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AN EXACT SOLUTION OF BLOCH'S PHENOMENOLOGICAL EQUATIONS

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ABSTRACT: The behavior of a system of magnetic dipoles in a magnetic field is studied using Bloch's phenomenological equations. It is found that, if a rotating field is applied, the transient behavior of the perpendicular component of the magnetization of the system is given by the sum of a decaying exponential and a sinusoidal function modulated by a decaying exponential. The theoretical expression derived gives amplitudes, frequencies, decay times, and phase in terms of parameters such as the relaxation times

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of the system. A convenient method is discussed whereby these values could be experimentally determined.

I. INTRODUCTION

In this paper we will analyze a system of N_0 magnetic dipoles per unit volume, each of which has a magnetic moment $\boldsymbol{m}$ and an intrinsic angular momentum $\boldsymbol{l}$ such that $\boldsymbol{m} = \gamma \boldsymbol{l}$ where γ is the gyromagnetic ratio. Each dipole is assumed to be located at a fixed point, and is allowed to perform small vibrations about this equilibrium position. Furthermore, the possible interactions between the dipoles are taken into account. A specific magnetic field is applied to this system, and the behavior of the system is studied.

It is well known[1] that if a magnetic field

$$\boldsymbol{H}' = \begin{pmatrix} 0 \\ 0 \\ H_0 \end{pmatrix}, \; H_0 = \text{constant}$$

is applied, the system eventually develops a magnetization

$$\boldsymbol{M}' = \begin{pmatrix} 0 \\ 0 \\ M \end{pmatrix}, \; M = (N_0 m^2/3kT_e)\, H_0 \;,$$

where T_e is the temperature of the system and k is Boltzmann's constant.

An additional field is applied to the system rotating with a frequency ω in the plane perpendicular to $\boldsymbol{H}'$ of magnitude H_1 so that the total magnetic field applied is

$$\boldsymbol{H} = \begin{pmatrix} H_1 \cos \omega t \\ -H_1 \sin \omega t \\ H_0 \end{pmatrix} .$$

We assume that the third component, H_0, has been applied for a sufficient time to allow the system to reach an initial magnetization of $\boldsymbol{M}'$.

Bloch's phenomenological equations[2] describe the behavior of the magnetization of this system, that is,

$$dM_3/dt = \gamma\,(\mathbf{M}\times\mathbf{H})_3 + (M - M_3)/T_m \tag{1}$$

$$dM_{1,2}/dt = \gamma\,(\mathbf{M}\times\mathbf{H})_{1,2} - (M_{1,2})/T_s \tag{2}$$

with

$$\mathbf{M} = \begin{pmatrix} M_1 \\ M_2 \\ M_3 \end{pmatrix},\ \mathbf{M}\big|_{t=0} = \mathbf{M}'\,;$$

and the meaning of the relaxation times T_s and T_m is discussed in the literature[3]. Defining

$$A \equiv \begin{pmatrix} 0 & \gamma H_0 & \gamma H_1\ \sin\omega t \\ -\gamma H_0 & 0 & \gamma H_1\ \cos\omega t \\ -\gamma H_1\ \sin\omega t & -\gamma H_1\ \cos\omega t & 0 \end{pmatrix},$$

$$\mathbf{N} \equiv \begin{pmatrix} 0 \\ 0 \\ -\delta_m M \end{pmatrix},\ \delta_s \equiv -1/T_s,\ \delta_m \equiv -1/T_m,$$

and

$$T \equiv \begin{pmatrix} \delta_s & 0 & 0 \\ 0 & \delta_s & 0 \\ 0 & 0 & \delta_m \end{pmatrix},$$

we can write eqs. (1) (2) in the matrix form

$$d(\mathbf{M})/dt = (A + T)\,\mathbf{M} + \mathbf{N}\ . \tag{3}$$

If we now make a coordinate transformation[4] to a rotating frame whose frequency of rotation is the same as that of the field in the 1-2 plane, by means of the matrix

$$B \equiv \begin{pmatrix} \cos \omega t & -\sin \omega t & 0 \\ \sin \omega t & \cos \omega t & 0 \\ 0 & 0 & 1 \end{pmatrix} ,$$

we obtain

$$d(B\mathbf{M})/dt = WB\mathbf{M} + B\mathbf{N} , \quad B\mathbf{M}\big|_{t=0} = \mathbf{M}\big|_{t=0} = \mathbf{M}' ,$$

since[5]

$$dB/dt = -\omega DB ,$$

where
$$W \equiv C - \omega D + T$$

and

$$D \equiv \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} ,$$

$$C \equiv \begin{pmatrix} 0 & \gamma H_0 & 0 \\ -\gamma H_0 & 0 & \gamma H_1 \\ 0 & -\gamma H_1 & 0 \end{pmatrix} .$$

A second coordinate transformation is now made to the frame of the eigenvectors of W by means of a constant matrix Q such that

$$Q^{-1} W Q = \Sigma$$

where

$$\Sigma \equiv \begin{pmatrix} \sigma & 0 & 0 \\ 0 & \sigma^{\times} & 0 \\ 0 & 0 & \sigma_0 \end{pmatrix} ,$$

σ, $\sigma^{\times}$, σ_0 designating the eigenvalues of W.

In general the eigenvalues of W are either three real numbers or one real number and a pair of complex conjugates. It can be shown (App., Ref. 5) that the real eigenvalues and the real part of any complex ones cannot be positive. This is a consequence of the fact that the coefficients of the third degree equations for the real eigenvalues and the real part of the possible complex eigenvalues are all positive..

In this eigenvector frame eq. (3) can be written as

$$d\mathbf{S}/dt = \Sigma\mathbf{S} + \mathbf{P}\,, \quad \mathbf{S}\big|_{t=0} = Q^{-1}\mathbf{M}' \equiv \mathbf{S}_0 \tag{4}$$

where

$$\mathbf{S} = Q^{-1}B\mathbf{M} \quad \text{and} \quad \mathbf{P} = Q^{-1}B\mathbf{N} = Q^{-1}\mathbf{N}\,.$$

We see that in this frame the system of differential equations (4) is uncoupled, since Σ is a diagonal matrix. The solution is then

$$\mathbf{S} = e^{\Sigma t}\left[\mathbf{S}_0 + \Sigma^{-1}\mathbf{P} - \Sigma^{-1}e^{-\Sigma t}\mathbf{P}\right]$$

where

$$e^{\Sigma t} = \begin{pmatrix} e^{\sigma t} & 0 & 0 \\ 0 & e^{\sigma^\times t} & 0 \\ 0 & 0 & e^{\sigma_0 t} \end{pmatrix} = \sum_{n=0}^{\infty} \frac{1}{N!}\,(\Sigma)^n\,.$$

Since two diagonal matrices commute, we can write

$$\mathbf{S} = e^{\Sigma t}\left[\mathbf{S}_0 + \Sigma^{-1}\mathbf{P}\right] - \Sigma^{-1}\mathbf{P}\,.$$

This solution can be split into two parts, $\mathbf{S}_t + \mathbf{S}_s$, where

$$\mathbf{S}_t = e^{\Sigma t}\left[\mathbf{S}_0 + \Sigma^{-1}\mathbf{P}\right] \quad \text{(time dependent)}$$

and

$$\mathbf{S}_s = -\,\Sigma^{-1}\mathbf{P} \qquad \text{(constant)}.$$

The matrix $e^{\Sigma t}$ tends to zero with increasing time, since the real part of the Σ matrix is negative. Thus $\mathbf{S}_t$ eventually vanishes and we can obtain the stationary solution by considering $S \sim S_s$ and transforming back to the laboratory frame. This solution is

$$\mathbf{M}_s = [\,\delta M/(\delta\alpha^2 + \beta^2 + \delta)] \begin{pmatrix} \alpha\beta \cos \omega t + \beta \sin \omega t \\ -\alpha\beta \sin \omega t + \beta \cos \omega t \\ 1 + \alpha^2 \end{pmatrix} ,$$

where

$$\delta \equiv \delta_m/\delta_s ,$$

$$\alpha \equiv (\gamma H_0 - \omega)/\delta_s ,$$

and

$$\beta \equiv \gamma H_1/\delta_s ,$$

which of course coincides with Bloch's solution[2].

We will be interested in analyzing the transient solution that has its origin in the term $\mathbf{S}_t^{6,7}$. Hence we transform back to the rotating frame obtaining

$$B\mathbf{M}_t = Q\mathbf{S}_t = Q\, e^{\Sigma t} (Q^{-1} Q) \left[\mathbf{S}_0 + \Sigma^{-1} \mathbf{P} \right] .$$

This we can write as

$$B\mathbf{M}_t = e^{Wt}\, \mathbf{V}$$

where the vector $\mathbf{V}$ is given by the relation

$$\mathbf{V} = B\mathbf{M}' + W^{-1} B\mathbf{N} = [-M\beta/(\delta\alpha^2 + \beta^2 + \delta)] \begin{pmatrix} \alpha\delta \\ \delta \\ -\beta \end{pmatrix}$$

since

$$QS_0 = \mathbf{M}' = B\mathbf{M}',\ Q\Sigma^{-1}\mathbf{P} = Q\Sigma^{-1}Q^{-1}B\mathbf{N} = W^{-1}B\mathbf{N},\ \text{and}\ Qe^{\Sigma t}Q^{-1} = e^{Wt}.$$

II. MATRIX ELEMENTS OF exp (Wt)

Our problem now is to find the matrix elements of exp (Wt). The particular case of $T_s = T_m$ can be taken care of immediately since

$$W = \delta_s \begin{pmatrix} 1 & \alpha & 0 \\ -\alpha & 1 & \beta \\ 0 & -\beta & 1 \end{pmatrix}$$

and defining $\lambda^2 \equiv \alpha^2 + \beta^2$ the matrix

$$Q_0 = \frac{1}{\sqrt{2}\lambda} \begin{pmatrix} -\alpha & -\alpha & \sqrt{2}\beta \\ -i\lambda & i\lambda & 0 \\ \beta & \beta & \sqrt{2}\alpha \end{pmatrix},\ Q_0^{-1} = \frac{1}{\sqrt{2}\lambda} \begin{pmatrix} -\alpha & i\lambda & \beta \\ -\alpha & -i\lambda & \beta \\ \sqrt{2}\beta & 0 & \sqrt{2}\alpha \end{pmatrix}$$

diagonalizes W, that is,

$$Q_0^{-1} W Q_0 = \delta_s I + \delta_s \begin{pmatrix} i\lambda & 0 & 0 \\ 0 & -i\lambda & 0 \\ 0 & 0 & 0 \end{pmatrix} = \Sigma_0\,,$$

where I is the identity matrix.

Then

$$e^{\Sigma_0 t} = \begin{pmatrix} e^{\delta_s t + i\lambda\delta_s t} & 0 & 0 \\ 0 & e^{\delta_s t - i\lambda\delta_s t} & 0 \\ 0 & 0 & e^{\delta_s t} \end{pmatrix}$$

and

$$e^{Wt} = \left[Q_0 e^{\Sigma_0 t} Q_0^{-1}/\lambda^2\right] \begin{pmatrix} \alpha^2 \cos\delta_s\lambda t + \beta^2 & \alpha\lambda \sin\delta_s\lambda t & -\alpha\beta\cos\delta_s\lambda t + \alpha\beta \\ -\alpha\lambda\sin\delta_s\lambda t & \lambda^2\cos\delta_s\lambda t & \beta\lambda\sin\delta_s\lambda t \\ -\alpha\beta\cos\delta_s\lambda t + \alpha\beta & -\beta\lambda\sin\delta_s\lambda t & \cdot\beta^2\cos\delta_s\lambda t + \alpha^2 \end{pmatrix}.$$

In the general case a convenient way of determining the required matrix elements is[8] by writing

$$\exp(Wt) = f(W) = 1/(2\pi i) \oint \frac{f(Z')}{Z' - W}\, dZ'$$

where $Z' = z'I$, z' being a complex variable. Thus the matrix elements are

$$f_{ij}(W) = 1/(2\pi i) \oint f(z')\, J_{ij}(Z')\, dz'$$

where the J_{ij} are the matrix elements of $J \equiv (Z' - W)^{-1}$. Defining the dimensionless matrices $Z \equiv Z'/\delta_s$ and $W' \equiv W/\delta_s$, we have

$$J = \left[1/\delta_s \det(Z - W')\right] \begin{pmatrix} (z-1)(z-\delta)+\beta^2 & \alpha(z-\delta) & \alpha\beta \\ -\alpha(z-\delta) & (z-1)(z-\delta) & \beta(z-1) \\ \alpha\beta & -\beta(z-1) & (z-1)^2+\alpha^2 \end{pmatrix} \tag{5}$$

or $J = [1/\delta_s \det(Z - W')]\, j$.

Using the formula[9]

$$(1/(2\pi i) \oint [p(z)/q(z)]\, dz = \sum_{m=1}^{n} p(a_m)/q'(a_m)$$

where a_m is one of the n poles of $p(z)/q(z)$ and $q'(z) = dq(z)/dA$, we can write

$$f_{ij} = \sum_{k=1}^{3} \frac{e^{\delta_s z_k t} j_{ij}(z_k)}{\frac{d}{dz}\left(\det(Z - W')\right)\Big|_{z=z_k}} \tag{6}$$

where the z_k are the zeros of the determinant, i.e., the eigenvalues of W'. These are $z_1 = \sigma/\delta_s$, $z_2 = \sigma^{\times}/\delta_s$, and $z_3 = \sigma_0/\delta_s$.

From the definition of W, it is readily seen that

$$\det(Z - W') = (z-1)\left[(z-1)(z-\delta) + \beta^2\right] + \alpha^2(z-\delta)$$

so that

$$d(z) \equiv d\left[\det(Z - W')\right]/dz = 2(z-1)(z-\delta) + (z-1)^2 + \alpha^2 + \beta^2 . \tag{7}$$

From eqs. (5), (6), (7) we have the matrix elements f_{ij}:

$$f_{11} = \sum_{k=1}^{3} e^{\delta_s z_k t} \frac{(z_k - 1)(z_k - \delta) + \beta^2}{d(z_k)} ,$$

$$f_{22} = \sum_k e^{\delta_s z_k t} \frac{(z_k - 1)(z_k - \delta)}{d(z_k)} ,$$

$$f_{33} = \sum_k e^{\delta_s z_k t} \frac{(z_k - 1)^2 + \alpha^2}{d(z_k)} ,$$

$$-f_{21} = f_{12} = \sum_k e^{\delta_s z_k t} \frac{\alpha(z_k - \delta)}{d(z_k)} ,$$

$$f_{31} = f_{13} = \sum_k e^{\delta_s z_k t} \frac{\alpha\beta}{d(z_k)} , \quad \text{and}$$

$$-f_{32} = f_{23} = \sum_k e^{\delta_s z_k t} \frac{\beta(z_k - 1)}{d(z_k)} .$$

III. ANALYSIS OF THE EIGENVALUE EQUATION

To find the explicit form of the z_k that appear in the matrix elements obtained in the preceding section we have to solve the following third degree algebraic equation:

$$\det(Z - W') = z^3 - (2 + \delta)\, z^2 + (2\delta + 1 + \alpha^2 + \beta^2)\, z - (\delta + \alpha^2\, \delta + \beta^2) = 0 . \tag{8}$$

The discriminant R of this equation is

$$R = (\alpha^6/(12)^3) \left[(8\eta^2/\alpha^2 + 8 - \beta^4/\alpha^4 - 20\,\beta^2/\alpha^2)^2 - (\beta^2/\alpha^2)(\beta^2/\alpha^2 - 8)^3 \right] \tag{9}$$

where $\eta \equiv \delta - 1$. Defining q as

$$q = -\eta/27\ \{2\eta^2 + 9(2\alpha^2 - \beta^2)\} \tag{10}$$

the roots z_k of eq. (8) for $R > 0$ are

$$z_1 = (-r + 1 + \eta/3) + i\sqrt{3}\, s ,$$

$$z_2 = (-r + 1 + \eta/3) - i\sqrt{3}\, s ,$$

and

$$z_3 = 2r + 1 + \eta/3 ,$$

where

$$r = 1/2\ \left[\sqrt[3]{-q/2 + \sqrt{R}} + \sqrt[3]{-q/2 - \sqrt{R}} \right]$$

$$s = 1/2\ \left[\sqrt[3]{-q/2 + \sqrt{R}} - \sqrt[3]{-q/2 - \sqrt{R}} \right] .$$

For $R < 0$ the three roots are real and distinct, and we will call them $\bar{z}_1$, $\bar{z}_2$ and $\bar{z}_3$.

We now want to determine the values of α, β, and δ that will make $R \geqslant 0$. We consider R as a function of two variables, α^2 and β^2, and a parameter δ. From eq. (9) we see that R is positive when $\beta^2/\alpha^2 - 8$ is negative. This means that, in a region of the α^2, β^2-plane contained between the α^2-axis and a line of slope 8, R is a positive quantity (see Fig. 1). Also, when $\beta^2 = 8\alpha^2$ we have

$$R = (\alpha^6/27)(\eta^2/\alpha^2 - 27)^2$$

and we see that in this case R is always positive except when $\alpha^2 = \eta^2/27$ where $R = 0$.

It now remains to examine the region $\beta^2 > 8\alpha^2$. We introduce a new variable p, by means of $\beta^2 = 8p\alpha^2$; $8p$ is then the slope of a line that goes through the origin contained in the region that we want to analyze provided that $p > 1$.

R can thus be written as

$$R = (\alpha^6/27)\,\{(\eta^2/\alpha^2 + 1 - 8p^2 - 20p)^2 - 64p(p-1)^3\}\ .$$

The expression in braces vanishes for two values of α^2, which we shall call $\alpha_\pm^2$, along a line of given p. These values are

$$\alpha_\pm^2 = \eta^2/(8p^2 + 20p - 1 \pm 8p^{1/2}(p-1)^{3/2})\ . \tag{11}$$

We notice that $\alpha_-^2 > \alpha_+^2$: If we choose a value of α^2 that lies between α_+^2 and α_-^2, for example $\overline{\alpha}^2 = \eta^2/(8p^2 + 20p - 1)$, we obtain that, for this particular value of α^2, $R < 0$. This means that:

$$1)\ \text{for}\ \ \alpha_+^2 < \alpha^2 < \alpha_-^2,\ R < 0\ , \tag{12}$$

$$2)\ \text{for}\ \ \alpha_-^2 < \alpha^2,\ \alpha_+^2 > \alpha^2,\ R > 0\ . \tag{13}$$

It can be shown, furthermore, that $\beta_-^2 = 8p\alpha_-^2$ obeys the condition $\beta_-^2 < 8\eta^2/27$; thus for $\beta^2 > 8\eta^2/27$ we have $R > 0$.

To summarize, we have now shown that the values of α^2 and β^2 that lie outside the triangular region bounded by the lines $\beta^2 = 8\alpha^2$ and $\beta^2 = 8\eta^2/27$ make R positive.

The expression

$$8\eta^2/\alpha^2 + 8 - \beta^4/\alpha^4 - 20\beta^2/\alpha^2 = \pm(\beta/\alpha)(\beta^2/\alpha^2 - 8)^{3/2}$$

gives the two curves on the α^2, β^2-plane on which $R = 0$. It can be shown that

$$d\beta^2_\pm / d\alpha^2_\pm > 0$$

which signifies that there are no extrema and secondly that the slopes of those curves are positive. Noticing that when $\alpha^2 = 0$ (resonance) $\beta^2_+ = 0$, $\beta^2_- = \eta^2/4$, we obtain the general shape of the $R = 0$ curves given in Figure 1. Conditions (12) and (13) tell us, of course, that in the region between these two curves and the β^2-axis, $R < 0$.

IV. TRANSIENT SOLUTION

For the particular case in which $T_s = T_m$ from the formulae given at the beginning of Section II it is easy to find that the third component of $B\mathbf{M}_t$ is

$$(B\mathbf{M}_t)_3 = \exp(Wt)\,\mathbf{V}_3 = \left[M\beta^2/(\lambda^2+1)\right] e^{\delta_s t} \left[(1+\lambda^2)^{1/2}/\lambda\right] \cos(\delta_s \lambda t + \phi_0) \tag{14}$$

where $\phi_0 = \arctan 1/\lambda$.

In the general case, from the formulae for the matrix elements f_{ij} given in Section II, we can write $\exp(Wt)$ in the following form

$$\exp(Wt) = e^{\delta_s z_1 t} A(z_1)/d(z_1) + e^{\delta_s z_2 t} A(z_2)/d(z_2) + e^{\delta_s z_3 t} A(z_3)/d(z_3) \tag{15}$$

where

$$A(z) = \begin{pmatrix} (z-1)(z-\delta)+\beta^2 & \alpha(z-\delta) & \alpha\beta \\ -\alpha(z-\delta) & (z-1)(z-\delta) & \beta(z-1) \\ \alpha\beta & -\beta(z-1) & (z-1)^2+\alpha^2 \end{pmatrix} .$$

Then defining the vector $\mathbf{K}(z) = A(z)\,\mathbf{V}/d(z)$ we can write

$$BM_t = \exp(Wt)\,\mathbf{V} = e^{\delta_s z_1 t}\mathbf{K}(z_1) + e^{\delta_s z_2 t}\mathbf{K}(z_2) + e^{\delta_s z_3 t}\mathbf{K}(z_3) . \quad (16)$$

When R is positive $z_1 = z_2^*$ and since $\mathbf{K}(z^*) = \mathbf{K}^*(z)$ we have, using the definition $z_1 \equiv \tilde{z}_1 + i\hat{z}_1$,

$$BM_t = e^{\delta_s \tilde{z}_1 t}\left\{[\mathbf{K}(z_1) + \mathbf{K}^*(z_1)]\cos\delta_s\hat{z}_1 t + i[\mathbf{K}(z_1) - \mathbf{K}^*(z_1)]\sin\delta_s\hat{z}_1 t\right\} + e^{\delta_s z_3 t}\mathbf{K}(z_3) . \quad (17)$$

However

$\mathbf{K}(z_1) + \mathbf{K}^*(z_1) = 2\,\mathrm{Re}\,\mathbf{K}(z_1)$ and $\mathbf{K}(z_1) - \mathbf{K}^*(z_1) = i\,2\,\mathrm{Im}\,\mathbf{K}(z_1)$, thus

$$BM_t = 2e^{\delta_s \tilde{z}_1 t}[\,\mathrm{Re}\,\mathbf{K}(z_1)\cos\delta_s\hat{z}_1 t - \mathrm{Im}\,\mathbf{K}(z_1)\sin\delta_s\hat{z}_1 t] + e^{\delta_s z_3 t}\mathbf{K}(z_3) . \quad (18)$$

The third component of this vector $(BM_t)_3$, which coincides with the third component of the transient solution $(\mathbf{M}_t)_3$ since B represents a rotation in the 1–2 plane, is then, making $\tan\phi = -$phase of $\{\mathbf{K}(z_1)\}_3$,

$$(BM_t)_3 = (\mathbf{M}_t)_3 = e^{\delta_s z_3 t}\{\mathbf{K}(z_3)\}_3 + 2e^{\delta_s \tilde{z}_1 t}\left|\{\mathbf{K}(z_1)\}_3\right|\cos(\delta_s\hat{z}_1 t + \phi). \quad (19)$$

Eq. (19) can be written as

$$(B\mathbf{M}_t)_3 = (\mathbf{M}_t)_3 = a\,e^{\sigma_0 t} + b\,e^{\tilde{\sigma} t}\cos(\hat{\sigma} t + \phi) \tag{20}$$

where

$$\sigma_0 = \delta_s z_3\,, \quad \tilde{\sigma} = \delta_s \tilde{z}_1\,, \quad \hat{\sigma} = \delta_s \hat{z}_1\,,$$

$$a = \{\mathbf{K}(z_3)\}_3 = [-M\beta^2/(\delta\alpha^2 + \beta^2 + \delta)]\,[\alpha^2\eta - \delta(z_3 - 1) - (z_3 - 1)^2]\,/d(z_3),$$

$$b = 2\,|\{\mathbf{K}(z_1)\}_3| = [2M\beta^2/(\delta\alpha^2 + \beta^2 + \delta)]\,[f_1^2 + f_2^2]^{1/2}/|d(z_1)|$$

$$\phi = -\,\text{phase}\;\{\mathbf{K}(z_1)\}_3 =$$

$$= \tan^{-1}\{(f_2\,\mathrm{Re}\,\{d(z_1)\} + f_1\,\mathrm{Im}\,\{d(z_1)\})/(f_1\,\mathrm{Re}\,\{d(z_1)\} - f_2\,\mathrm{Im}\,\{d(z_1)\})$$

$$f_1 = \alpha^2\eta + (1 - \tilde{z}_1)(\eta + \tilde{z}_1) + \hat{z}_1^2\,,$$

and

$$f_2 = \delta\hat{z}_1 + 2\,\hat{z}_1(1 - \tilde{z}_1)\,.$$

From eq. (20) it can be seen that $\mathbf{M}_t$ is written as the sum of three terms. The first one is simply a decaying exponential of amplitude a and characteristic time σ_0, decaying because σ_0 is real and negative. The second term is a cosine function of frequency $\hat{\sigma}$, phase ϕ, and amplitude b modulated by a decaying exponential of characteristic time $\tilde{\sigma}$, decaying also because $\tilde{\sigma}$ is real and negative.

When $R = 0$ the frequency of the periodic function vanishes, and $\mathbf{M}_t$ loses its periodic behavior. For $R < 0$ we know that there is also no periodic behavior since the roots $\sigma = \delta_s \bar{z}_1$, $\sigma^\times = \delta_s \bar{z}_2$, $\sigma_0 = \delta_s \bar{z}_3$ are real, distinct, and negative. We have in these two cases that $\mathbf{M}_t$ is just a linear combination of decaying exponentials.

We notice that in the particular case of $T_s = T_m$ the purely decaying term is not present, and that the behavior is just a periodic function modulated by a decaying exponential.

V. NUMERICAL RESULTS

To compare the order of magnitude of the transient part of the solution with that of the stationary part, it is convenient to calculate numerically the values of the quantities $(\mathbf{M}_s)_3$, a, b, and ϕ for a given set of values of the parameters γ, ω, k, T_e, N_0, m and several values of the relaxation times T_m, T_s and of the applied fields H_0 and H_1. The parameters were chosen as $\gamma = 1.7618 \times 10^7$ rad/gauss-sec, $\omega = 5.9690 \times 10^9$ rad/sec, $k = 1.3804 \times 10^{-16}$ erg/°K, $T_e = 300$ °K, $N_0 = 10^{15}$, $m = 9.2732 \times 10^{-21}$ erg/gauss, T_m and T_s were varied between 10^{-2} and 10^{-6} seconds, H_0 from 2.7 to 3.9 kilogauss, and H_1 from 10 to 40 gauss. Some results are shown in Tables I - X. The following trends appear:

1. Table I shows that $\hat{\sigma}$ does not appreciably depend on the relaxation times, the order of magnitude being in all cases 10^9 to 10^{10} Hz.

2. Tables II - IV show that σ_0 is approximately equal to δ_m, its dependence on δ_s and the applied fields being quite small.

3. Tables V - VII show that $\tilde{\sigma}$ is approximately equal to δ_s, its dependence on δ_m and the applied fields being very small.

4. In Table VIII it is seen that coefficient b is esentially independent of the relaxation times, its value basically determined by the applied fields.

5. Table IX shows that coefficient a has a noticeable dependence on the applied fields and that its strong dependence on the relaxation times is mainly through the quantity δ, rather than δ_m or δ_s.

6. A comparison of Tables VIII - X shows that a and b are only a few orders of magnitude smaller than $(\mathbf{M}_s)_3$.

Thus the transient solution is seen to be either of the same order of magnitude as the stationary solution or down to a few orders smaller and that it decays with characteristic times approximately equal to the relaxation times of the system, and that its frequency of oscillation is determined by the applied fields, being of the order of magnitude of 10^9 Hz.

TABLE I

H_0 (Kgauss)	$-\hat{\sigma} \times 10^9$ rad/sec		
	H_1 = 10 gauss	H_1 = 25 gauss	H_1 = 40 gauss
2.7	12.123	12.129	12.142
3.1	5.0776	5.0937	5.1233
3.5	1.9804	2.0212	2.0947
3.9	9.0215	9.0306	9.0473

all δ_m, δ_s

TABLE II

H_0 (Kgauss)	$-\sigma_0 \times 10^5$ rad/sec		
	H_1 = 10 gauss	H_1 = 25 gauss	H_1 = 40 gauss
2.7	3.3479	3.3490	3.3185
3.1	3.3274	3.3135	3.2699
3.5	3.3036	3.1763	2.9572
3.9	3.3224	3.3240	3.3035

$\delta_m = -3.333 \times 10^5\ Hz$, $\delta_s = -1.111 \times 10^2\ Hz$

TABLE III

H_0 (Kgauss)	$-\sigma_0 \times 10^5$ rad/sec		
	$H_1 = 10$ gauss	$H_1 = 25$ gauss	$H_1 = 40$ gauss
2.7	3.3283	3.3289	3.3295
3.1	3.3332	3.3249	3.3173
3.5	3.3257	3.2938	3.2377
3.9	3.3369	3.3308	3.3253

$\delta_m = -3.333 \times 10^5\ Hz$, $\delta_s = -2.500 \times 10^5\ Hz$

TABLE IV

H_0 (Kgauss)	$-\sigma_0 \times 10^4$ rad/sec		
	$H_1 = 10$ gauss	$H_1 = 25$ gauss	$H_1 = 40$ gauss
2.7	4.9951	4.9951	4.9890
3.1	4.9676	5.0592	5.1690
3.5	5.0653	5.3537	5.8633
3.9	4.9707	4.9707	5.0134

$\delta_m = -5.000 \times 10^4\ Hz$, $\delta_s = -1.250 \times 10^5\ Hz$

TABLE V

H_0 (Kgauss)	$-\tilde{\sigma} \times 10^5$ rad/sec		
	$H_1 = 10$ gauss	$H_1 = 25$ gauss	$H_1 = 40$ gauss
2.7	2.5025	2.5022	2.5019
3.1	2.5001	2.5041	2.5080
3.5	2.5038	2.5198	2.5478
3.9	2.4982	2.5013	2.5040

$\delta_m = -3.333 \times 10^5\ Hz,\ \delta_s = -2.500 \times 10^5\ Hz$

TABLE VI

H_0 (Kgauss)	$-\tilde{\sigma} \times 10^5$ rad/sec		
	$H_1 = 10$ gauss	$H_1 = 25$ gauss	$H_1 = 40$ gauss
2.7	2.5008	2.4971	2.4934
3.1	2.4974	2.4924	2.4821
3.5	2.4917	2.4523	2.3862
3.9	2.4977	2.4986	2.4934

$\delta_m = -5.000 \times 10^4\ Hz,\ \delta_s = -2.500 \times 10^5\ Hz$

TABLE VII

H_0 (Kgauss)	$-\tilde{\sigma} \times 10^4$ rad/sec		
	$H_1 = 10$ gauss	$H_1 = 25$ gauss	$H_1 = 40$ gauss
2.7	2.3526	3.3526	3.3136
3.1	3.3266	3.3120	3.3022
3.5	3.3233	3.2558	3.1492
3.9	3.3136	3.3810	3.3461

$\delta_m = -1.250 \times 10^2\ Hz,\ \ \delta_s = -3.333 \times 10^4\ Hz$

TABLE VIII

H_0 (Kgauss)	$b \times 10^{-2}$ erg/gauss		
	$H_1 = 10$ gauss	$H_1 = 25$ gauss	$H_1 = 40$ gauss
2.7	0.39	2.46	6.29
3.1	2.58	16.0	40.6
3.5	19.2	115.	274.
3.9	1.02	6.42	16.4

all δ_m, δ_s

TABLE IX

δ	H_0 (Kgauss)	$\lvert a \rvert \times 10^{-12}$ erg/gauss $H_1 = 10$ gauss	$H_1 = 25$ gauss	$H_1 = 40$ gauss
	2.7	0.39	2.46	6.29
3×10^3	3.1	2.58	16.0	40.5
	3.5	19.1	115.	274.
	3.9	1.02	6.49	16.3
	2.7	0.36	2.22	5.66
10	3.1	2.32	14.4	36.4
	3.5	17.2	103.	243.
	3.9	0.92	5.77	14.7
	2.7	0.59	3.68	9.36
4×10^{-1}	3.1	3.86	23.6	58.0
	3.5	28.1	153.	311.
	3.9	1.54	9.57	24.1
	2.7	325.	1060.	1435.
10^{-3}	3.1	1170.	1878.	1999.
	3.5	2135.	2259.	2129.
	3.9	744.	1895.	2303.

a is positive for $\delta < 1$, negative for $\delta > 1$

TABLE X

H_0 (Kgauss)	$(M_s)_3 \times 10^{-9}$ erg/gauss
2.7	1.7 – 1.9
3.1	1.5 – 2.1
3.5	0.6 – 2.4
3.9	2.6 – 2.7

(characteristic values)

VI. CONCLUSIONS

We have obtained the full solution of Bloch's phenomenological equations that describe the behavior of the magnetization of a system of N_0 dipoles per unit volume under the influence of a magnetic field that has two parts; a constant one and a rotating one perpendicular to the first. The dipoles are allowed to interact and to perform small vibrations around their equilibrium positions. This full solution contains, of course, the stationary solution given by Bloch[2], but also contains a transient part[1]. This transient part (eq. (20)) is composed of two terms: an exponentially decaying term, and a cosine function modulated by a decreasing exponential.

The existence of the transient part could be used to obtain information about the relaxation times of this system. If, for a given material, we set a value of H_0 near to resonance, in other words fixing a small value of α^2, and commence with a reasonably high value of the magnitude of the rotating field (a reasonably high value of β^2), we should observe a transient that has periodic behavior, since presumably this point is in the region where $R > 0$ (see Fig. 1). We then decrease the value of H_1 until we find the first value of H for which this periodic behavior does not appear. These values of α^2 and β^2 give a point which is now on the curve $R = 0$. This gives us the value of β_-^2, from which we can obtain the value of δ, from eq. (11). If we continue reducing H_1 we eventually will arrive at a point where the periodic transient behavior reappears. This is the value β_+^2, from which we can obtain a check on the value of δ previously obtained. If for all values of H_1 the periodic behavior persists we know that the chosen value of α^2 is too large, and we must repeat the above procedure with a smaller value of α^2.

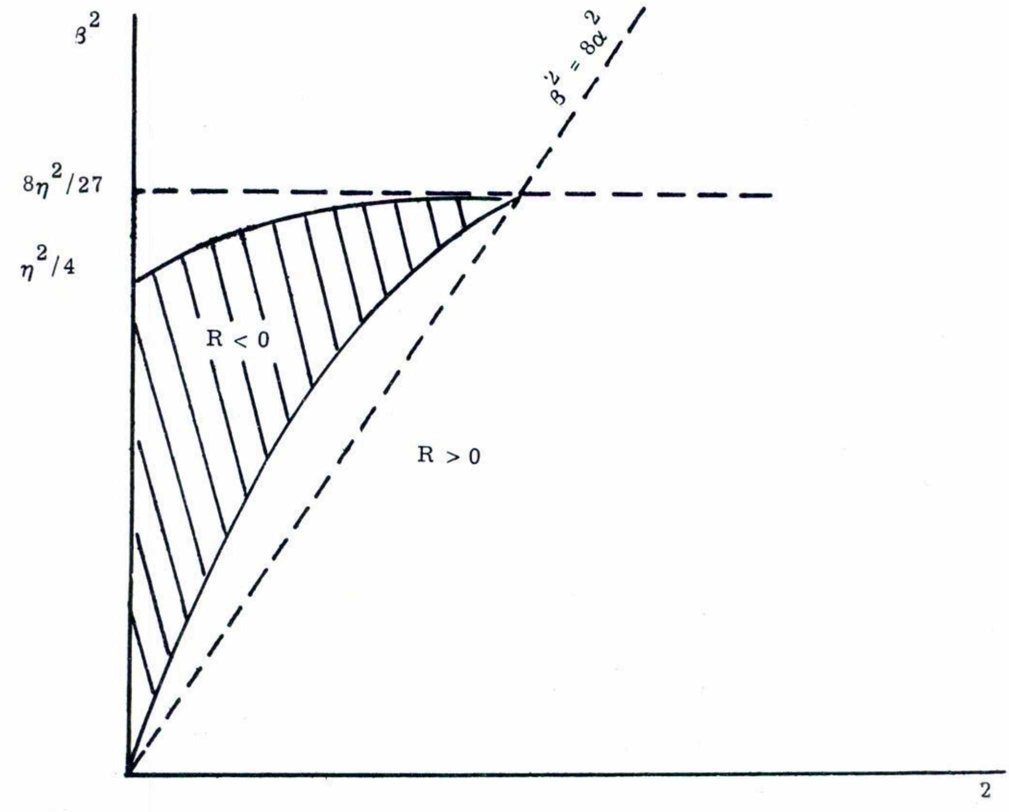

Fig. 1. Sign of R in the α^2, β^2-plane. $R < 0$ in the shaded region, $R = 0$ on the solid curves, and $R > 0$ elsewhere.

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RESUMEN

Se estudia el comportamiento de un sistema de dipolos magnéticos en un campo magnético, usando las ecuaciones fenomenológicas de Bloch. Se encuentra que, si se aplica un campo rotativo, el comportamiento transitorio de la componente perpendicular de la magnetización del sistema está dado por la suma de una exponencial decreciente y una función senoidal modulada por una exponencial decreciente. La expresión teórica que se obtiene proporciona las amplitudes, frecuencias, tiempos de decaimiento y fase en términos de parámetros como los tiempos de relajamiento del sistema. Se discute un método apropiado para determinar experimentalmente estos valores.