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TRANSFORMATION BRACKETS FOR FOUR-NUCLEON PROBLEMS*

V. C. Aguilera-Navarro

Instituto de Física Teórica, São Paulo 3, Brazil

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ABSTRACT: We give the explicit expression for the transformation brackets between four-particle translationally invariant states in symmetric and Jacobi relative coordinates.

1. INTRODUCTION

Translationally-invariant four-nucleon states with definite angular momentum and arbitrary symmetry in configurationspace were derived by using general projection techniques[1]. They are given as linear combinations of harmonic-oscillator states of the type

$$|n_1 l_1, n_2 l_2(\Lambda); n_3 l_3, \lambda\mu) \equiv$$

$$[[<y_1|n_1 l_1><y_2|n_2 l_2>]^{\Lambda} <y_3|n_3 l_3>]^{\lambda}_{\mu} \qquad (1)$$

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where the pairs of brackets [,] stand for angular momentum coupling.

It was proved to be convenient for the projection operations to define the relative normalized coordinates

$$
\begin{aligned}
y_1 &= \frac{1}{2}(x_1 - x_2 - x_3 + x_4) \\
y_2 &= \frac{1}{2}(x_2 - x_1 - x_3 + x_4) \qquad (2) \\
y_3 &= \frac{1}{2}(x_3 - x_1 - x_2 + x_4)
\end{aligned}
$$

since they carry the fundamental representation of S(3)[1] and D(2)[2] is diagonal in this basis.

The use of this system of coordinates is indicated by the round ket in left-hand side of (1).

Once the states are available they can be used, together with their corresponding spin-isospin part, for a systematic analysis of four-nucleon systems.

On the other hand, for the calculations of both the matrix elements of the Hamiltonianand of form factors it is much more convenient[3] to use the relative Jacobi coordinates defined by

$$
\begin{aligned}
x_a &= \sqrt{\frac{1}{2}}(x_1 - x_2) \\
x_b &= \sqrt{\frac{1}{6}}(x_1 + x_2 - 2x_3) \qquad (3) \\
x_c &= \sqrt{\frac{1}{12}}(x_1 + x_2 + x_3 - 3x_4)
\end{aligned}
$$

Thus we construct the states

$$|n_a l_a, n_b l_b (\Lambda'); n_c l_c, \lambda\mu> \equiv$$

$$[[<x_a|n_a l_a><x_b|n_b l_b>]^{\Lambda'} <x_c|n_c l_c>]^{\lambda}_{\mu} \quad (4)$$

(where the angular ket in the left-hand side of (4) indicates the use of Jacobi coordinates), and look for an explicity expression for the coefficients

$$<n_a l_a, n_b l_b (\Lambda'); n_c l_c, \lambda | n_1 l_1, n_2 l_2 (\Lambda); n_3 l_3, \lambda) \quad (5)$$

connecting the states (1) and (4) through the definition

$$|n_1 l_1, n_2 l_2 (\Lambda); n_3 l_3, \lambda\mu) =$$

$$\Sigma |n_a l_a, n_b l_b (\Lambda'); n_c l_c, \lambda\mu><n_a l_a, n_b l_b (\Lambda'); n_c l_c, \lambda | n_1 l_1, n_2 l_2 (\Lambda); n_3 l_3, \lambda) \quad (6)$$

The sum is extended over all the quantum numbers in the angular ket with exception of λ and μ which are fixed.

2. PRELIMINARY REMARKS AND PROPERTIES

Let us introduce the center-of-mass coordinate

$$x_d = y_4 = \frac{1}{2}(x_1 + x_2 + x_3 + x_4) \quad (7)$$

and notice that

$$\lambda = l_1 + l_2 + l_3 + l_4 = l_a + l_b + l_c + l_d \quad (8)$$

where $l_i = y_i \times p_{yi}$, $i = 1, 2, 3, 4$ and $l_\alpha = x_\alpha \times p_\alpha$, $\alpha = a, b, c, d$ ($\hbar = 1$). This indicates that λ is the same in both the bra and ket parts of the coefficient (5). The relation (8) also indicates that the transformation brackets (5) is independent of the angular-momentum projection μ since we can obtain in both sides of (6) states with μ increased by unit by means of the corresponding raising operator. This justifies the absence of μ in (5).

The relative coordinates (2) and (3) are related by the orthogonal transformation

$$y = Mx \tag{9}$$

where y and x are supervectors whose components are (y_1, y_2, y_3) and (x_a, x_b, x_c), respectively, and M is given by

$$M = \begin{bmatrix} \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{6}} & -\sqrt{\frac{1}{3}} \\ -\sqrt{\frac{1}{2}} & \sqrt{\frac{1}{6}} & -\sqrt{\frac{1}{3}} \\ 0 & -\sqrt{\frac{2}{3}} & -\sqrt{\frac{1}{3}} \end{bmatrix} \equiv M_1 M_2 M_3$$

$$= \begin{bmatrix} \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} & 0 \\ -\sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} \\ 0 & -\sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{10}$$

This decomposition of M will be useful later. The sums in (6) are restricted by the energy condition

$$2n_a + l_a + 2n_b + l_b + 2n_c + l_c = 2n_1 + l_1 + 2n_2 + l_2 + 2n_3 + l_3 \tag{11}$$

which follows from the invariance of the harmonic-oscillator Hamiltonian under the orthogonal transformation (9). Notice that spurious states are eliminated by putting zero excitation on the center-of-mass coordinate (7);

i.e., $n_4 = l_4 = n_d = l_d = 0$. On the other hand the values of Λ' are restricted by the triangular condition

$$|l_a - l_b| \leqslant \Lambda' \leqslant l_a + l_b \ . \tag{12}$$

The transformation brackets vanishes whenever (11) or (12) is not satisfied.

3. THE EXPLICITY EXPRESSION FOR THE TRANSFORMATION BRACKETS

Now we are ready to obtain the explicity expression for the coefficients (5). All we have to do is to analyse the effect on (1) of each matrix factor in the decomposition (10) of M and then to compare the resulting expression with the definition (6). The effect of M on (1) is, by convention, the same as the effect of M^{-1} on the coordinates. So we start with M_1. This transformation induces a rotation by $\pi/4$ in the plane $(y_1 - y_2)$ and then we obtain a linear combination in terms of Moshinsky-brackets[4, 5], i.e.,

$$M_1 : |n_1 l_1, n_2 l_2(\Lambda); n_3 l_3, \lambda\mu) \rightarrow$$

$$\sum_{\substack{n \ l \\ n_a \ l_a}} |n_a l_a, nl(\Lambda); n_3 l_3, \lambda\mu >< n_a l_a, nl, \Lambda | n_1 l_1, n_2 l_2, \Lambda > \ . \tag{13}$$

The transformation M_2 induces a rotation in the plane defined by the new second and old third coordinates by an angle such that

$$\cos \tfrac{1}{2}\beta = \sqrt{\frac{1}{3}}\,, \qquad \sin \tfrac{1}{2}\beta = \sqrt{\frac{2}{3}} \ . \tag{14}$$

To get the effect of M_2 is then convenient to do some Racah algebra to recouple the states in (13) as follows[6]

$$|n_a l_a, nl(\Lambda); n_3 l_3, \lambda\mu> = \sum_{\Lambda''} |n_a l_a; nl, n_3 l_3 (\Lambda''), \lambda\mu> \times$$

$$[(2\Lambda+1)(2\Lambda''+1)]^{\frac{1}{2}} \; W(l_a l \lambda l_3; \Lambda\Lambda'') \quad . \tag{15}$$

Now the effect of M_2 is clear and given by

$$M_2: \; |n_a l_a; nl, n_3 l_3 (\Lambda''), \lambda\mu> \; \rightarrow$$

$$\sum_{\substack{n_b l_b \\ n_c l_c}} |n_a l_a; n_b l_b, n_c l_c (\Lambda''), \lambda\mu><n_b l_b, n_c l_c, \Lambda'' | nl, n_3 l_3, \Lambda''>_\beta \tag{16}$$

where β is defined by (14) and the coefficients in the expansion are generalized Moshinsky brackets[7] which can be put in terms of the usual and tabulated[5] Moshinsky brackets.

Finally the effect of M_3 is simply to introduce the phase factor $(-)^{l_c}$.

From the relations (13), (15) and (16) (where we have partially relaxed the convention previously adopted for angular and round kets, in order to avoid cumbersome notation) we obtain the result

$$|n_1 l_1, n_2 l_2 (\Lambda); n_3 l_3, \lambda\mu) = \sum_{\Lambda''} \sum_{\substack{n \; l \\ n_a l_a}} \sum_{\substack{n_b l_b \\ n_c l_c}} (-)^{l_c} [(2\Lambda+1)(2\Lambda''+1)]^{\frac{1}{2}} \times$$

$$|n_a l_a; n_b l_b, n_c l_c (\Lambda''), \lambda\mu> W(l_a l \lambda l_3; \Lambda\Lambda'') <n_b l_b, n_c l_c, \Lambda'' | nl, n_3 l_3, \Lambda''>_\beta \times$$
$$<n_a l_a, nl, \Lambda | n_1 l_1, n_2 l_2 \Lambda>$$

$$= \sum_{\Lambda', \Lambda''} \sum_{\substack{n \; l \\ n_a l_a}} \sum_{\substack{n_b l_b \\ n_c l_c}} \; n_a l_a, n_b l_b (\Lambda'); n_c l_c, \lambda\mu> (-)^{l_c} (2\Lambda''+1) \times$$

$$[(2\Lambda+1)(2\Lambda'+1)]^{\frac{1}{2}} W(l_a l \lambda l_3; \Lambda\Lambda'') W(l_a l_b \lambda l_c; \Lambda'\Lambda) \times$$

$$<n_a l_a, nl, \Lambda | n_1 l_1, n_2 l_2, \Lambda><n_b l_b, n_c l_c, \Lambda'' | nl, n_3 l_3, \Lambda''>_\beta \tag{17}$$

where the last expression follows from an angular-momentum recoupling.

Comparing (17) and the definition (6) we get the following final expression for the transformation brackets

$$< n_a l_a , n_b l_b (\Lambda') ; n_c l_c , \lambda | n_1 l_1 , n_2 l_2 (\Lambda) ; n_3 l_3 , \lambda) =$$

$$(-)^{l_c} [(2\Lambda + 1)(2\Lambda' + 1)]^{\frac{1}{2}} \sum_{nl\Lambda''} (2\Lambda'' + 1) W(l_a l_b \lambda l_c ; \Lambda' \Lambda'') \times$$

$$W(l_a l \lambda l_3 ; \Lambda \Lambda'') < n_a l_a , nl , \Lambda | n_1 l_1 , n_2 l_2 , \Lambda >< n_b l_b , n_c l_c , \Lambda'' | nl , n_3 l_3 , \Lambda'' >_\beta \quad (18)$$

In this expression the sum over Λ'' is restricted in the usual way while n and l assume values such that

$$2n + l = 2n_1 + l_1 + 2n_2 + l_2 - 2n_a - l_a \quad (19)$$

and

$$| \Lambda - l_a | \leqslant l \leqslant \Lambda + l_a \quad (20)$$

where the condition (19) follows from the energy condition implicit in the Moshinsky brackets[4, 5] and (20) from the rule for addition of angular momenta.

On the other hand we could have written (6) under the form

$$| n_1 l_1 ; n_2 l_2 , n_3 l_3 (\Lambda) , \lambda\mu) = \Sigma | n_a l_a ; n_b l_b , n_c l_c (\Lambda') , \lambda\mu > \times$$

$$< n_a l_a , n_b l_b , n_c l_c (\Lambda') , \lambda | n_1 l_1 ; n_2 l_2 , n_3 l_3 (\Lambda) , \lambda) \quad (21)$$

where we have now a different order in the coupling of the three angular momenta.

It is a matter of playing a little with Racah algebra to see that the new coefficients defined in (21) are linear combination of (18). In gact we have

$$< n_a l_a ; n_b l_b , n_c l_c (\Lambda') , \lambda | n_1 l_1 ; n_2 l_2 , n_3 l_3 (\Lambda) , \lambda) =$$

$$\sum_{\bar{\Lambda} \bar{\Lambda}'} [(2\Lambda + 1)(2\Lambda' + 1)(2\bar{\Lambda} + 1)(2\bar{\Lambda}' + 1)]^{\frac{1}{2}} W(l_a l_b \lambda l_c ; \bar{\Lambda}' \Lambda') \times$$

$$W(l_1 l_2 \lambda l_3 ; \bar{\Lambda}\Lambda) < n_a l_a , n_b l_b (\bar{\Lambda}') ; n_c l_c , \lambda | n_1 l_1 , n_2 l_2 (\bar{\Lambda}) ; n_3 l_3 , \lambda) \quad (22)$$

We can obtain an explicity form for these coefficients by means of the orthogonality relation for the Racah coefficients. We insert (18) into (22) and sum over $\bar{\Lambda}'$ and then over Λ''. In the resulting expression we replace $\bar{\Lambda}$ by Λ'' obtaining

$$< n_a l_a ; n_b l_b , n_c l_c (\Lambda') , \lambda | n_1 l_1 ; n_2 l_2 , n_3 l_3 (\Lambda) , \lambda) =$$

$$(-)^{l_c} [(2\Lambda + 1)(2\Lambda' + 1)]^{\frac{1}{2}} \sum_{n l \Lambda''} (2\Lambda'' + 1) W(l_1 l_2 \lambda l_3 ; \Lambda''\Lambda) \times$$

$$W(l_a l \lambda l_3 ; \Lambda'' \Lambda') < n_a l_a , nl , \Lambda'' | n_1 l_1 , n_2 l_2 , \Lambda'' >< n_b l_b , n_c l_c , \Lambda' | nl , n_3 l_3 , \Lambda' >_{\beta}$$

(23)

It is easy to see from (22) that the coefficients (18) and (23) coincide in the important particular case $\lambda = 0$. They are given by

$$< n_a l_a , n_b l_b , n_c l_c | n_1 l_1 , n_2 l_2 , n_3 l_3) \equiv$$

$$< n_a l_a , n_b l_b (l_c) ; n_c l_c , 0 | n_1 l_1 , n_2 l_2 (l_3) ; n_3 l_3 , 0) =$$

$$(-)^{l_c} \sum_{nl} < n_a l_a , nl , l_3 | n_1 l_1 , n_2 l_2 , l_3 >< n_b l_b , n_c l_c , l_a | nl , n_3 l_3 , l_a >_{\beta}$$

(24)

where the absence of the semi-colon indicates that $\lambda = 0$ and the order of coupling the angular momenta is irrelevant.

4. APPLICATION

As a simple application we shall discuss briefly the determination of the ground state of the alpha particle for a given Hamiltonian and a fixed number of quanta N in the approximation.

We shall take $\lambda = 0$ and completely symmetric states in configuration space. The wave function will be given by[3]

$$\phi \equiv \Sigma\, a(n_1 l_1, n_2 l_2, n_3 l_3) \left| n_1 l_1, n_2 l_2, n_3 l_3 \right)_S \tag{25}$$

where the sum is extended over all n_i, l_i $(i = 1, 2, 3)$ which satisfy the condition

$$2n_1 + l_1 + 2n_2 + l_2 + 2n_3 + l_3 \leqslant N \; . \tag{26}$$

The index S in (25) stands for symmetrization and the coefficients a can be determined, for instance, by diagonalization of the Hamiltonian which, for simplicity, will be assumed to be given in the form

$$\mathcal{H} = \mathcal{H}_0 + \sum_{ij} v_{ij} \tag{27}$$

where $\mathcal{H}_0$ is an oscillator term and v_{ij} depends only on $|\mathbf{x}_i - \mathbf{x}_j|^2$.

As $\mathcal{H}_0$ is diagonal in the representation $|n_1 l_1, n_2 l_2, n_3 l_3)$ we are left with the calculation of

$${}_S(n_1' l_1', n_2' l_2', n_3' l_3' \,|\, \Sigma v_{ij} \,|\, n_1 l_1, n_2 l_2, n_3 l_3)_S$$

which is equal to

$${}_S(n_1' l_1', n_2' l_2', n_3' l_3' \,|\, v_{12} \,|\, n_1 l_1, n_2 l_2, n_3 l_3)_S$$

as the states $|n_1 l_1, n_2 l_2, n_3 l_3)_S$ are symmetric.

Then we have to calculate in general 36 matrix elements of the type (notice the absence of the index S)

$$(n_i' l_i', n_j' l_j', n_k' l_k' \,|\, f(r^2) \,|\, n_p l_p, n_q l_q, n_s l_s), \qquad 1 \leqslant i, j, k \leqslant 3$$
$$1 \leqslant p, q, s \leqslant 3 \tag{28}$$

where

$$r^2 \equiv \frac{1}{2} |\mathbf{x}_1 - \mathbf{x}_2|^2 = |\mathbf{x}_a|^2 . \tag{29}$$

Now taking the transformation brackets (24) to express the states in terms of states in the Jacobi coordinates (3) it is easy to see that

$$(n_i' l_i', n_j' l_j', n_k' l_k' \,|\, f \,|\, n_p l_p, n_q l_q, n_s l_s) =$$

$$\delta(n_k', n_s)\, \delta(l_k', l_s) \sum_{n_a' n_a l_a} \langle n_a' l_a \| f \| n_a l_a \rangle \times$$

$$\sum_{nl} \langle n_a' l_a, nl, l_s \,|\, n_i' l_i', n_j' l_j', l_s \rangle \langle n_a l_a, nl, l_s \,|\, n_p l_p, n_q l_q, l_s \rangle \tag{30}$$

where use was made of the orthogonality property of the generalized Moshinsky brackets[7].

We see then that the matrix elements are given in terms of Moshinsky brackets and a reduced matrix element that can be expressed in terms of Talmi integrals[8].

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RESUMEN

Damos la expresión explícita para los paréntesis de transformación entre estados translacionalmente invariantes en coordenadas relativas simétricas y las de Jacobi.