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MATRIX ELEMENTS OF GENERATORS OF THE GROUP $U(3) \times U(3)$ IN A $\mathfrak{U}(3)$ BASIS

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ABSTRACT: The group $U(3) \times U(3)$ has recently become of importance in elementary particle physics. The relevant irreducible representations of this group are labeled by four non-negative integer numbers $[h'_1 h'_2][h'_3 h'_4]$, and its physically significant basis states are classified by a chain of groups $\mathfrak{U}(3) \supset \mathfrak{U}(2) \supset \mathfrak{U}(1)$. It is known that this classification scheme involves a "multiplicity problem", but when $h'_4 = 0$ this problem does not arise. In this paper we determine the matrix elements of the generators of $U(3) \times U(3)$ with respect to the previously mentioned

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basis states, restricting ourselves to those irreducible representations with $h'_4 = 0$. We also give a brief discussion of the representation coefficients of $U(3) \times U(3)$.

1. INTRODUCTION

The problem we want to discuss in this paper is the generalization to $n = 3$ of another problem, corresponding to $n = 2$, which is considered nowadays a standard topic in atomic spectroscopy[1], namely: the determination of the matrix elements (ME) of the generators of a group $SU(2) \times SU(2)$ with respect to a basis that diagonalizes the Casimir operator of a definite $\mathfrak{SU}(2)$ subgroup. If we denote the generators of $SU(2) \times SU(2)$ by $J_q^{(s)}$, $s = 1, 2$; $q = 1, 2, 3$; obeying the commutation relations

$$\left[J_1^{(s)}, J_2^{(s')} \right] = i J_3^{(s)} \delta_{ss'} \qquad 1, 2, 3 \text{ cyclical},$$

then the $\mathfrak{SU}(2)$ subgroup is that one with generators $J_q = J_q^{(1)} + J_q^{(2)}$, and the problem consists in the evaluations of the ME

$$< j_1 j_2 j' m' \mid J_q^{(s)} \mid j_1 j_2 j m > ,$$

where $j_1 j_2$ are the labels of the irreducible representation (IR) of $SU(2) \times SU(2)$, and jm are the labels of the IR of the subgroups $\mathfrak{SU}(2) \supset R(2)$. This problem was originally solved in 1931 by Güttinger and Pauli[1], who used in their analysis only the commutation rules of the generators $J_q^{(s)}$. The same problem, at present, can be trivially solved by using the powerful techniques of the Algebra of $SU(2)$ irreducible tensor operators, i.e. use of Wigner-Eckart theorem and Racah coefficients of $SU(2)$.

In our generalization to the case $n = 3$, i.e. the case of the group $SU(3) \times SU(3)$, we have not attempted an analysis similar to that of Güttinger and Pauli; on the other hand we cannot fully apply the techniques of the algebra of tensor operators in this case, as the necessary Racah coefficients of $SU(3)$ are not available at the present time. So we based our analysis on a combined use of the Wigner-Eckart theorem and use of the explicit expressions of the $U(3) \times U(3)$ basis states, which already have appeared in the literature[2].

Our main motivation in undertaking this problem was the fact that the group $U(3) \times U(3)$ plays an important role in the recent applications of higher symmetry schemes to elementary particle physics[1]. It thus seems to be a relevant problem to explore the general properties of this group, such as the bases for its I R, the M E of its generators with respect to these bases, the representation coefficients, the Clebsch-Gordan coefficients of the group and so forth. To our knowledge, the only of the aforementioned properties which has been studied in some detail is the first one. Moshinsky[2] has given the explicit expressions of basis states belonging to an IR, of $U(3) \times U(3)$ and having the highest weight in a $\mathcal{U}(3)$ subgroup, this being the classification of physical interest[1]. From the states of reference 2, and the lowering operators of the unitary group[3], we can construct the complete basis for an IR of $U(3) \times U(3)$. In this paper we make use of the basis states of reference 2 in order to evaluate the ME of the generators of $U(3) \times U(3)$ with respect to them. As far as we can tell, these ME formerly had been explicitly determined only for some particular IR of $U(3) \times U(3)$ or for particular IR of the $\mathcal{U}(3)$ subgroup. Our own analysis is also somewhat restricted owing to our desire of avoiding the "multiplicity problem" that will be mentioned three paragraphs below; nevertheless it goes further than the previous attempts[4,5]. We also give in this paper a brief discussion of the representation coefficients of $U(3) \times U(3)$.

The generators of $U(3) \times U(3)$ are 18 operators M^{ν}_{μ}, N^{ν}_{μ}; $\mu, \nu = 1, 2, 3$ obeying the commutation rules

$$[M^{\nu}_{\mu}, M^{\nu'}_{\mu'}] = M^{\nu'}_{\mu} \delta^{\nu}_{\mu'} - M^{\nu}_{\mu'} \delta^{\nu'}_{\mu}$$

$$[N^{\nu}_{\mu}, N^{\nu'}_{\mu'}] = N^{\nu'}_{\mu} \delta^{\nu}_{\mu'} - N^{\nu}_{\mu'} \delta^{\nu'}_{\mu}$$

$$[M^{\nu}_{\mu}, N^{\nu'}_{\mu'}] = 0 \tag{1}$$

The M^{ν}_{μ} are the generators of the first $U(3)$ group, and the N^{ν}_{μ} are those of the second one. An IR of $U(3) \times U(3)$ will be labeles by the indices $[h'_1 h'_2][h'_3 h'_4]$ with the h'_s being non-negative integers obeying $h'_3 \geqslant h'_4 \geqslant 0$; $h'_3 \geqslant h'_4 \geqslant 0$; i.e., each partition $[h'_s h'_t]$ specifies a Young pattern of two rows. As a matter of fact the most general IR of $U(3)$ is labeled by a Young pattern with 3 rows[2]; however, we are mostly interested on IR of $SU(3)$, all of which are equivalent to IR of $U(3)$ with only two rows. In the notation commonly used in elementary particle physics[1] and IR of $SU(3) \times SU(3)$ would be denoted by $[\lambda\mu][\lambda'\mu']$, the connection with our notation being: $\lambda = h_1' - h_2'$, $\mu = h_2'$, $\lambda' = h_3' - h_4'$, $\mu' = h_4'$.

The rows of an IR of $U(3) \times U(3)$ can be classified in several ways; we shall mention two of them. In the first classification scheme one gives the 6 indices specifying the canonical subgroups $U(2) \supset U(1)$ of each $U(3)$ group; i.e., one is classifying the rows of the IR according to the canonical chain of groups

$$U(3) \times U(3) \supset U(2) \times U(2) \supset U(1) \times U(1) .$$

The basis states of an IR of $U(3) \times U(3)$ in this classification scheme are simply the product of the Gelfand basis states[2,6] associated with each $U(3)$ group, and are denoted as

$$\left| \begin{matrix} h_1' & & h_2' & & 0 \\ & q_1' & & q_2' & \\ & & r_1' & & \end{matrix} \quad \begin{matrix} h_3' & & h_4' & & 0 \\ & q_3' & & q_4' & \\ & & r_3' & & \end{matrix} \right\rangle \equiv |[h_1' h_2'][h_3' h_4']; q_1' q_2' r_1', q_3' q_4' r_3' > . \tag{2}$$

The operators M_1^1, M_2^2, N_1^1, N_2^2 and the quadratic Casimir operator of each $U(2)$ subgroup, are diagonal with respect to the basis states (2).

Now, for the physical applications[1] we rather need a different set of basis states, with respect to which all the Casimir operators of a chain of groups $\mathcal{U}(3) \supset \mathcal{U}(2) \supset \mathcal{U}(1)$ are diagonal, the group $\mathcal{U}(3)$ having as generators

$$C_\mu^\nu \equiv M_\mu^\nu + N_\mu^\nu ; \quad \mu, \nu = 1, 2, 3 \tag{3a}$$

It is then natural to define

$$K_\mu^\nu \equiv M_\mu^\nu - N_\mu^\nu ; \quad \mu, \nu = 1, 2, 3 \tag{3b}$$

and consider C_μ^ν and K_μ^ν as generators of $U(3) \times U(3)$; according to (1) they satisfy the commutation rules

$$[C_\mu^\nu , C_{\mu'}^{\nu'}] = C_\mu^{\nu'} \delta_{\mu'}^\nu - C_{\mu'}^\nu \delta_\mu^{\nu'} \tag{4a}$$

$$[C^{\nu}_{\mu} ; K^{\nu'}_{\mu'}] = K^{\nu'}_{\mu} \delta^{\nu}_{\mu'} - K^{\nu}_{\mu'} \delta^{\nu'}_{\mu} \tag{4b}$$

$$[K^{\nu}_{\mu} , K^{\nu'}_{\mu'}] = C^{\nu'}_{\mu} \delta^{\nu}_{\mu'} - C^{\nu}_{\mu'} \delta^{\nu'}_{\mu} \tag{4c}$$

The physical states mentioned above, thus correspond to a classification scheme by means of the chain of groups

$$U(3) \times U(3) \supset \mathcal{U}(3) \supset \mathcal{U}(2) \supset \mathcal{U}(1) ,$$

and they will be denoted as

$$\left| [h_1' h_2'] [h_3' h_4'] ; \gamma, \begin{matrix} h_1 & & h_2 & & h_3 \\ & q_1 & & q_2 & \\ & & r_1 & & \end{matrix} \right\rangle \equiv | [h_1' h_2'] [h_3' h_4'] ; \gamma, h_1 h_2 h_3 , q_1 q_2 r_1 > \tag{5}$$

In ref. (2) the problem of construction of the states (5) has been solved; it is shown there that the identity $h_1' + h_2' + h_3' + h_4' = h_1 + h_2 + h_3$ holds, so we have only 5 independent indices to label the rows of the IR of $U(3) \times U(3)$ in this classification scheme. The sixth index needed to complete the classification, namely γ, has been exhibited in the notation of the states (5); γ serves to distinguish between equivalent IR of the $\mathcal{U}(3)$ subgroup when these appear more than once; this is the "multiplicity problem". Again from reference (2), it is known that when $h_4' = 0$ there is no need for the index γ; i.e. no IR $(h_1 h_2 h_3)$ of $\mathcal{U}(3)$ is contained more than once in the IR $[h_1' h_2'] [h_3' 0]$ of $U(3) \times U(3)$. In this paper we restrict our analysis to these particular IR of $U(3) \times U(3)$; our basis states will be denoted as

$$| [h_1' h_2'] [h_3' 0] ; h_1 h_2 h_3 , q_1 q_2 r_1 > \tag{6}$$

and as the indices $[h_1' h_2'] [h_3' 0]$ will always be the same in all states, we shall frequently keep them in mind and write our $U(3) \times U(3)$ states simply as $| h_1 h_2 h_3 , q_1 q_2 r_1 >$.

Even though in ref. (2) a method has been presented to deal with the "multiplicity problem", we make the restriction indicated in (6) in order to keep the problem within manageable proportions, as its complexity greatly increases when one considers $h_4' \neq 0$. There are two former determinations of the ME of the generators of $U(3) \times U(3)$ known to us[4,5]. In ref. (4) S.K. Bose has solved the problem in the special case $h_1' = h_2' = h_3'$, $h_4' = 0$, and in ref. (5) A. Bincer has described a method to perform the calculation in the general case ($h_4' \neq 0$). The approach of ref. (5) is based, as ours is, on the use of the Wigner-Eckart theorem[7] applied to the tensor of K^{ν}_{μ}, but the reduced ME of K^{ν}_{μ} were explicitly determined by Bincer only for some special IR of $\mathcal{U}(3)$ of interest in current algebra theory; namely the case when, in the notation of equation (19) of our paper, $(h_1 h_2 h_3)$ and/or $(\overline{h}_1 \overline{h}_2 \overline{h}_3)$ is an octet representation of $\mathcal{U}(3)$.

The ME of the generators $\mathcal{C}^{\nu}_{\mu}$ of (3a) with respect to the states (6) are, of course, well known[8]. The remaining task is the evaluation of the ME of the K^{ν}_{μ} of (3b) with respect to the states (6). This will be done in Section 3, after the ME of an auxiliary tensor operator have been evaluated in Section 2. In Section 4 we make some considerations about the representation coefficients of $U(3) \times U(3)$.

2. MATRIX ELEMENTS OF THE CREATION OPERATOR Δ^{3}_{μ}

We shall adopt the notation of ref. (2). The generators of $U(3) \times U(3)$ will be realized in terms of a set of creation and annihilation operators $a^{+}_{\mu s}$, $a^{+}_{\mu s}$, $a^{\mu s}$; μ, $s = 1, 2, 3$, obeying the usual commutation rules of boson operators[2]. The explicit form of the generators (3a, b) is

$$\mathcal{C}^{\nu}_{\mu} = \sum_{s=1}^{3} a^{+}_{\mu s}\, a^{\nu s} \tag{7a}$$

$$K^{\nu}_{\mu} = a^{+}_{\mu 1}\, a^{\nu 1} + a^{+}_{\mu 2}\, a^{\nu 2} - a^{+}_{\mu 3}\, a^{\nu 3} \equiv \mathcal{C}^{\nu}_{\mu} - 2a^{+}_{\mu 3}\, a^{\nu 3} \tag{7b}$$

The states (6) are given in terms of some determinants in creation operators acting on the ground state $|0>$. The determinants that appear are

$$\Delta^{s}_{\mu} \equiv a^{+}_{\mu s},\; \Delta^{st}_{\mu\nu} \equiv a^{+}_{\mu s}\, a^{+}_{\nu t} - a^{+}_{\nu s}\, a^{+}_{\mu t},\; \Delta^{123}_{123} \equiv \sum_{ijk} \epsilon_{ijk}\, a^{+}_{i1}\, a^{+}_{j2}\, a^{+}_{k3} \tag{8}$$

ϵ_{ijk} being the completely antisymmetric tensor. In particular, the state (6) having the highest weight in the IR $(h_1 h_2 h_3)$ of the $\mathcal{U}(3)$ subgroup is[2]

$$|[h_1' h_2'][h_3' 0]; h_1 h_2 h_3, h_1 h_2 h_1> =$$

$$= N(h_1 h_2 h_3)(\Delta_1^1)^{h_1' - h_2}(\Delta_1^3)^{h_1 - h_1'}(\Delta_{12}^{12})^{h_2' - h_3}(\Delta_{12}^{13})^{h_2 - h_2'}(\Delta_{123}^{123})^{h_3} |0> \tag{9a}$$

$$N(h_1 h_2 h_3) = \left[\frac{(h_1 - h_3 + 2)!(h_2 - h_3 + 1)!(h_1' - h_2' + 1)!(h_1 - h_2 + 1)!}{(h_1 + 2)!(h_2 + 1)!h_3!(h_2' - h_3)!(h_1' - h_3 + 1)!(h_2 - h_2')!(h_1 - h_2' + 1)!(h_1' - h_2)!(h_1 - h_1')!}\right]^{\frac{1}{2}} \tag{9b}$$

All the other states of the IR of $\mathcal{U}(3)$ can be generated from this one by means of some functions of the generators $\mathcal{C}_\mu^\nu$ called lowering operators[3]. From the fact that the exponent of each variable in the polynomial (9a) must be non-negative we find that the IR $(h_1 h_2 h_3)$ of $\mathcal{U}(3)$ contained in the IR $[h_1' h_2'][h_3' 0]$ of $U(3) \times U(3)$ are those which satisfy $h_1 \geqslant h_1' \geqslant h_2 \geqslant h_2' \geqslant h_3 \geqslant 0$; $h_1 + h_2 + h_3 = h_1' + h_2' + h_3'$.

Our next step will be the evaluation of the ME of the creation operator Δ_μ^3 with respect to the states (6):

$$<\overline{h}_1 \overline{h}_2 \overline{h}_3, \overline{q}_1 \overline{q}_2 \overline{r}_1 | \Delta_\mu^3 | h_1 h_2 h_3, q_1 q_2 r_1> \quad . \tag{10}$$

(Notice that we are using now the abbreviated notation for the $U(3) \times U(3)$ states*. We shall evaluate (10) by means of the Wigner-Eckart theorem[7] in $\mathcal{U}(3)$. From the explicit expression of the Gelfand states[2] we find that Δ_μ^3, $\mu = 1, 2, 3$ are the components of a $\mathcal{U}(3)$ irreducible tensor

$$T_{q_1 q_2 r_1}^{[h_1 h_2 h_3]}$$

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*As a matter of fact, the $U(3) \times U(3)$ labels of the bra-state in (10) are $[h_1' h_2'][h_3' + 1, 0]$; however this change in h_3' is irrelevant for the calculation as from (9) the index h_3' occurs explicitly nowhere.

with indices $h_1 = 1$, $h_2 = h_3 = 0$; i.e. a triplet in the language of elementary particles, the detailed classification being

$$\Delta_1^3 = T_{101}^{[100]}, \ \Delta_2^3 = T_{100}^{[100]}, \ \Delta_3^3 = T_{000}^{[100]}. \qquad (11)$$

In order to be able to apply the Wigner-Eckart theorem we must know the $\mathcal{U}(3)$ Clebsch-Gordan coefficients (CGC)

$$\left\langle \begin{matrix} h_1 h_2 h_3 & 1\ 0\ 0 & \bar{h}_1 \bar{h}_2 \bar{h}_3 \\ q_1 q_2 & ; \ q_1'' \ 0 & \bar{q}_1 \bar{q}_2 \\ r_1 & r_1 & \bar{r}_1 \end{matrix} \right\rangle \equiv \langle h_1 h_2 h_3, q_1 q_2 ; 1 q_1'') \bar{h}_1 \bar{h}_2 \bar{h}_3, \bar{q}_1 \bar{q}_2 \rangle$$

$$\times \left\langle \frac{1}{2}(q_1 - q_2), r_1 - \frac{1}{2}(q_1 + q_2); \frac{1}{2} q_1'', r_1'' - \frac{1}{2} q_1'' \,\middle|\, \frac{1}{2}(\bar{q}_1 - \bar{q}_2), \bar{r}_1 - \frac{1}{2}(\bar{q}_1 + \bar{q}_2) \right\rangle \qquad (12)$$

where on the right hand side the first term is the reduced CGC or isoscalar factor, and the second term is the ordinary $S\mathcal{U}(2)$ CGC in the notation $\langle j_1 m_1 j_2 m_2 | jm \rangle$. We give in Table I the reduced CGC that appear in (12). This table was constructed by setting $h_1'' = 1$ in the closed algebraic expression for the $\mathcal{U}(3)$ reduced CGC

$$\langle h_1 h_2 h_3, q_1 q_2 ; h_1'' q_1'') \bar{h}_1 \bar{h}_2 \bar{h}_3, \bar{q}_1 \bar{q}_2 \rangle$$

calculated by Moshinsky[9]. The ME in (10) can thus be written, according to the Wigner-Eckart theorem[7], as

$$\langle \bar{h}_1 \bar{h}_2 \bar{h}_3, \bar{q}_1 \bar{q}_2 \bar{r}_1 | T_{q_1'' 0 r_1''}^{[100]} | h_1 h_2 h_3, q_1 q_2 r_1 \rangle$$

$$= \left\langle \begin{matrix} h_1 h_2 h_3 & 1\ 0\ 0 & \bar{h}_1 \bar{h}_2 \bar{h}_3 \\ q_1 q_2 & ; \ q_1'' \ 0 & \bar{q}_1 \bar{q}_2 \\ r_1 & r_1'' & \bar{r}_1 \end{matrix} \right\rangle \langle \bar{h}_1 \bar{h}_2 \bar{h}_3 \ T^{[100]} \| h_1 h_2 h_3 \rangle \qquad (13)$$

the last factor being the reduced ME of the tensor $T^{[100]}$.

From the Clebsch-Gordan series of $\mathcal{U}(3)$ we know that the IR $(\bar{h}_1 \bar{h}_2 \bar{h}_3)$ in (10) can be either $(h_1+1, h_2 h_3)$ or (h_1, h_2+1, h_3) or $(h_1 h_2, h_3+1)$. We have then to evaluate 3 reduced ME of the tensor $T^{[100]}$. In the appendix we give the details of the determination of these reduced ME and quote now the results:

$$< h_1+1, h_2 h_3 \| T^{[100]} \| h_1 h_2 h_3 > = \left[\frac{(h_1+3)(h_1-h_2'+2)(h_1-h_1'+1)}{(h_1-h_3+3)(h_1-h_2+2)} \right]^{\frac{1}{2}} \tag{14a}$$

$$< h_1, h_2+1, h_3 \| T^{[100]} \| h_1 h_2 h_3 > = \left[\frac{(h_2+2)(h_2-h_2'+1)(h_1'-h_2)}{(h_1-h_2)(h_2-h_3+2)} \right]^{\frac{1}{2}} \tag{14b}$$

$$< h_1 h_2, h_3+1 \| T^{[100]} \| h_1 h_2 h_3 > = \left[\frac{(h_3+1)(h_2'-h_3)(h_1'-h_3+1)}{(h_1-h_3+1)(h_2-h_3)} \right]^{\frac{1}{2}} \tag{14c}$$

With these formulas, plus Table I and a table of $S\mathcal{U}(2)$ CGC we can evaluate all the ME in (10).

3. MATRIX ELEMENTS OF K_μ^ν

We shall consider two separate cases for the ME of K_μ^ν with respect to the $U(3) \times U(3)$ basis states (6), namely: ME diagonal in $(h_1 h_2 h_3)$ and ME non-diagonal in $(h_1 h_2 h_3)$. The analysis in the latter case will be done using the Wigner-Eckart theorem, but in the first case we shall follow a more direct analysis. We prefer not to use the Wigner-Eckart approach in the first case because of the involved nature of the 2 reduced ME of that case.

a) ME diagonal in $(h_1 h_2 h_3)$.

In this case we first determine explicitly the ME

$$< h_1 h_2 h_3, q_1' q_2' r_1' | K_3^3 | h_1 h_2 h_3, q_1 q_2 r_1 > . \tag{15}$$

From the commutation relations (4b) we have:

$$K^3_2 = \mathcal{C}^3_2 K^3_3 - K^3_3 \mathcal{C}^3_2 , \quad K^3_1 = \mathcal{C}^3_1 K^3_3 - K^3_3 \mathcal{C}^3_1 ;$$

taking the ME of these identities with respect to the same states as in (15) and introducing a complete set of intermediate states between a $\mathcal{C}$ and a K, we deduce the ME of K^3_2 and K^3_1 in terms of the ME in (15) plus the known[8] ME of the $\mathcal{U}(3)$ generators $\mathcal{C}^\nu_\mu$. We obtain, for instance, using equation (16) below, and recalling that the ME of a $\mathcal{C}$ are diagonal in $(h_1 h_2 h_3)$:

$$\langle h_1 h_2 h_3, q'_1 q'_2 r'_1 | K^3_2 | h_1 h_2 h_3 , q_1 q_2 r_1 \rangle = \langle h_1 h_2 h_3 , q'_1 q'_2 r'_1 | \mathcal{C}^3_2 | h_1 h_2 h_3 , q_1 q_2 r_1 \rangle$$

$$\times [\langle h_1 h_2 h_3, q_1 q_2 r_1 | K^3_3 | h_1 h_2 h_3, q_1 q_2 r_1 \rangle - \langle h_1 h_2 h_3, q'_1 q'_2 r'_1 | K^3_3 | h_1 h_2 h_3 , q'_1 q'_2 r'_1 \rangle] .$$

Then the ME of K^2_3 and K^1_3 follow from the hermiticity properties $K^2_3 = (K^3_2)^+$ and $K^1_3 = (K^3_1)^+$. As for the ME of the K^ν_μ ; $\mu , \nu = 1, 2$ they can be obtained by an analogous method using the commutators

$$[\mathcal{C}^3_1 , K^2_3] = K^2_1 \text{ and } [\mathcal{C}^3_\mu , K^\mu_3] = K^\mu_\mu - K^3_3 ; \ \mu = 1,2 .$$

Thus we see that the essential step is the evaluation of the ME in (15). This can easily be done, as from (7b) we have $K^3_3 = \mathcal{C}^3_3 - 2\Delta^3_3 \Delta^{3+}_3$ and we know from the last section the ME of Δ^3_3. It is found that the ME of K^3_3 have the value

$$<h_1 h_2 h_3, q_1' q_2' r_1' | K_3^3 | h_1 h_2 h_3, q_1 q_2 r_1> = \delta_{q_1 q_1'} \delta_{q_2 q_2'} \delta_{r_1 r_1'} \times$$

$$\times \left\{ h_1 + h_2 + h_3 - q_1 - q_2 - 2 \frac{(h_1 - q_2 + 1)(h_1 - q_1)(h_1 + 2)(h_1 - h_2' + 1)(h_1 - h_1')}{(h_1 - h_2)(h_1 - h_3 + 1)(h_1 - h_3 + 2)(h_1 - h_2 + 1)} \right.$$

$$- 2 \frac{(q_1 - h_2 + 1)(h_2 - q_2)(h_2 + 1)(h_2 - h_2')(h_1' - h_2 + 1)}{(h_1 - h_2 + 1)(h_1 - h_2 + 2)(h_2 - h_3)(h_2 - h_3 + 1)}$$

$$\left. - 2 \frac{(q_1 - h_3 + 2)(q_2 - h_3 + 1)(h_2' - h_3 + 1)(h_1' - h_3 + 2)\, h_3}{(h_1 - h_3 + 2)(h_1 - h_3 + 3)(h_2 - h_3 + 1)(h_2 - h_3 + 2)} \right\} \qquad (16)$$

b) ME non-diagonal in $(h_1 h_2 h_3)$.

The classification of the K_μ^ν as $\mathcal{U}(3)$ irreducible tensors $T_{q_1 q_2 r_1}^{[100]}$ has been given by Kuriyan et al[10]. Except for an overall multiplicative factor they found that

$$K_1^3 = T_{212}^{[210]}, \qquad K_2^3 = T_{211}^{[210]}, \qquad K_1^2 = -T_{202}^{[210]}$$

$$K_3^1 = T_{100}^{[210]}, \qquad K_3^2 = -T_{101}^{[210]}, \qquad K_2^1 = T_{200}^{[210]}$$

$$K_1^1 = \frac{1}{\sqrt{3}} T_{111}^{[111]} - \frac{1}{\sqrt{6}} T_{111}^{[210]} + \frac{1}{\sqrt{2}} T_{201}^{[210]} \qquad (17)$$

$$K_2^2 = \frac{1}{\sqrt{3}} T_{111}^{[111]} - \frac{1}{\sqrt{6}} T_{111}^{[210]} - \frac{1}{\sqrt{2}} T_{201}^{[210]}$$

$$K_3^3 = \frac{1}{\sqrt{3}} T_{111}^{[111]} + \frac{2}{\sqrt{6}} T_{111}^{[210]} .$$

We thus see that the K^{ν}_{μ} are a mixture of a singlet and an octet operators, in the language of elementary particles. As

$$\sqrt{3}\, T^{[111]}_{111} = K^1_1 + K^2_2 + K^3_3 = C^1_1 + C^2_2 - C^3_3 ,$$

with

$$C^s_s = \sum_{\mu=1}^{3} \Delta^s_\mu \Delta^{s+}_\mu \; ; \quad s = 1, 2, 3 \tag{18}$$

and the states (6) are eigenstates[2] of C^s_s with eigenvalue h'_s , then the tensor $T^{[111]}$ is diagonal with respect to the states (6) with eigenvalue $\frac{1}{\sqrt{3}} (h'_1 + h'_2 - h'_3)$.

It remains only to evaluate the ME of the octet operator $T^{[210]}$. We shall do this by means of the Wigner-Eckart theorem:

$$\langle \bar{h}_1 \bar{h}_2 \bar{h}_3 , \bar{q}_1 \bar{q}_2 \bar{r}_1 | T^{[210]}_{q'_1 q'_2 r'_1} | h_1 h_2 h_3 , q_1 q_2 r_1 \rangle =$$

$$\langle h_1 h_2 h_3 , q_1 q_2 ; 210, q'_1 q'_2) \bar{h}_1 \bar{h}_2 \bar{h}_3 , \bar{q}_1 \bar{q}_2 \rangle \times$$

$$\times \langle \tfrac{1}{2}(q_1 - q_2) , r_1 - \tfrac{1}{2}(q_1 + q_2) ; \tfrac{1}{2}(q'_1 - q'_2), r'_1 - \tfrac{1}{2}(q'_1 + q'_2) | \tfrac{1}{2}(\bar{q}_1 - \bar{q}_2) , \bar{r}_1 - \tfrac{1}{2}(\bar{q}_1 + \bar{q}_2) \rangle \times$$

$$\times \langle \bar{h}_1 \bar{h}_2 \bar{h}_3 \| T^{[210]} \| h_1 h_2 h_3 \rangle \tag{19}$$

where, on the right hand side we have : the reduced $\mathcal{U}(3)$ CGC (or isoscalar factor), the ordinary $S\mathcal{U}(2)$ CGC and the reduced ME of the tensor $T^{[210]}$. As we are now interested in those ME ***non-diagonal*** in $\mathcal{U}(3)$, we know[10] that the IR $(\bar{h}_1 \bar{h}_2 \bar{h}_3)$ in (19) can be any of the six IR

$$(h_1 \pm 1, h_2 \mp 1, h_3) , \;\; (h_1 \pm 1, h_2 , h_3 \mp 1) , \;\; (h_1 , h_2 \pm 1, h_3 \mp 1) , \tag{20}$$

and as each one of these appears at most once in the Clebsch-Gordan series, there is no need for multiplicity labels[7] in (19). The reduced CGC of (19) are of a special type, tables of which are available in the literature[10,11].

In the Appendix we briefly explain how the 6 reduced ME of the octet tensor $T^{[210]}$ are evaluated, and we give now the results:

$$\langle h_1+1, h_2-1, h_3 \| T^{[210]} \| h_1 h_2 h_3 \rangle$$

$$= -2 \left[\frac{(h_1+3)(h_2+1)(h_1-h_1'+1)(h_1'-h_2+1)(h_1-h_2'+2)(h_2-h_2')}{(h_1-h_2+2)(h_1-h_2+3)(h_1-h_3+3)(h_2-h_3)} \right]^{\frac{1}{2}}, \tag{21a}$$

$$\langle h_1-1, h_2+1, h_3 \| T^{[210]} \| h_1 h_2 h_3 \rangle$$

$$= -2 \left[\frac{(h_1+2)(h_2+2)(h_1-h_1')(h_1'-h_2)(h_1-h_2'+1)(h_2-h_2'+1)}{(h_1-h_2-1)(h_1-h_2)(h_1-h_3+1)(h_2-h_3+2)} \right]^{\frac{1}{2}}, \tag{21b}$$

$$\langle h_1+1, h_2, h_3-1 \| T^{[210]} \| h_1 h_2 h_3 \rangle$$

$$= -2 \left[\frac{h_3(h_1+3)(h_1-h_1'+3)(h_1-h_2'+2)(h_2'-h_3+1)(h_1'-h_3+2)}{(h_1-h_2+2)(h_1-h_3+3)(h_1-h_3+4)(h_2-h_3+2)} \right]^{\frac{1}{2}}, \tag{21c}$$

$$\langle h_1-1, h_2, h_3+1 \| T^{[210]} \| h_1 h_2 h_3 \rangle$$

$$= +2 \left[\frac{(h_1+2)(h_3+1)(h_1-h_1')(h_1-h_2'+1)(h_2'-h_3)(h_1'-h_3+1)}{(h_1-h_2)(h_1-h_3)(h_1-h_3+1)(h_2-h_3)} \right]^{\frac{1}{2}}, \tag{21d}$$

$$\langle h_1 h_2+1, h_3-1 \| T^{[210]} \| h_1 h_2 h_3 \rangle$$

$$= -2 \left[\frac{(h_2+2)\, h_3 (h_1'-h_2)(h_1'-h_3+2)(h_2-h_2'+1)(h_2'-h_3+1)}{(h_1-h_2)(h_1-h_3+3)(h_2-h_3+2)(h_2-h_3+3)} \right]^{\frac{1}{2}}, \tag{21e}$$

$$< b_1\, , b_2 - 1, b_3 + 1 \,\| \, T^{[210]} \,\| \, b_1 b_2 b_3 >$$

$$= + 2 \left[\frac{(b_2 + 1)(b_3 + 1)(b_1' - b_2 + 1)(b_1' - b_3 + 1)(b_2 - b_2')(b_2' - b_3)}{(b_1 - b_2 + 2)(b_1 - b_3 + 1)(b_2 - b_3 - 1)(b_2 - b_3)} \right]^{\frac{1}{2}} \tag{21f}$$

Combining (19) and (21) together with the tables of references 10 or 11, we obtain the ME of K^{ν}_{μ} non-diagonal in $(b_1 b_2 b_3)$.

4. REPRESENTATION COEFFICIENTS OF $U(3) \times U(3)$

The representation coefficients (RC) of the group

$$SU(3): \; \mathcal{D}^{[b_1 b_2]}_{q_1' q_2' r_1',\, q_1 q_2 r_1}(\alpha_1, \ldots, \alpha_8)$$

can be expressed in terms of the familiar RC of $SU(2)$: $\mathcal{D}^{j}_{mm'}(\alpha, \beta, \gamma)$; an explicit expression of this fact has been given by one of the present authors[12] in the form

$$\mathcal{D}^{[b_1 b_2]}_{q_1 q_2 r_1,\, q_1' q_2' r_1'}(\alpha_1, \ldots, \gamma_3) = \sum_{\sigma = 0}^{b_2} \sum_{\tau = b_2}^{b_1} [(q_1 - q_2 + 1)(q_1' - q_2' + 1)]^{\frac{1}{2}} (\tau - \sigma + 1)$$

$$\cdot W\left[\tfrac{1}{2}(b_1 + b_2 - q_1 - q_2), \tfrac{1}{2}(q_1 - b_1 + \tau), \tfrac{1}{2}(b_1 - q_2 - \sigma), \tfrac{1}{2}(b_1 + b_2 - \sigma - \tau); \tfrac{1}{2}(q_2 - b_2 + \tau), \tfrac{1}{2}(q_1 - b_2 + \sigma)\right]$$

$$\cdot W\left[\tfrac{1}{2}(b_1 + b_2 - q_1' - q_2'), \tfrac{1}{2}(q_1' - b_1 + \tau), \tfrac{1}{2}(b_1 - q_2' - \sigma), \tfrac{1}{2}(b_1 + b_2 - \sigma - \tau); \tfrac{1}{2}(q_2' - b_2 + \tau), \tfrac{1}{2}(q_1' - b_2 + \sigma)\right]$$

$$\cdot \mathcal{D}^{\frac{1}{2}(q_1 q_2)}_{r_1 - \frac{1}{2}(q_1 + q_2),\, \sigma + \tau - b_1 - b_2 + \frac{1}{2}(q_1 + q_2)}(\alpha_1, \beta_1, \gamma_1)$$

$$\cdot \mathcal{D}^{\frac{1}{2}(q_1' - q_2')}_{\sigma + \tau - b_1 - b_2 + \frac{1}{2}(q_1' + q_2'),\, r_1' - \frac{1}{2}(q_1' + q_2')}(\alpha_2, \beta_2, \gamma_2)$$

$$\cdot \mathcal{D}^{\frac{1}{2}(\tau - \sigma)}_{q_1 + q_2 - b_1 - b_2 + \frac{1}{2}(\sigma + \tau),\, q_1' + q_2' - b_1 - b_2 + \frac{1}{2}(\sigma + \tau)}(\alpha_2, \beta_2, \alpha_2) \tag{22}$$

The RC of the group $SU(3)\times SU(3)$ would be simply a product of two RC of $SU(3)$ if the basis states of the IR of $SU(3)\times SU(3)$ were classified as in (2). Now, of course, we are rather interested in the RC of $SU(3)\times SU(3)$ when the basis states of the IR are classified as in (6). One way to obtain the latter RC is to transform from the basis in (6) to that in (2)(with $h_4'=0$); the transformation brackets between these 2 bases are, as is well known, are the CGC

$$\left\langle \begin{matrix} h_1' & h_2' & 0 \\ q_1' & q_2' & \\ & r_1' & \end{matrix} ; \begin{matrix} h_3' & 0 & 0 \\ q_3' & 0 & \\ & r_3' & \end{matrix} \middle| \begin{matrix} h_1 & h_2 & h_3 \\ q_1 & q_2 & \\ & r_1 & \end{matrix} \right\rangle$$

whose isoscalar factors were algebraically determined by Moshinsky[9]. Hence the RC of $SU(3)\times SU(3)$ in a basis classified by $\mathcal{U}(3)$ are

$$\mathcal{D}^{[h_1'h_2'][h_3'0]}_{\bar h_1\bar h_2\bar h_3\bar q_1\bar q_2\bar r_1;\,h_1h_2h_3q_1q_2r_1}(u,v)=$$

$$=\sum_{q'r'\bar q'\bar r'}\left\langle \begin{matrix} h_1' & h_2' & 0 \\ \bar q_1' & \bar q_2' & \\ & \bar r_1' & \end{matrix} ; \begin{matrix} h_3' & 0 & 0 \\ \bar q_3 & 0 & \\ & \bar r_3' & \end{matrix} \middle| \begin{matrix} \bar h_1 & \bar h_2 & \bar h_3 \\ \bar q_1 & \bar q_2 & \\ & \bar r_1 & \end{matrix} \right\rangle \mathcal{D}^{[h_1'h_2']}_{\bar q_1'\bar q_2'\bar r_1',\,q_1'q_2'r_1'}(u)$$

$$\cdot\,\mathcal{D}^{[h_3'0]}_{\bar q_3'0\bar r_3',\,q_3'0r_3'}(v)\left\langle \begin{matrix} h_1' & h_2' & 0 \\ q_1' & q_2' & \\ & r_1' & \end{matrix} ; \begin{matrix} h_3' & 0 & 0 \\ q_3' & 0 & \\ & r_3' & \end{matrix} \middle| \begin{matrix} h_1 & h_2 & h_3 \\ q_1 & q_2 & \\ & r_1 & \end{matrix} \right\rangle \tag{23}$$

where u, v stand for the 8 parameters of each $SU(3)$ group, respectively.

A formula similar to (23) is valid for the ME of the operator $\exp\,(i\alpha K_3^3)$ with respect to the states (6); in that case the product of two $\mathcal{D}$ on the right hand side of (23) is replaced by

$$\exp\,[i\alpha(h_1'+h_2'-h_3'-q_1'-q_2'+q_3')]\,\delta_{q_1'\bar q_1'}\,\delta_{q_2'\bar q_2'}\,\delta_{q_3'\bar q_3'}\,\delta_{r_1'\bar r_1'}\,\delta_{r_3'\bar r_3'}\,.$$

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RESUMEN

El grupo $U(3) \times U(3)$ recientemente ha adquirido gran importancia en la física de las partículas elementales. Las representaciones irreducibles relevantes de este grupo están caracterizadas por cuatro números enteros no negativos $[h_1' h_2'] [h_3' h_4']$, y sus estados básicos con significado físico se clasifican por medio de una cadena de grupos $\mathfrak{U}(3) \supset \mathfrak{U}(2) \supset \mathfrak{U}(1)$. Es sabido que este esquema de clasificación involucra un "problema de multiplicidad", pero cuando $h_4' = 0$ este problema no se presenta. En este trabajo determinamos los elementos de matriz de los generadores del grupo $U(3) \times U(3)$ con respecto a los estados básicos mencionados arriba, restringiéndonos a representaciones irreducibles con $h_4' = 0$. También damos una breve discusión de los coeficientes de representación de $U(3) \times U(3)$.

APPENDIX

We shall describe in this appendix the method by which the reduced ME given in (14) and (21) were determined.

First, in the notation of (9ab) we have*

$$\Delta_1^3 \mid h_1 h_2 h_3 \,;\, h_1 h_2 h_1 > \; = \frac{N(h_1 h_2 h_3)}{N(h_1 + 1,\, h_2 h_3)} \mid h_1 + 1,\, h_2 h_3 \,;\, h_1 + 1,\, h_2 h_1 + 1 > \,, \tag{A.1}$$

from here we deduce the ME

$$< h_1 + 1,\, h_2 h_3 \,;\; h_1 + 1,\, h_2 ,\, h_1 + 1 \mid \Delta_1^3 \mid h_1 h_2 h_3 \,;\, h_1 h_2 h_1 >$$

and through the use of (13) we obtain the value of the reduced ME given in (14a). Next, we have from (9a, b):

$$\Delta_2^{3+} \mid h_1 ,\, h_2 + 1, h_3 \,; h_1 ,\, h_2 + 1, h_1 > \; =$$

$$= \frac{N(h_1 ,\, h_2 + 1,\, h_3)}{N(h_1 h_2 h_3)} \, (h_2 - h_2' + 1) \mid h_1 h_2 h_3 \,;\, h_1 h_2 h_1 >$$

$$- h_3 N(h_1 ,\, h_2 + 1,\, h_3)(\Delta_1^1)^{h_1' - h_2 - 1}(\Delta_1^3)^{h_1 - h_1'}(\Delta_{12}^{12})^{h_2' - h_3}(\Delta_{12}^{13})^{h_2 - h_2' + 1}\Delta_{13}^{12}(\Delta_{123}^{123})^{h_3 - 1} \mid 0 > . \tag{A.2}$$

But from the explicit form of the lowering operators[9] we have

$$\mid h_1 ,\, h_2 + 1, h_3 - 1 \,;\, h_1 h_2 h_1 > \; = \frac{1}{\sqrt{h_2 - h_3 + 2}} \, C_3^2 \mid h_1 ,\, h_2 + 1, h_3 - 1 \,; h_1 ,\, h_2 + 1, h_1 > \,;$$

applying C_3^2 on the state on the right hand side as given by (9a) and using the identity

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*The remark in the footnote of page 283 applies to this appendix.

$\Delta^{12}_{12}\Delta^{13}_{13} \equiv \Delta^{12}_{13} + \Delta^{1}_{1}\Delta^{123}_{123}$, we find that the second term on the right hand side of (A.2) is equal to

$$- \frac{h_3 N(h_1, h_2+1, h_3)}{\sqrt{h_2 - h_3 + 2}\, N(h_1, h_2+1, h_3-1)} \,|\, h_1, h_2+1, h_3-1; h_1 h_2 h_1 >$$

$$\cdot \frac{h_3 (h_2 - h_2' + 1)\, N(h_1, h_2+1, h_3)}{(h_2 - h_3 + 2)\, N(h_1 h_2 h_3)} \,|\, h_1 h_2 h_3; h_1 h_2 h_3 > . \qquad (A.3)$$

From (A.2) and (A.3) we deduce the ME

$$< h_1, h_2+1, h_3; h_1, h_2+1, h_1 \,|\Delta^3_2|\, h_1 h_2 h_3; h_1 h_2 h_1 >$$

and again through eq. (13) we obtain the reduced ME given in (14b). Finally we have from (9a)

$$\Delta^{3+}_{3} \,|\, h_1 h_2, h_3+1; h_1 h_2 h_1 > = \frac{N(h_1 h_2, h_3+1)}{N(h_1 h_2 h_3)} (h_3+1) \,|\, h_1 h_2 h_3; h_1 h_2 h_1 > , \qquad (A.4)$$

this permits us to calculate the ME

$$< h_1 h_2, h_3+1; h_1 h_2 h_1 \,|\Delta^3_3|\, h_1 h_2 h_3; h_1 h_2 h_1 >$$

and through eq. (13) we obtain the reduced ME given in (14c).

As for the reduced ME of the tensor $T^{[210]}$, again their evaluation depends on the complete determination of some particular ME conveniently chosen. From formulas (17) we have

$$T^{[210]}_{111} = - \frac{1}{\sqrt{2}} T^{[111]}_{111} + \frac{\sqrt{6}}{2} K^3_3 ,$$

and it was shown in the text that the tensor $T^{[111]}_{111}$ is diagonal with respect to the states (6) with eigenvalue $\frac{1}{\sqrt{3}}(h'_1 + h'_2 - h'_3)$. Furthermore, $K^3_3 = C^3_3 - 2\Delta^3_3(\Delta^3_3)^+$, so the determination of the ME of $T^{[210]}_{111}$ is essentially equivalent to the calculation of the ME of $\Delta^3_3(\Delta^3_3)^+$, and this latter problem can be easily solved using the results of section 2. In this way we can determine explicitly the ME

$$< h_1 + 1, h_2 - 1, h_3; h_1, h_2 - 1, h_1 | T^{[210]}_{111} | h_1 h_2 h_3 ; h_1, h_2 - 1, h_1 > ,$$

$$< h_1 - 1, h_2 + 1, h_3 ; h_1 - 1, h_2, h_1 - 1 | T^{[210]}_{111} | h_1 h_2 h_3 ; h_1 - 1, h_2, h_1 - 1 > ,$$

$$< h_1 + 1, h_2, h_3 - 1 ; h_1 h_2 h_1 | T^{[210]}_{111} | h_1 h_2 h_3 ; h_1 h_2 h_1 > ,$$

$$< h_1 - 1, h_2, h_3 + 1 ; h_1 - 1, h_2, h_1 - 1 | T^{[210]}_{111} | h_1 h_2 h_3 ; h_1 - 1, h_2, h_1 - 1 > ,$$

$$< h_1, h_2 + 1, h_3 - 1 ; h_1 h_2 h_1 | T^{[210]}_{111} | h_1 h_2 h_3 ; h_1 h_2 h_1 > ,$$

$$< h_1, h_2 - 1, h_3 + 1 ; h_1, h_2 - 1, h_1 | T^{[210]}_{111} | h_1 h_2 h_3 ; h_1, h_2 - 1, h_1 > ,$$

from which, through the use of eq. (19) we deduce successively the reduced ME given in (21a, b, c, d, e, f).

TABLE I

Reduced Clebsch-Gordan Coefficients

$$\langle b_1 b_2 b_3 , q_1 q_2 ; 1 q'') \bar{b}_1 \bar{b}_2 \bar{b}_3 , \bar{q}_1 \bar{q}_2 \rangle .$$

$$\langle b_1 b_2 b_3 , q_1 q_2 ; 11) b_1 + 1, b_2 , b_3 , q_1 + 1, q_2 \rangle = \left[\frac{(q_1 - b_3 + 2)(b_1 - q_2 + 2)(q_1 - b_2 + 1)}{(b_1 - b_2 + 1)(b_1 - b_3 + 2)(q_1 - q_2 + 2)} \right]^{1/2}$$

$$\langle b_1 b_2 b_3 , q_1 q_2 ; 11) b_1 + 1, b_2 , b_3 , q_1 , q_2 + 1 \rangle = \left[\frac{(q_2 - b_3 + 1)(b_2 - q_2)(b_1 - q_1 + 1)}{(b_1 - b_2 + 1)(b_1 - b_3 + 2)(q_1 - q_2)} \right]^{1/2}$$

$$\langle b_1 b_2 b_3 , q_1 q_2 ; 11) b_1 , b_2 + 1, b_3 , q_1 + 1, q_2 \rangle = - \left[\frac{(b_2 - q_2 + 1)(q_1 - b_3 + 2)(b_1 - q_1)}{(b_1 - b_2 + 1)(b_2 - b_3 + 1)(q_1 - q_2 + 2)} \right]^{1/2}$$

$$\langle b_1 b_2 b_3 , q_1 q_2 ; 11) b_1 , b_2 + 1, b_3 , q_1 , q_2 + 1 \rangle = \left[\frac{(b_1 - q_2 + 1)(q_1 - b_2)(q_2 - b_3 + 1)}{(q_1 - q_2)(b_1 - b_2 + 1)(b_2 - b_3 + 1)} \right]^{1/2}$$

$$\langle b_1 b_2 b_3 , q_1 q_2 ; 11) b_1 , b_2 , b_3 + 1, q_1 + 1, q_2 \rangle = - \left[\frac{(q_1 - b_2 + 1)(b_1 - q_1)(q_2 - b_3)}{(q_1 - q_2 + 2)(b_1 - b_3 + 2)(b_2 - b_3 + 1)} \right]^{1/2}$$

$$\langle b_1 b_2 b_3 , q_1 q_2 ; 11) b_1 , b_2 , b_3 + 1, q_1 , q_2 + 1 \rangle = - \left[\frac{(q_1 - b_3 + 1)(b_1 - q_2 + 1)(b_2 - q_2)}{(q_1 - q_2)(b_1 - b_3 + 2)(b_2 - b_3 + 1)} \right]^{1/2}$$

$$\langle b_1 b_2 b_3 , q_1 q_2 ; 10) b_1 + 1, b_2 , b_3 , q_1 q_2 \rangle = \left[\frac{(b_1 - q_2 + 2)(b_1 - q_1 + 1)}{(b_1 - b_2 + 1)(b_1 - b_3 + 2)} \right]^{1/2}$$

$$\langle b_1 b_2 b_3 , q_1 q_2 ; 10) b_1 , b_2 + 1, b_3 , q_1 q_2 \rangle = \left[\frac{(q_1 - b_2)(b_2 - q_2 + 1)}{(b_1 - b_2 + 1)(b_2 - b_3 + 1)} \right]^{1/2}$$

$$\langle b_1 b_2 b_3 ; q_1 q_2 ; 10) b_1 , b_2 , b_3 + 1, q_1 q_2 \rangle = \left[\frac{(q_1 - b_3 + 1)(q_2 - b_3)}{(b_1 - b_3 + 2)(b_2 - b_3 + 1)} \right]^{1/2}$$