Integrals of the motion and Green functions for time-dependent mass harmonic oscillators

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The application of the integrals of the motion of a quantum system in deriving Green function or propagator is established. The Green function is shown to be the eigenfunction of the integrals of the motion which described initial points of the system trajectory in the phase space. The explicit expressions for the Green functions of the damped harmonic oscillator, the harmonic oscillator with strongly pulsating mass, and the harmonic oscillator with mass growing with time are obtained in co-ordinate representations. The connection between the integrals of the motion method and other method such as Feynman path integral and Schwinger method are also discussed.

Keywords: Integrals of the motion; Green function; Time-dependent mass harmonic oscillators

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1. Introduction

In non-relativistic quantum mechanics, the propagator is represented as the transition probability amplitude for a particle to motion from initial space-time configuration to final space-time configuration. The Feynman path integral [1] and the Schwinger action principle [2] are the well-known methods in calculating the propagator. The aim of this paper is to present the connection between the integrals of the motion of a quantum system and its Green function or propagator.

As reveal by V.V. Dodonov *et al*. [3] that the Green function is the eigenfunction of the integrals of the motion describing initial points of the system trajectory in the phase space. D.B. Lemeshevskiy and V.I. Man'ko [4] constructed the Green functions for the driven harmonic oscillator with the aid of integrals of the motion. In the present paper we want to calculate the Green functions or propagators for the damped harmonic oscillator [5-7], the harmonic oscillator with strongly pulsating mass, [8] and the harmonic oscillator with mass growing with time [9] by the method developed by V.V. Dodonov *et al*. [3]

This paper is organized as follows. In Sec. 2, the Green function for the damped harmonic oscillator is derived. In Section 3, the calculation of the Green function for the harmonic oscillator with strongly pulsating mass is presented. The Green function for the harmonic oscillator with mass growing with time is evaluated in Sec. 4. Finally, the conclusion is given in Sec. 5.

2. The Green function for a damped harmonic oscillator

The Hamiltonian operator for a damped harmonic oscillator is described by [5-7]

$$\hat{H}(t) = e^{-rt}\frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2 e^{rt}\hat{q}^2, \tag{1}$$

where r is the damping constant coefficient.

The aim of this section is to drive the Green function $G(x,x',t)$ of the Schrodinger equation by the method of integrals of motion [3-4]. The classical correspondence of the Hamiltonian operator in Eq. (1) is

$$H(q,p,t) = e^{-rt}\frac{p^2}{2m} + \frac{1}{2}m\omega^2 e^{rt}q^2, \tag{2}$$

The Hamilton equation of motion for position and momentum are [10]

$$\dot{q} = \frac{p}{m}e^{-rt}, \qquad \dot{p} = --m\omega^2 e^{rt}q. \tag{3}$$

The classical paths in the phase space under the initial conditions $q(0) = q_0$ and $p(0) = p_0$ are given by

$$q(t) = q_0\left(e^{-rt}\cos\Omega t + \frac{re^{-rt/2}}{2\Omega}\right) + p_0\left(\frac{re^{-rt/2}}{m\Omega}\sin\Omega t\right), \tag{4}$$

$$p(t) = p_0\left(e^{rt/2}\cos\Omega t - \frac{re^{rt/2}}{2\Omega}\sin\Omega t\right) - q_0\left(\frac{m\omega^2}{\Omega}re^{-rt/2}\sin\Omega t\right), \tag{5}$$

where $\Omega^2 = \omega^2 - r^2/4$. Now we consider the system of Eqs. (4) and (5) as an algebraic system for unknown initial position q_0 and momentum p_0, respectively. The variables

q, p, and t are taken as the parameters. The solution of this system are given as

$$q_0(q,p,t) = q\left(e^{rt/2}\cos\Omega t - \frac{r}{2\Omega}e^{rt/2}\sin\Omega t\right) - p\left(\frac{e^{-rt/2}}{m\Omega}\sin\Omega t\right), \tag{6}$$

$$p_0(q,p,t) = q\left(\frac{m\omega^2}{\Omega}e^{rt/2}\sin\Omega t\right) + p\left(e^{-rt/2}\cos\Omega t + \frac{re^{-rt/2}}{2\Omega}\sin\Omega t\right). \tag{7}$$

We define operators acting in the Hilbert space as follows

$$\hat{q}_0(\hat{q},\hat{p},t) = \hat{q}\left(e^{rt/2}\cos\Omega t - \frac{r}{2\Omega}e^{rt/2}\sin\Omega t\right) - \hat{p}\left(\frac{e^{-rt/2}}{m\Omega}\sin\Omega t\right), \tag{8}$$

$$\hat{p}_0(\hat{q},\hat{p},t) = \hat{q}\left(\frac{m\omega^2}{\Omega}e^{rt/2}\sin\Omega t\right) + \hat{p}\left(e^{-rt/2}\cos\Omega t + \frac{re^{-rt/2}}{2\Omega}\sin\Omega t\right). \tag{9}$$

Calculating the total derivative of the operator $\hat{q}_0(\hat{q},\hat{p},t)$ with respect to time t, we obtain

$$\frac{d\hat{q}_0}{dt} = \frac{\partial\hat{q}_0}{\partial t} + \frac{i}{\hbar}[\hat{H},\hat{q}_0]. \tag{10}$$

Similarly, the total time-derivative of the operator $\hat{p}_0(\hat{q},\hat{p},t)$ is

$$\frac{d\hat{p}_0}{dt} = \frac{\partial\hat{p}_0}{\partial t} + \frac{i}{\hbar}[\hat{H},\hat{p}_0]. \tag{11}$$

Thus, operators in Eqs. (8) and (9) are integrals of the motion and correspond to the initial position and momentum. Then these operators must satisfy equations for the Green function $G(x,x',t)$, [3-4]

$$\hat{q}_0(x)G(x,x',t) = \hat{q}(x')G(x,x',t), \tag{12}$$

$$\hat{p}_0(x)G(x,x',t) = -\hat{p}(x')G(x,x',t), \tag{13}$$

where the operators on the left-hand sides of the equations act on variable x, and on the right- hand sides, on x'. Now we write Eqs. (12) and (13) explicitly,

$$\left(x\left(e^{rt/2}\cos\Omega t - \frac{r}{2\Omega}e^{rt/2}\sin\Omega t\right) + \frac{i\hbar}{m\Omega}e^{-rt/2}\sin\Omega t\frac{\partial}{\partial x}\right)G(x,x',t) = x'G(x,x',t), \tag{14}$$

$$\left(x\left(\frac{m\omega^2}{\Omega}e^{rt/2}\sin\Omega t\right) - i\hbar\left(e^{-rt/2}\cos\Omega t + \frac{re^{-rt/2}}{2\Omega}\sin\Omega t\right)\frac{\partial}{\partial x}\right)G(x,x',t) = i\hbar\frac{\partial G(x,x',t)}{\partial x'}. \tag{15}$$

By modifying Eqs. (14) and (15), the system of equations for deriving the Green function $G(x,x',t)$ are

$$\frac{\partial G(x,x',t)}{\partial x} = -\frac{im\Omega}{\hbar}\left(\frac{e^{rt/2}}{\sin\Omega t}x' - \left(e^{rt}\cot\Omega t - \frac{re^{rt}}{2\Omega}\right)x\right)G(x,x',t), \tag{16}$$

$$\frac{\partial G(x,x',t)}{\partial x'} = -\frac{i}{\hbar}\left(\frac{m\Omega e^{rt/2}}{\sin\Omega t}x - \left(m\Omega\cot\Omega t + \frac{mr}{2}\right)x'\right)G(x,x',t). \tag{17}$$

Now one can integrate Eq. (16) with respect to the variable x to obtain

$$G(x,x',t) = C(x',t)\exp\left(\frac{i}{\hbar}\left\{\left(\frac{m\Omega}{2}e^{rt}\cot\Omega t - \frac{mr}{4}e^{rt}\right)x^2 - \frac{m\Omega}{\sin\Omega t}e^{rt/2}xx'\right\}\right), \tag{18}$$

where $C(x',t)$ is the function of x' and t.

Substituting Eq. (18) into Eq. (17), we obtain the differential equation for $C(x',t)$ as

$$\frac{\partial C(x',t)}{\partial x'} = \frac{i}{\hbar}\left(m\Omega\cot\Omega t + \frac{mr}{2}\right)x'C(x',t). \tag{19}$$

Solving Eq. (19), the function $C(x',t)$ can be expressed as

$$C(x',t) = C(t)\exp\left(\frac{i}{\hbar}\left(\frac{m\Omega}{2}\cot\Omega t + \frac{mr}{4}\right)x'^2\right), \tag{20}$$

where $C(t)$ is the pure function of time.

So, the Green function in Eq. (18) can be written as

$$G(x,x',t) = C(t)\exp\left(\frac{i}{\hbar}\left\{\left(\frac{m\Omega}{2}e^{rt}\cot\Omega t - \frac{mr}{4}e^{rt}\right)x^2 + \left(\frac{m\Omega}{2}\cot\Omega t + \frac{mr}{4}\right)x'^2 - \frac{m\Omega e^{rt/2}}{\sin\Omega t}xx'\right\}\right). \tag{21}$$

To find $C(t)$, we must substitute the Green function of Eq. (21) into the Schrodinger equation

$$i\hbar\frac{\partial G(x,x',t)}{\partial t} = -\frac{\hbar^2}{2m}e^{-rt}\frac{\partial^2 G(x,x',t)}{\partial x^2} + \frac{1}{2}m\omega^2 e^{rt}x^2 G(x,x',t). \tag{22}$$

After some algebra, we obtain an equation that does not contain the variables x and x',

$$\frac{dC(t)}{dt} = C(t)\left(\frac{r}{2} - \frac{\Omega\cot\Omega t}{2}\right) \tag{23}$$

Eq. (23) can be simply integrated with respect to time t, and one obtains

$$C(t) = \frac{c}{\sqrt{\sin\Omega t}} e^{rt/4}, \tag{24}$$

where C is a constant.

Substituting Eq. (24) into Eq. (21) and applying the initial condition

$$G(x, x', t = 0) = \delta(x - x'), \tag{25}$$

we get

$$C = \sqrt{\frac{m\Omega}{2\pi i\hbar}}. \tag{26}$$

So, the Green function or propagator for a damped harmonic oscillator can be written as

$$G(x,x',t) = \sqrt{\frac{m\Omega e^{rt/2}}{2\pi i\hbar\sin\Omega t}} \exp\left(\frac{i}{\hbar}\left\{\left(\frac{m\Omega}{2}e^{rt}\cot\Omega t - \frac{mre^{rt}}{4}\right)x^2 + \left(\frac{m\Omega}{2}\cot\Omega t + \frac{mr}{4}\right)x'^2 - \frac{m\Omega e^{rt/2}}{\sin\Omega t}xx'\right\}\right), \tag{27}$$

which is the same form as the result of S. Pepore *et al.* [5] calculating from Feynman path integral.

3. The Green function for a harmonic oscillator with strongly pulsating mass

The Hamiltonian operator for a harmonic oscillator with strongly pulsating mass can be expressed as [9]

$$\hat{H}(t) = \frac{p^2}{2m\cos^2 vt} + \frac{1}{2}m\cos^2 vt\omega^2\hat{q}^2, \tag{28}$$

where v is the frequency of mass. The classical analog of the Hamiltonian operator in Eq. (28) is

$$H(q, p, t) = \frac{p^2}{2m\cos^2 vt} + \frac{1}{2}m\cos^2 vt\omega^2 q^2. \tag{29}$$

The classical equations of motion determining the oscillator position and momentum are

$$\ddot{q} - 2v\tan vt\dot{q} + \omega^2 q = 0 \tag{30}$$

The classical trajectories in the phase space under the initial conditions $q(0) = q_0$ and $p(0) = p_0$ can be written as

$$q(t) = q_0 \sec vt\cos\Omega t + p_0/m\Omega\sec vt\sin\Omega t, \tag{31}$$

$$p(t) = q_0(mv\cos vt\tan vt\cos\Omega t - m\Omega\cos vt\sin\Omega t) + p_0(\cos vt\cos\Omega t + \frac{v}{\Omega}\cos vt\tan vt sin\Omega t), \tag{32}$$

where $\Omega^2 = \omega^2 + v^2$.

By eliminating p_0 in Eq. (31) and q_0 in Eq. (32), the solutions are

$$q_0(q, p, t) = q\left(\cos vt\cos\Omega t + \frac{v}{\Omega}\sin vt\sin\Omega t\right) - p\left(\frac{\sec vt\sin\Omega t}{m\Omega}\right), \tag{33}$$

$$p_0(q, p, t) = q(m\Omega\cos vt\sin\Omega t - mv\sin vt\cos\Omega t) + p(\sec vt\cos\Omega t). \tag{34}$$

The Hilbert space operators of q_0 and p_0 are

$$\hat{q}_0(\hat{q}, \hat{p}, t) = \hat{q}\left(\cos vt\cos\Omega t + \frac{v}{\Omega}\sin vt\sin\Omega t\right) - \hat{p}\left(\frac{\sec vt\sin\Omega t}{m\Omega}\right), \tag{35}$$

$$\hat{p}(\hat{q}, \hat{p}, t) = \hat{q}(m\Omega\cos vt\sin\Omega t - mv\sin vt\cos\Omega t) + \hat{p}(\sec vt\cos\Omega t). \tag{36}$$

We can determine that $\hat{q}_0$ and $\hat{p}_0$ are integrals of the motion by finding total time derivatives of

$$\frac{d\hat{q}_0}{dt} = \frac{\partial\hat{q}_0}{\partial t} + \frac{i}{\hbar}[\hat{H}, \hat{q}_0] = 0, \tag{37}$$

$$\frac{d\hat{p}_0}{dt} = \frac{\partial\hat{p}_0}{\partial t} + \frac{i}{\hbar}[\hat{H}, \hat{p}_0] = 0. \tag{38}$$

Then these operators must satisfy the equations for the Green function $G(x, x', t)$ [3-4]

$$\hat{q}_0(x)G(x, x', t) = \hat{q}(x')G(x, x', t), \tag{39}$$

$$\hat{p}_0(x)G(x, x', t) = -\hat{p}(x')G(x, x', t). \tag{40}$$

Now we write Eqs. (39) and (40) explicitly,

$$\left(x\left(\cos vt\cos\Omega t + \frac{v}{\Omega}\sin vt\sin\Omega t\right) + \frac{i\hbar\sec vt\sin\Omega t}{m\Omega}\frac{\partial}{\partial x}\right)G(x, x', t) = x'G(x, x', t), \tag{41}$$

$$\left(x\left(m\Omega\cos vt\sin\Omega t - mv\sin vt\cos\Omega t\right) - i\hbar\sec vt\cos\Omega t\frac{\partial}{\partial x}\right)G(x, x', t) = i\hbar\frac{\partial}{\partial x'}G(x, x', t) \tag{42}$$

The system of equations for defining the Green function $G(x, x', t)$ are

$$\begin{aligned}\frac{\partial G(x,x',t)}{\partial x} &= -\frac{i}{\hbar}[x'(m\Omega\cos vt\csc\Omega t)\\&\quad - x(m\Omega\cos^2 vt\cot\Omega t\\&\quad + mv\sin vt\cos vt)]G(x,x',t), \end{aligned}\tag{43}$$

$$\begin{aligned}\frac{\partial G(x,x',t)}{\partial x'} &= -\frac{i}{\hbar}[x(m\Omega\cos vt\csc\Omega t)\\&\quad - x'(m\Omega\cot\Omega t)]G(x,x',t).\end{aligned}\tag{44}$$

Now we can integrate Eq. (43) with respect to the variable x to get

$$\begin{aligned}G(x,x',t) = C(x',t)\exp\Bigg[\frac{i}{\hbar}\Bigg(\Bigg(\frac{1}{2}m\Omega\cos^2 vt\cot\Omega t\\ + \frac{1}{2}mv\sin vt\cos vt\Bigg)x^2\\ - m\Omega\cos vt\csc\Omega t x x'\Bigg)\Bigg].\end{aligned}\tag{45}$$

Substituting Eq. (45) into Eq. (44), we obtain the differential equation for $C(x',t)$ as

$$\frac{\partial C(x',t)}{\partial x'} = \frac{i}{\hbar}(m\Omega\cot\Omega t)x'C(x',t) \tag{46}$$

Solving Eq.(46), we obtain

$$C(x',t) = C(t)\exp\left(\frac{i}{2\hbar}m\Omega\cot\Omega t x'^2\right). \tag{47}$$

After substituting Eq. (47) into Eq. (45), we arrive at

$$\begin{aligned}G(x,x',t) = C(t)\exp\Bigg[\frac{i}{\hbar}\Bigg(\Bigg(\frac{1}{2}m\Omega\cos^2 vt\cot\Omega t\\ + \frac{1}{2}mv\sin vt\cos vt\Bigg)x^2\\ \frac{1}{2}m\Omega\cot\Omega t x'^2 - m\Omega\cos vt\csc\Omega t x x'\Bigg)\Bigg].\end{aligned}\tag{48}$$

To find $C(t)$, we must substitute the Green function of Eq. (48) into the Schrödinger equation

$$\begin{aligned}i\hbar\frac{\partial}{\partial t}G(x,x',t) &= -\frac{\hbar^2}{2m\cos^2 vt}\frac{\partial^2 G(x,x',t)}{\partial x^2}\\&\quad + \frac{1}{2}m\cos^2 vt\omega^2 x^2 G(x,x',t).\end{aligned}\tag{49}$$

After some algebra, we get

$$\frac{dC(t)}{dt} = -\frac{1}{2}(\Omega\cot\Omega t + v\tan vt)C(t). \tag{50}$$

Integrating Eq. (50) with respect to time, we obtain

$$C(t) = C\left(\frac{\cos vt}{\sin\Omega t}\right)^{1/2}. \tag{51}$$

Substituting Eq. (51) into Eq. (48) and applying the initial condition in Eq. (25), the constant C is

$$C = \sqrt{\frac{m\Omega}{2\pi i\hbar}} \tag{52}$$

Thus, the Green function for a harmonic oscillator with strongly pulsating mass can be expressed as

$$\begin{aligned}G(x,x',t) = \sqrt{\frac{m\Omega\cos vt}{2\pi i\hbar\sin\Omega t}}\exp\Bigg[\frac{i}{\hbar}\Bigg(\Bigg(\frac{1}{2}m\Omega\cos^2 vt\cot\Omega t\\ + \frac{1}{2}mv\sin vt\cos vt\Bigg)x^2 + \frac{1}{2}m\Omega\cot\Omega t x'^2\\ - m\Omega\cos vt\csc\Omega t x x'\Bigg)\Bigg],\end{aligned}\tag{53}$$

which is the same result as M. Sabir and S. Rajagopalan [9] by Feynman path integral method.

4. The Green function for a harmonic oscillator with mass growing with time

The Hamiltonian operator for a harmonic oscillator with mass growing with time can be written as

$$\hat{H}(t) = \frac{\hat{p}^2}{2m(1+\alpha t)^2} + \frac{1}{2}m(1+\alpha t)^2\omega^2\hat{q}^2, \tag{54}$$

where α is a constant.

$$H(q,p,t) = \frac{p^2}{2m(1+\alpha t)^2} + \frac{1}{2}m(1+\alpha t)^2\omega^2 q^2 \tag{55}$$

The equation of motion for this oscillator is

$$\ddot{q} + \frac{2\alpha}{(1+\alpha t)}\dot{q} + \omega^2 q = 0. \tag{56}$$

The classical paths in the space under the initial conditions $q(0) = q_0$ and $p(0) = p_0$ can be expressed as

$$\begin{aligned}q(t) = q_0\left(\frac{\alpha\sin\omega t + \omega\cos\omega t}{\omega(1+\alpha t)}\right)\\ + p_0\left(\frac{\sin\omega t}{m\omega(1+\alpha t)}\right),\end{aligned}\tag{57}$$

$$\begin{aligned}p(t)=q_0\left(m\alpha^2 t\cos\omega t - m\omega(1-\alpha t)\sin\omega t - \frac{m\alpha^2}{\omega}\sin\omega t\right)\\ +p_0\left((1+\alpha t)\cos\omega t - \frac{\alpha}{\omega}\sin\omega t\right).\end{aligned}\tag{58}$$

We can express q_0 and p_0 in terms of q, p, and t by

$$q_0(q,p,t) = q\left((1+\alpha t)\cos\omega t - \frac{\alpha}{\omega}\sin\omega t\right) - p\left(\frac{\sin\omega t}{m\omega(1+\alpha t)}\right), \quad (59)$$

$$p_0(q,p,t) = p\left(\frac{\alpha\sin\omega t + \omega\cos\omega t}{\omega(1+\alpha t)}\right) - q\left(m\alpha^2 t\cos\omega t - m\omega(1+\alpha t)\sin\omega t - \frac{m\alpha^2\sin\omega t}{\omega}\right). \quad (60)$$

We define operators acting in the Hilbert space as follows

$$\hat{q}_0(\hat{q},\hat{p},t) = \hat{q}\left((1-\alpha t)\cos\omega t - \frac{\alpha}{\omega}\sin\omega t\right) - \hat{p}\left(\frac{\sin\omega t}{m\omega(1+\alpha t)}\right), \quad (61)$$

$$\hat{q}_0(\hat{q},\hat{p},t) = \hat{q}\left(\frac{\alpha\sin\omega t + \omega\cos\omega t}{\omega(1+\alpha t)}\right) \hat{q}\left(m\alpha^2 t\cos\omega t - m\omega(1+\alpha t)\sin\omega t - \frac{m\alpha^2\sin\omega t}{\omega}\right). \quad (62)$$

Calculating the total derivatives of the operators $\hat{q}_0$ and $\hat{q}_0$ with respect to time, we obtain

$$\frac{d\hat{q}_0}{dt} = \frac{\partial\hat{q}_0}{\partial t} + \frac{i}{\hbar}[\hat{H},\hat{q}_0] = 0, \quad (63)$$

$$\frac{d\hat{p}_0}{dt} = \frac{\partial\hat{p}_0}{\partial t} + \frac{i}{\hbar}[\hat{H},\hat{p}_0] = 0. \quad (64)$$

Hence, operators in Eqs. (61) and (62) are integrals of motion and correspond to the initial position and momentum. Then these operators must satisfy the equations for the Green function $G(x,x',t)$ [3-4]

$$\hat{q}_0(x)G(x,x',t) = \hat{q}(x')G(x,x',t), \quad (65)$$

$$\hat{p}_0(x)G(x,x',t) = -\hat{p}(x')G(x,x',t). \quad (66)$$

Writing Eqs. (65) and (66) explicitly, it can be shown that

$$\left(x\left[(1+\alpha t)\cos\omega t - \frac{\alpha}{\omega}\sin\omega t\right] + \frac{i\hbar\sin\omega t}{m\omega(1+\alpha t)}\frac{\partial}{\partial x}\right) \times G(x,x',t) = x'G(x,x',t), \quad (67)$$

$$\left(-i\hbar\left(\frac{\alpha\sin\omega t + \omega\cos\omega t}{\omega(1+\alpha t)}\right)\frac{\partial}{\partial x} - x\left(m\alpha^2 t\cos\omega t - m\omega(1+\alpha t)\sin\omega t - \frac{m\alpha^2\sin\omega t}{\omega}\right)G(x,x',t) = i\hbar\frac{\partial G(x,x',t)}{\partial x'} \quad (68)$$

The system of equations for calculating the Green function $G(x,x',t)$ are

$$\frac{\partial G(x,x',t)}{\partial x} = \left(-\frac{im\omega(1+\alpha t)x'}{\sin\omega t} + \frac{ix}{\hbar}[m\omega(1+\alpha t)^2 \times \cot\omega t - m\alpha(1+\alpha t)]\right)G(x,x',t), \quad (69)$$

$$\frac{\partial G(x,x',t)}{\partial x'} = \left(\frac{im}{\hbar}(\alpha+\omega\cot\omega t)x' - \frac{im\omega(1+\alpha t)}{\hbar\sin\omega t}x\right) \times G(x,x',t). \quad (70)$$

Now we can integrate Eq. (70) with respect to the variable x to obtain

$$G(x,x',t) = C(x',t)\exp\left[\frac{i}{2\hbar}\left((m\omega(1+\alpha t)^2\cot\omega t - m\alpha(1+\alpha t))x^2 - \frac{2m\omega(1+\alpha t)}{\sin\omega t}xx'\right)\right]. \quad (71)$$

Substituting Eq. (71) into Eq. (70), we obtain the differential equation for $C(x',t)$ as

$$\frac{\partial C(x',t)}{\partial x'} = \frac{im}{\hbar}(\alpha+\omega\cot\omega t)x'C(x',t). \quad (72)$$

Solving Eq. (72), we obtain

$$C(x',t) = C(t)\exp\left(\frac{i}{2\hbar}(m\alpha + m\omega\cot\omega t)x'^2\right) \quad (73)$$

After substituting Eq. (73) into Eq. (71), we obtain

$$G(x,x',t) = C(x',t)\exp\left(\frac{i}{2\hbar}\left[(m\omega(1+\alpha t)^2\cot\omega t - m\alpha(1+\alpha t))x^2 + (m\alpha + m\omega\cot\omega t)x'^2 - \frac{2m\omega(1+\alpha t)}{\sin\omega t}xx'\right]\right). \quad (74)$$

To get $C(t)$, we must substitute the Green function of Eq. (74) into the Schrödinger equation

$$i\hbar\frac{\partial G(x,x',t)}{\partial t} = -\frac{\hbar^2}{2m(1+\alpha t)^2}\frac{\partial^2 G(x,x',t)}{\partial x^2} + \frac{1}{2}m(1+\alpha t)^2\omega^2 G(x,x',t). \quad (75)$$

After some algebra, we obtain

$$\frac{dC(t)}{dt} = \left(\frac{\alpha}{2(1+\alpha t)} - \frac{1}{2}\omega\cot\omega t\right)C(t). \quad (76)$$

Integrating Eq. (76) with respect to time, we get

$$C(t) = C\left(\frac{1+\alpha t}{\sin\omega t}\right)^{1/2}. \quad (77)$$

Substituting Eq. (77) into Eq. (74) and employing the initial condition in Eq. (25), the constant C becomes

$$C = \sqrt{\frac{m\omega}{2\pi i\hbar}}. \tag{78}$$

So, the Green function for a harmonic oscillator with mass growing with time can be written as

$$G(x,x',t) = \left(\frac{m\omega(1+\alpha t)}{2\pi i\hbar \sin\omega t}\right)^{1/2} \exp\left(\frac{i}{2\hbar}\Big[(m\omega(1+\alpha t)^2 \times \cot\omega t - m\alpha(1+\alpha t))x^2 + (m\alpha + m\omega\cot\omega t)x'^2 - \frac{2m\omega(1+\alpha t)}{\sin\omega t}xx'\Big]\right), \tag{79}$$

which is agreement with the result of S. Pepore and B. Sukbot [11] calculating by Schwinger method.

5. Conclusion

The method for deriving the Green functions with the helping of integrals of the motion presented in this paper can be successfully applied in solving time-dependent mass harmonic oscillator problems. This method has the important steps in finding the constant of motions q_0 and p_0 and implying that the Green functions $G(x,x',t)$ is the eigenfunctions of the operators $\hat{q}_0(x)$ and $\hat{p}_0(x)$.

In fact, this method has many common features with the Schwinger method, [11-14] but the Schwinger method requires the operator $\hat{q}_0(x)$ and $\hat{p}_0(x)$ in calculating the matrix element of Hamiltonian operator in the Green function

$$G(x,x',t) = C(x,x')\exp\left\{-\frac{i}{\hbar} \times \int_0^t \frac{\langle x(t) \mid \hat{H}(\hat{x}(t),\hat{x}(0) \mid x'(0)\rangle \mid\rangle}{\langle x(t) \mid x'(0)\rangle}dt\right\}. \tag{80}$$

In Feynman path integral, the pre-exponential function $C(t)$ comes from sum over all historical paths that depend on the calculation of functional integration while in the integrals of motion method this term appears in solving Schrodinger equation of Green function. In my opinion the method in this article seems to be more simple from the viewpoint of calculation.

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