] (continued)

	40,0	30,1	21,1	20,2
4-2	1/7	25/42	5/42	1/7
3-2	1/3	1/18	-5/18	-1/3
2^2-2	-1/6	1/9	-5/9	1/6
2^1-2	5/14	-5/21	-1/21	5/14

	30,0	20,1
4-3	1/2	1/2
3-3	1/2	-1/2

	20,0
4-4	1

[33]

	33,3
33	1

	32,3
32	1

[33]

	32,2	31,3
31	1/5	4/5
11	-4/5	1/5

	31,2	30,3
30	3/5	2/5
10	-2/5	3/5

	31,1	30,2
3-1	1/5	4/5
1-1	-4/5	1/5

	30,1
3-2	1

	30,0
3-3	1

REFERENCES

1. A.A. Makarov, J.A. Smorodinsky, Kh. Valiev and P. Winternitz, I.A.E.A. preprint IC/67/8.
2. V. Bargmann, Z. Physik 99 (1936) 576
   M. Bander and C. Itzykson, Rev. Mod. Phys. 38 (1966) 330.
3. B.H. Flowers, Proc. Roy. Soc. (London) A212 (1952) 248.
   J.P. Elliott, Proc. Roy. Soc. (London) A245 (1958) 128.
   B. Kaufmann and C. Noack, Journ. Math. Phys. 6 (1965) 142.
4. V. Bargmann and M. Moshinsky, Nucl. Phys. 18 (1960) 697, ibid. 23 (1961) 177.
   G. Racah, Lectures at the Istambul Physics Summer School (1962).
   R. Sen, Ph.D. thesis, Jerusalem, Hebrew University (1963).
5. J. Flores, E. Chacón, P.A. Mello and M. de Llano, Nucl. Phys. 72 (1965) 352; ibid. 72 (1965) 379.
6. For a review on the subject, see the lecture by R.H. Dalitz at the Second Hawaii Topical Conference in Particle Physics, Honolulu (1967).
7. M. Moshinsky, Rev. Mod. Phys. 34 (1962) 813 eq. (4.13).
   E. Chacón and M. de Llano, Rev. Mex. Fís. 12 (1963) 57.
   J. Flores, P.A. Mello and M. Moshinsky have generated transformation brackets by diagonalizing the L^2 operator matrix elements between Gel'fand states, by computer.
8. K.B. Wolf, Journ. Comp. Phys. 2 (1968) 334.
9. M. Moshinsky, Journ. Math. Phys. 4 (1963) 1128.
   J. Nagel and M. Moshinsky, Journ. Math. Phys. 6 (1965) 682.
10. I.C. Biedernharn, Lecture notes CERN 65-41 (1965).

RESUMEN

Las matrices de transformación entre las bases de representaciones irreducibles del grupo U_3 clasificado por la cadena de subgrupos canónica $U_3 \supset U_2 \supset U_1$ y las clasificadas por la cadena orbital $U_3 \supset O_3 \supset O_2$ se obtienen directamente. Los resultados, para todas las representaciones de orden menor o igual a 6, se presentan en una tabla, como raíces de números racionales.