SUM RULES FOR THE MOSHINSKY BRACKETS*

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ABSTRACT: In this note we derive sum rules and symmetry relations for the generalized Moshinsky transformation coefficients. A number of relations valid only for the ordinary Moshinsky brackets are also derived.

INTRODUCTION

The harmonic-oscillator transformation brackets[1,2] have proved to be extremely useful in nuclear theory[3]. Lately, use has been made of them, as well as of their generalization introduced by Gal[4], to construct translationally invariant states of three and four particles[5,6,7,8] and to calculate binding energies[8] and electric form factors[6,9] of the triton and the α-particle. In the course of these investigations we have found some symmetry properties of the generalized

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Moshinsky brackets and sum rules for the usual transformation brackets which can be proved in a simple manner and, up to our knowledge, have not been reported before.

The generalized Moshinsky brackets[4],

$$\langle n_a l_a, n_b l_b, \lambda | n_1 l_1, n_2 l_2, \lambda \rangle_\beta \equiv \langle ab | 12 \rangle_\beta \qquad (1)$$

which arise when one considers particles of different masses, are used to express the two-particle harmonic-oscillator states, when given in terms of the coordinates $\mathbf{x}_1$, $\mathbf{x}_2$, in terms of similar states in the coordinates $\mathbf{x}_a$ and $\mathbf{x}_b$, related to the previous vectors by a rotation through an angle $\frac{1}{2}\beta$ in the following form

$$\begin{pmatrix} \mathbf{x}_a \\ \mathbf{x}_b \end{pmatrix} = \begin{pmatrix} \cos \frac{1}{2}\beta & -\sin \frac{1}{2}\beta \\ \sin \frac{1}{2}\beta & \cos \frac{1}{2}\beta \end{pmatrix} \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} . \qquad (2)$$

In (1) the pairs $n_a l_a$ are the single-particle harmonic-oscillator quantum numbers and λ is the total orbital angular momentum. The usual Moshinsky brackets, providing the transformation coefficients from center-of-well to center-of-mass and relative coordinates, correspond to the special value $\beta = \frac{1}{2}\pi$ (equal mass oscillators); we use for them the same notation as in (1), but suppressing the index β. The following closed expression has been obtained[4] for the brackets (1) in terms of those corresponding to $\beta = \frac{1}{2}\pi$, which have been tabulated[2]

$$\begin{aligned} \langle ab | 12 \rangle_\beta &= (-i)^{g_b} i^{g_2} \sum_{cd} \exp\left[\tfrac{1}{2} i\beta (g_d - g_c)\right] \langle cd | ab \rangle \langle cd | 12 \rangle \\ &\equiv (-i)^{g_b} i^{g_2} \sum_{cd} \exp\left[\tfrac{1}{2} i\beta (g_d - g_c)\right] \langle ab | cd \rangle \langle 12 | cd \rangle \end{aligned} \qquad (3)$$

where

$$g_i \equiv 2n_i + l_i .$$

These coefficients have the following properties. First, as elements of a unitary transformation, they fulfill the following orthogonality relation.

$$\sum_{ab} \langle ab|12\rangle_{\beta} \langle ab|1'2'\rangle_{\beta}^{*} = \delta_{1'1}\,\delta_{2'2} \tag{4}$$

where

$$\delta_{i'i} \equiv \delta_{n_i' n_i}\,\delta_{l_i' l_i} ,$$

which can also be derived from their explicit form (3). Second, it can be proved[11] that the generalized coefficients are real, so that (4) can be written in the following way

$$\sum_{ab} \langle ab|12\rangle_{\beta} \langle ab|1'2'\rangle_{\beta} = \delta_{1'1}\,\delta_{2'2} . \tag{5a}$$

Similarly, one gets

$$\sum_{12} \langle ab|12\rangle_{\beta} \langle a'b'|12\rangle_{\beta} = \delta_{a'a}\,\delta_{b'b} . \tag{5b}$$

From the expression (3) we have derived in a direct way the following symmetry relations which turn out to be a generalization of the corresponding relations valid for the Moshinsky brackets[10]

$$\langle ab|12\rangle_{\beta} = (-1)^{l_b-\lambda} \langle ab|21\rangle_{\pi-\beta} \tag{6a}$$

$$= (-1)^{l_1-\lambda} \langle ba|12\rangle_{\pi-\beta} \tag{6b}$$

$$= (-1)^{l_1+l_a} \langle ba|21\rangle_{\beta} \tag{6c}$$

$$= (-1)^{l_2+l_b} \langle 12|ab\rangle_{\beta} \tag{6d}$$

$$= (-1)^{l_2+l_b} \langle ab|12\rangle_{-\beta} \tag{6e}$$

Notice that in (6d) the values but not the physical meaning of the bra and ket quantum numbers have been interchanged and that in both (6c) and (6d) the angle β remains the same.

The symmetry relations (6) can be proved straightforwardly using the previously derived relations[10] for the special case $\beta = \frac{1}{2}\pi$ and the "energy condition" for the brackets, which implies that

$$g_1 + g_2 = g_a + g_b = g_c + g_d \,. \tag{7}$$

One can now obtain from the orthonormality and symmetry relations a large number of sum rules, some of them valid for β arbitrary, others valid only for the special case $\beta = \frac{1}{2}\pi$.

Using Eqs. (3) and (5) one can prove directly that

$$\sum_{11'\,22'\ldots kk'} \langle 11'|22'\rangle_{\beta_1} \langle 22'|33'\rangle_{\beta_2} \ldots \langle kk'|11'\rangle_{\beta_k} =$$

$$= \sum_{12} \cos\left[\tfrac{1}{2}(g_1 - g_2)(\beta_1 + \beta_2 + \ldots + \beta_k)\right] \tag{8}$$

where the quantum numbers $1, 1', 2, 2', \ldots, k'$ vary over all the combinations compatible with (7) and the angular-momentum triangular conditions. The case $k = 1$ is of some interest:

$$\sum_{12} \langle 12|12\rangle_\beta = \sum_{12} e^{\frac{1}{2}i\beta(g_1 - g_2)} = \sum_{12} \cos\left[\tfrac{1}{2}(g_1 - g_2)\beta\right] . \tag{9}$$

Further sum rules may be derived from Eq. (3). For $\beta = \frac{1}{2}\pi$ one obtains a sum rule for the Moshinsky brackets which may be written as

$$\sum_{56} \langle 12|56\rangle\langle 34|56\rangle\, e^{\frac{1}{2}i\beta(g_5 - g_6)} = i^{g_2}(-i)^{g_4} \langle 12|34\rangle \tag{10}$$

or, equivalently,

$$\sum_{56} \langle 12|56\rangle\langle 34|56\rangle \cos\left[\tfrac{1}{4}\pi(g_5 - g_6)\right] =$$

$$= (-1)^{n_2+n_4+\frac{1}{2}(l_4-l_2)} \quad \text{if } l_2+l_4 \text{ is even,} \tag{11}$$

$$\sum_{56} \langle 12|56\rangle\langle 34|56\rangle \sin\left[\tfrac{1}{4}\pi(g_5 - g_6)\right]$$

$$= (-1)^{n_2+n_4+\frac{1}{2}(l_4-l_2+1)} \quad \text{if } l_2+l_4 \text{ is odd.} \tag{12}$$

Another sum rule is obtained from (3) by noting that if $l_2 + l_4$ is even, the imaginary part of the sum must be zero, while if $l_2 + l_4$ is odd, the real part is zero; this is so because the generalized transformation brackets are easily shown to be real.[11] Hence

$$\sum_{56} \sin\left[\tfrac{1}{2}\beta(g_5 - g_6)\right]\langle 12|56\rangle\langle 34|56\rangle = 0 \qquad l_2+l_4 \text{ even,} \tag{13}$$

$$\sum_{56} \cos\left[\tfrac{1}{2}\beta(g_5 - g_6)\right]\langle 12|56\rangle\langle 34|56\rangle = 0 \qquad l_2+l_4 \text{ odd.} \tag{14}$$

These two relations may be differentiated with respect to β, say p times. Putting

$$\beta = \begin{cases} 0 & \text{if} \quad l_2+l_4+p \quad \text{is even} \\ \pi & \text{if} \quad l_2+l_4+p \quad \text{is odd} \end{cases}$$

one obtains the result

$$\sum_{56} (g_5 - g_6)^p \langle 12|56\rangle\langle 34|56\rangle = 0, \quad p = 1, 2, \ldots \,. \tag{15}$$

Using (5a) and (6) we can now derive some other sum rules valid only for the special case $\beta = \frac{1}{2}\pi$:

$$\sum_{\substack{ab \\ l_a \text{ even}}} \langle ab|12\rangle\langle ab|1'2'\rangle =$$

$$= \frac{1}{2}\left[\delta_{1'1}\delta_{2'2} + (-1)^{l_1+l_2+\lambda}\delta_{1'2}\delta_{2'1}\right] \tag{16}$$

and

$$\sum_{\substack{ab \\ l_a \text{ odd}}} \langle ab|12\rangle\langle ab|1'2'\rangle =$$

$$= \frac{1}{2}\left[\delta_{1'1}\delta_{2'2} - (-1)^{l_1+l_2+\lambda}\delta_{1'2}\delta_{2'1}\right]. \tag{17}$$

The way they are proved is the following: one splits the sum over l_a in (5a) into two parts, one containing the even and the other the odd values of l_a. A similar expression is obtained using (5a) again but now interchanging $n_1' l_1'$ with $n_2' l_2'$. One then uses the symmetry relation (6a) for $\beta = \frac{1}{2}\pi$ and adds and subtracts these two expressions after multiplying the second one by a convenient phase factor.

The sum rules (16) and (17) are important in checking the normalization of the translationally-invariant three and four-body harmonic-oscillator states mentioned above.

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REFERENCES

1. M. Moshinsky, Nucl. Phys. 13 (1959) 104.
2. T. A. Brody and M. Moshinsky, "Tables of Transformation Brackets", Gordon and Breach, New York, 1967.
3. See, for example, K. T. R. Davies, S. J. Krieger and M. Baranger, Nucl. Phys. 84 (1966) 545; T. T. S. Kuo and G. E. Brown, Nucl. Phys. 35 (1966) 87.

4. A. Gal, Ann. of Phys. (N.Y.) 49 (1968) 341.
5. V.C. Aguilera-Navarro, M. Moshinsky and W.W. Yeh, Rev. Mex. Fís. 17 (1968) 241.
6. V.C. Aguilera-Navarro, M. Moshinsky and W.W. Yeh, Ann. Phys. (N.Y.) 51 (1969) 312.
7. V.C. Aguilera-Navarro, M. Moshinsky and P. Kramer, Ann. Phys. (N.Y.) 54 (1969) 379.
8. V.C. Aguilera-Navarro, Ph.D. Thesis, Univ. México (1969).
9. G. Cocho and J. Flores, Rev. Mex. Fís. 19 (1970) 67.
10. T.A. Brody and M. Moshinsky, Rev. Mex. Fís. 9 (1960) 181.
11. M. Moshinsky, "The Harmonic Oscillator in Modern Physics", Gordon and Breach, London 1969.

RESUMEN

En esta nota se obtienen reglas de suma y relaciones de simetría para los paréntesis de transformación de Moshinsky generalizados. También se obtienen algunas relaciones válidas sólo para los paréntesis de transformación de Moshinsky ordinarios.