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Soliton operators in the quantum equivalence of the CP_1 and $O(3) - \sigma$ models

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We discuss some interesting aspects of the well known quantum equivalence between the $O(3) - \sigma$ and CP_1 models in $3D$, working in the canonical and in the path integral formulations. We show first that the canonical quantization in the hamiltonian formulation is free of ordering ambiguities for both models. We use the canonical map between the fields and momenta of the two models and compute the relevant functional determinant to verify the equivalence between the phase-space partition functions and the quantum equivalence in all the topological sectors. We also use the explicit form of the map to construct the soliton operator of the $O(3) - \sigma$ model starting from the representation of the operator in the CP_1 model, and discuss their properties.

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1. Introduction

The non-linear $O(3) - \sigma$ model is defined by the action

$$\mathcal{I}_\sigma = \frac{1}{2G^2} < [\partial_\mu n_p \partial^\mu n_p - \lambda(n_p n_p - 1)] > , \qquad (1)$$

where n_p are components of a $O(3)$ vector field. The CP_1 model is defined in terms of the complex fields z_a, $a = 1, 2$, by the action

$$\mathcal{I}_{CP} = g^{-2} < \partial_\mu \mathbf{z}^* \cdot \partial^\mu \mathbf{z} - (\mathbf{z}^* \cdot \partial_\mu \mathbf{z})(\partial^\mu \mathbf{z}^* \cdot \mathbf{z}) > - < \Lambda(|z_1|^2 + |z_2|^2 - 1) > , \qquad (2)$$

(λ and Λ are Lagrange multipliers, $< >$ denotes space time integration, G and g are coupling constants). They provide in three dimensions, an interesting example of classical and quantum equivalence between two field theories [1,2]. Each one of these models has interest by its applications in high energy physics, condensed matter physics and statistical mechanics. This interest rests partially in the topological properties of these models, notably, the existence of soliton solutions and identically conserved topological currents. For the sigma model the topological current is given by,

$$j^\mu_\sigma = \frac{1}{8\pi} \vec{n} \cdot (\epsilon^{\mu\nu\rho} \partial_\nu \vec{n} \times \partial_\rho \vec{n}) , \qquad (3)$$

and the charge by

$$Q_\sigma = \frac{1}{8\pi} \int \vec{n} \cdot (\epsilon_{ij} \partial_i \vec{n} \times \partial_j \vec{n}) d^2x , \qquad (4)$$

where we introduced vector like notation $\vec{n}(x)$ for the σ model variables. To define the topological current for the CP_1 model one makes use of its gauge invariance. This is made explicit writing the Lagrangian in terms of a composite gauge field

$$A_\mu = \frac{\partial^\mu \mathbf{z}^* \cdot \mathbf{z} - \mathbf{z}^* \cdot \partial^\mu \mathbf{z}}{2i} = \mathrm{Im}(\partial^\mu \mathbf{z}^* \cdot \mathbf{z}) , \qquad (5)$$

and the corresponding covariant derivative $D_\mu \mathbf{z} = \partial_\mu \mathbf{z} + iA_\mu \mathbf{z}$, as

$$\mathcal{L}_{CP} = g^{-2} \left[(D_\mu \mathbf{z})^* \cdot (D^\mu \mathbf{z}) - \Lambda(|z_1|^2 + |z_2|^2 - 1) \right] . \qquad (6)$$

The topological current is then,

$$j^\mu_{CP} = \frac{1}{2\pi i} \epsilon^{\mu\nu\rho} (D_\nu \mathbf{z})^* \cdot (D_\rho \mathbf{z}) , \qquad (7)$$

and the charge

$$Q_{CP} = \frac{1}{2\pi i} \int \epsilon_{ij} (D_i \mathbf{z})^* \cdot (D_j \mathbf{z}) d^2x . \qquad (8)$$

Classically, the equivalence between the models is provided by the map

$$\vec{n} = \mathbf{z}^\dagger \vec{\sigma} \mathbf{z} , \qquad (9)$$

where $\vec{\sigma}$ are Pauli's matrices. Due to the identity $\sigma_{pab}\sigma_{pcd} = \delta_{ab}\delta_{cd} - 2\epsilon_{ac}\epsilon_{bd}$, one finds that $n_p n_p = (\mathbf{z} \cdot \mathbf{z})^2 = 1$ so that the constraints are equivalent. One may also identify the Lagrangians, the topological currents, the charges and the solutions of both models. In particular the relation between the solitonic solutions in each model has been discussed thoroughly in the literature [3].

Quantum equivalence of these models has also been studied in detail and used routinely in applications to critical phenomena and condensed matter physics. This is done usually [4] in the Lagrangian path integral approach, where the equivalence of the partitions functions can be easily asserted up to an arbitrary factor. Although if one is careful this does not affect the analysis of the physics of the systems, it is worthwhile to improve the analysis working in the Hamiltonian formulation which have been shown useful in the case of other topologically non trivial models [5,6].

In the canonical approach the structure of the constraints and the quantum equivalence of the systems are more involved, since for the $O(3) - \sigma$ model one has three real fields with one constraint, and for the CP_1 model two complex fields and only a real constraint. The analysis of both systems using Dirac's method [7] was presented in Ref. 8 (see also [9] for a discussion of the more general $CP(n-1)$ model) . It was shown that to establish the canonical equivalence between them, Eq. (9) should be complemented with a corresponding relation for the momenta, which emerge from the procedure. This is reviewed in the next section, where we show how after quantization, the hermiticity requirement solves the operator ordering ambiguities. Since some of the constraints in both models are of second class, a rigorous approach for the equivalence of the partitions functions should be pursued starting from the Senjanovic-Fadeev-Popov path integral [10]. We develop this point of view and present the details of this computation in Sec. 3, which of course confirms the result of the Lagrangian approach. Finally in Sec. 4 we discuss how the canonical equivalence between the phase space variables of the two models can also be used to establish the equivalence of the disorder soliton like operators of each formulation.

2. Canonical Quantization

Let us first consider the quantization of the $O(3)-\sigma$ model. The momenta computed from (1) are given by,

$$\pi_p(x) = \frac{\delta L}{\delta \dot{n}_p} = \dot{n}_p \, . \tag{10}$$

We use vector like notation $\{\vec{n}(x), \vec{\pi}(y)\}$ for the phase space variables, take $G=1$ and write the Hamiltonian in the form

$$H_\sigma = \int \left(\frac{1}{2}\|\vec{\pi}\|^2 + \frac{1}{2}\|\partial_i \vec{n}\|^2 + \frac{1}{2}\lambda[\|\vec{n}\|^2 - 1] \right) d^2x \, . \tag{11}$$

Time conservation of the constraint

$$\theta_1 = \|\vec{n}(x)\|^2 - 1 = 0 \;\; , \tag{12}$$

implies

$$\theta_2 = \vec{n}\cdot\vec{\pi} = 0 \;\; . \tag{13}$$

Conservation of this constraint allows to fix the Lagrange multiplier $\lambda = -|\vec{\pi}|^2 - \vec{n}\cdot\nabla^2\vec{n}$. These constraints are second class. Dirac Brackets between phase space functions ξ and η of a system with second class constraints θ_α are defined by $\{\xi,\eta\}_D = \{\xi,\eta\} - \{\xi,\theta_\alpha\}c_{\alpha\beta}\{\theta_\beta,\eta\}$ with $c_{\alpha\beta}\{\theta_\beta,\theta_\delta\} = \delta_{\alpha\delta}$. The relevant matrix necessary to compute the Dirac brackets is given by,

$$\begin{aligned} c_{\alpha\beta} &= \{\theta_\alpha(x), \theta_\beta(y)\}^{-1} \\ &= \begin{pmatrix} 0 & -\delta^2(\vec{x}-\vec{y}) \\ \delta^2(\vec{x}-\vec{y}) & 0 \end{pmatrix} \, , \end{aligned} \tag{14}$$

with $\alpha,\beta = 1,2$. The Dirac algebra for the system is [8],

$$\begin{aligned} &\{n_p(x), n_q(y)\}^D = 0 \, , \\ &\{n_p(x), \pi_q(y)\}^D = [\delta_{pq} - n_p(x)n_q(y)]\delta^2(\vec{x}-\vec{y}) \, , \\ &\{\pi_p(x), \pi_q(y)\}^D = [\pi_p(x)n_q(y) - \pi_q(y)n_p(x)]\delta^2(\vec{x}-\vec{y}). \end{aligned}$$

At the quantum level we face ordering ambiguities. Checking for consistence we obtain for the commutators of the quantum operators two possible orderings

$$[\Pi_p(x), \Pi_q(y)] = \begin{cases} i[\Pi_p(x)N_q(y) - \Pi_q(y)N_p(x)]\delta^2(\vec{x}-\vec{y}) \\ i[N_q(y)\Pi_p(x) - N_p(x)\Pi_q(y)]\delta^2(\vec{x}-\vec{y}) \end{cases} ,$$

which (using $[N_p(x), N_q(y)] = 0$), are equivalent. Also it is derived that

$$\vec{N}(x)\cdot\vec{\Pi}(y) - \vec{\Pi}(y)\cdot\vec{N}(x) = 2i\delta^2(\vec{x}-\vec{y}) \, , \tag{15}$$

which implies an ambiguity in the order of the constraint $\vec{n}\cdot\pi = 0$. Using hermiticity of N_p and Π_q the constraint is fixed to be

$$\vec{N}\cdot\vec{\Pi} + \vec{\Pi}\cdot\vec{N} = 0 \, . \tag{16}$$

The constraint $N_pN_p = \mathbb{I}$ presents no ordering problems.

We now turn our attention to the CP_1 model. Associated to gauge invariance, the system has a first class constraint. Taking $g=1$, the canonical momenta are

$$\pi_{z_a} = \dot{z}_a^* - (\dot{\mathbf{z}}^*\cdot\mathbf{z})z_a^* \, , \quad \pi_{z_a^*} = \dot{z}_a - (\dot{\mathbf{z}}\cdot\mathbf{z}^*)z_a \, . \tag{17}$$

Observe that since $\pi_{z_a^*} = \pi_{z_a}^*$, we may represent the variables in the compact form $\{\mathbf{z}, \mathbf{z}^*, \boldsymbol{\pi}, \boldsymbol{\pi}^*\}$, where $\mathbf{z} = \{z_a\}$ and $\boldsymbol{\pi} = \{\pi_a\}$, $a = 1,2$. The latter are distinguished from σ model momenta by the indices which are taken from the first letters of the alphabet. When necessary as in Eq. (37) an explicit superscript is used. Writing the equation for π_a^* in the form $\pi_a^* = (\delta_{ab} - z_az_b^*)\dot{z}_b$ and taking into account that $(\delta_{ab} - z_az_b^*)z_b = 0$ we obtain for consistency the constraints $\boldsymbol{\pi}\cdot\mathbf{z} = 0$ or equivalently $\boldsymbol{\pi}^*\cdot\mathbf{z}^* = 0$. Choosing real combinations of these we have the constraints

$$\begin{aligned} &\Theta_1 = |z_a|^2 - 1 = 0, \quad \Theta_2 = \frac{1}{2}(z_a\pi_a + z_a^*\pi_a^*) = 0 \, . \\ &\varphi = z_a\pi_a - z_a^*\pi_a^* \, . \end{aligned} \tag{18}$$

The Hamiltonian is

$$H_{CP} = \int \left(|\boldsymbol{\pi}|^2 + |\partial_i\mathbf{z}|^2 - |\mathbf{z}^*\cdot\partial_i\mathbf{z}|^2 \right) d^2x \, . \tag{19}$$

No further constraints are obtained from Dirac's procedure. φ is found to be the required first class constraint. For the second class constraints, the matrix $\{\Theta_\alpha, \Theta_\beta\}$ is given by the right hand side of (14).

The Dirac algebra is given by [8],

$$\begin{aligned} &\{z_a(x), z_b(y)\}^D = 0 \, , \;\; \{z_a(x), z_b^*(y)\}^D = 0 \, , \\ &\{z_a(x), \pi_b(y)\}^D = [\delta_{ab} - \frac{1}{2}z_a(x)z_b^*(y)]\delta^2(\vec{x}-\vec{y}) \, , \\ &\{z_a(x), \pi_b^*(y)\}^D = -\frac{1}{2}z_a(x)z_b(y)\delta^2(\vec{x}-\vec{y}) \, , \\ &\{\pi_a(x), \pi_b(y)\}^D = \frac{1}{2}[\pi_a(x)z_b^*(y) - \pi_b(y)z_a^*(x)]\delta^2(\vec{x}-\vec{y}) \, , \\ &\{\pi_a(x), \pi_b^*(y)\}^D = \frac{1}{2}[\pi_a(x)z_b(y) - z_a^*(x)\pi_b^*(y)]\delta^2(\vec{x}-\vec{y}) \, . \end{aligned}$$

The commutators associated to the first three relations above present no ordering problems. For the fourth, checking for consistency and hermiticity we are lead to the following two equivalent options

$$[\Pi_a(x),\Pi_b(x)]=\begin{cases}\frac{i}{2}[\Pi_a(x)Z^*_b(y)-\Pi_b(y)Z^*_a(x)]\delta^2(\vec{x}-\vec{y})\\ \frac{i}{2}[Z^*_b(y)\Pi_a(x)-Z^*_a(x)\Pi_b(y)]\delta^2(\vec{x}-\vec{y})\end{cases},$$

Also, since Z^*_a and Π^*_a are the hermitian conjugates of Z_a and Π_a using the identities $[\Pi_a(x),\Pi^\dagger_b(y)]^\dagger = [\Pi_b(y),\Pi^\dagger_a(x)]$, the last commutator is written in the alternative forms

$$[\Pi_a(x),\Pi^\dagger_b(y)]=\begin{cases}i[\Pi_a(x)Z_b(y)-Z^\dagger_a(x)\Pi^\dagger_b(y)]\delta^2(\vec{x}-\vec{y})\\ i[Z_b(y)\Pi_a(x)-\Pi^\dagger_b(y)Z^\dagger_a(x)]\delta^2(\vec{x}-\vec{y})\end{cases}.$$

Observing that this relations imply

$$Z_a(x)\Pi_a(y)-\Pi_a(y)Z_a(x)=\frac{3}{2}i\delta^2(\vec{x}-\vec{y})\ , \qquad (20)$$

the quantum constraints should be taken as combinations of the symmetric ordered terms

$$Z_a\Pi_a+\Pi_aZ_a=0 \qquad Z^*_a\Pi^*_a+\Pi^*_aZ^*_a=0\ . \qquad (21)$$

The constraint $Z^\dagger_aZ_a=\mathbb{I}$, the topological charge and the gauge fields $A_i=iZ^\dagger_a\partial_iZ_a$ which are hermitian are free of ambiguities.

To establish the canonical quantum equivalence of the systems it is necessary to complement the map of Eq. (9) between the fields with a corresponding relation for the momenta [8]. This is obtained classically taking the time derivative of (9) and using (17) and the fact that $\dot{\mathbf{z}}^*\cdot\mathbf{z}+\mathbf{z}^*\cdot\dot{\mathbf{z}}=0$. It reads,

$$\pi_i=\dot{n}_i=\pi_a\sigma_{iab}z_b+z^*_a\sigma_{iab}\pi^*_b\ . \qquad (22)$$

With these relations it can be verified that the Poisson and Dirac Brackets of any two expressions in one model, maps into the corresponding ones in the other.

At the quantum level we have to take care of the order ambiguity present in (22). This is done as before, to end up with the following equivalent maps between the momentum operators,

$$\Pi_i=\begin{cases}\frac{1}{2}(\Pi_a\sigma_{iab}Z_b+Z^\dagger_a\sigma_{iab}\Pi^\dagger_b)\\ \frac{1}{2}(Z_a\sigma_{iba}\Pi_b+\Pi^\dagger_a\sigma_{iba}Z^\dagger_b)\end{cases}. \qquad (23)$$

They fulfill the commutation relations. The quantum models are canonically equivalent.

3. Equivalence of the partitions functions

The path integral of a system described by coordinates q_i subject to s second class constraints θ_α, r first class constraints φ_m and r gauge fixing conditions χ_m constructed by Senjanovic [10] takes the form

$$Z_\sigma=\int e^{i/\hbar\int_0^T(p_i\dot{q}_i-H(p,q))dt}d\mu\ , \qquad (24)$$

where the measure is given by

$$d\mu=\mathscr{D}p\mathscr{D}q\prod_{n=1}^{r}\delta(\chi_n)\delta(\varphi_n)|\det\{\chi_m,\varphi_p\}|$$
$$\times\prod_{c=1}^{s}\delta(\theta_c)|\det\{\theta_\alpha,\theta_\beta\}|^{1/2}\ . \qquad (25)$$

Let us show that this expression gives the same result for the two models. For the σ model the path integral is affected by the factor $\det\{\theta_\alpha,\theta_\beta\}$ with the constraints given by (12) and (13) and the Poisson matrix by (14). The eigenvalues of the matrix are $\pm i$ and the determinant is 1. The partition function is,

$$Z_\sigma=\int\mathscr{D}\vec{n}\mathscr{D}\vec{\pi}\delta(\|\vec{n}\|^2-1)\delta(\vec{n}\cdot\vec{\pi})e^{<\vec{\pi}\cdot\dot{\vec{n}}-H_\sigma>}\ . \qquad (26)$$

For the CP_1 model we have to choose a gauge condition in order to determine the Fadeev-Popov [11] term. One suitable condition is the radiation gauge $\chi=\partial_iA_i=0$. This is rewritten as

$$\chi=\frac{\nabla^2z^*_az_a-z^*_a\nabla^2z_a}{2i}=0\ . \qquad (27)$$

The factor of the second class constraints $\det\{\Theta_\alpha,\Theta_\beta\}$ is again 1. The Poisson brackets of φ with Θ_1 and Θ_2 vanish and the remaining Poisson bracket is computed using $\nabla^2(|z|^2-1)=0$, which implies that

$$\nabla^2z^*_a(x)z_a(y)+z^*_a(y)\nabla^2z_a(x)=-2|\partial_iz|^2.$$

The bracket is given by,

$$\{\chi(x),\varphi(y)\}=\frac{1}{2}[|\partial_iz|^2+\nabla^2_{\vec{x}}]\delta(\vec{x}-\vec{y})\ . \qquad (28)$$

The partition function for the CP_1 model is

$$Z_{CP_1}=\int\mathscr{D}\mathbf{z}\mathscr{D}\mathbf{z}^*\mathscr{D}\boldsymbol{\pi}\mathscr{D}\boldsymbol{\pi}^*\delta(|z|^2-1)$$
$$\times\delta\left(\frac{\nabla^2\mathbf{z}^*\cdot\mathbf{z}-\mathbf{z}^*\cdot\nabla^2\mathbf{z}}{2i}\right)\delta\left(\frac{\mathbf{z}\cdot\boldsymbol{\pi}+\mathbf{z}^*\cdot\boldsymbol{\pi}^*}{2}\right)$$
$$\times\delta\left(\frac{\mathbf{z}\cdot\boldsymbol{\pi}-\mathbf{z}^*\cdot\boldsymbol{\pi}^*}{2i}\right)\left|\det\left(\frac{1}{2}[|\partial_iz|^2+\nabla^2]\right)\right|$$
$$\times\exp\ i\langle\boldsymbol{\pi}\cdot\dot{\mathbf{z}}+\boldsymbol{\pi}^*\cdot\dot{\mathbf{z}}^*-H_{CP}\rangle\ . \qquad (29)$$

To compare we modify the expression (26) introducing two auxiliary variables s and π_s,

$$Z_\sigma=\int\mathscr{D}\vec{n}\mathscr{D}\vec{\pi}\mathscr{D}\vec{\pi}_s\mathscr{D}\pi_s\delta(|\vec{n}|^2-1)$$
$$\times\delta(\vec{n}\cdot\pi)\delta(s)\delta(\pi_s)e^{\langle\vec{\pi}\cdot\dot{\vec{n}}-H_\sigma\rangle}\ , \qquad (30)$$

and perform the change of variables $M\ (\vec{n},\vec{\pi},s,\pi_s)\leftrightarrow(\mathbf{z},\boldsymbol{\pi})$ defined by,

$$n_i=z^*_a\sigma_{iab}z_b\ , \qquad \pi_i=\frac{\pi_a\sigma_{iab}z_b+z^*_a\sigma_{iab}\pi^*_b}{2} \qquad (31)$$

$$s=\frac{\nabla^2z^*_az_a-z^*_a\nabla^2z_a}{2i}\ , \qquad \pi_s=\frac{z_a\pi_a-z^*_a\pi^*_a}{2i}\ . \qquad (32)$$

The δ functions in (30) map onto δ functions of the partition function of the CP_1 model (29) and the Hamiltonian actions map into each other. The Jacobian of the transformation is $J = \det M$ with

$$M = \begin{pmatrix} z_2^* & z_1^* & z_2 & z_1 & 0 & 0 & 0 & 0 \\ iz_2^* & -iz_1^* & -iz_2 & iz_1 & 0 & 0 & 0 & 0 \\ z_1^* & -z_2^* & z_1 & -z_2 & 0 & 0 & 0 & 0 \\ [\nabla^2, z_1^*]/2i & [\nabla^2, z_2^*]/2i & [z_1, \nabla^2]/2i & [z_2, \nabla^2]/2i & 0 & 0 & 0 & 0 \\ \pi_2/2 & \pi_1/2 & \pi_2^*/2 & \pi_1^*/2 & z_2/2 & z_1/2 & z_2^*/2 & z_1^*/2 \\ i\pi_2/2 & -i\pi_1/2 & -i\pi_2^*/2 & i\pi_1^*/2 & -iz_2/2 & iz_1/2 & iz_2^*/2 & -iz_1^*/2 \\ \pi_1/2 & -\pi_2/2 & \pi_1^*/2 & -\pi_2^*/2 & z_1/2 & -z_2/2 & z_1^*/2 & -z_2^*/2 \\ \pi_1/2i & \pi_2/2i & -\pi_1^*/2i & -\pi_2^*/2i & z_1/2i & z_2/2i & -z_1^*/2i & -z_2^*/2i \end{pmatrix} \delta^2(\vec{x}-\vec{y}).$$

Using the block structure of M we have

$$|J| = \frac{1}{2i} \det \begin{pmatrix} z_2^* & z_1^* & z_2 & z_1 \\ iz_2^* & -iz_1^* & -iz_2 & iz_1 \\ z_1^* & -z_2^* & z_1 & -z_2 \\ \nabla^2 z_1^* - z_1^* \nabla^2 & \nabla^2 z_2^* - z_2^* \nabla^2 & z_1 \nabla^2 - \nabla^2 z_1 & z_2 \nabla^2 - \nabla^2 z_2 \end{pmatrix}$$
$$\times \frac{1}{2i} \left[\frac{1}{2}\right]^3 \det \begin{pmatrix} z_2 & z_1 & z_2^* & z_1^* \\ -iz_2 & iz_1 & iz_2^* & -iz_1^* \\ z_1 & -z_2 & z_1^* & -z_2^* \\ z_1 & z_2 & -z_1^* & -z_2^* \end{pmatrix}.$$

Using the identity $(\nabla^2 z_a^*) z_a + z_a^* (\nabla^2 z_a) = -2|\partial_i z|^2$ we finally obtain

$$|J| = \left| \det \left[-\frac{1}{8.(4)} (-4i).4i(|\partial_i z|^2 + \nabla^2) \right] \right|$$
$$= \left| \det \left(\frac{1}{2} [|\partial_i z|^2 + \nabla^2] \right) \right|, \qquad (33)$$

which is the Fadeev Popov determinant in (29). This establishes the quantum equivalence of the theories in the sector of zero topological charge. The identification of the topological charges which is preserved in the quantum theory by the canonical map, guarantees the quantum equivalence of the models in all the sectors.

4. Soliton operators

Topological solitons in field theory models are the signature of a non trivial phase structure of the quantum theory, with the phase transition being driven by the condensation of the quantum solitons. Accordingly, soliton operators may be constructed in quantum field theory [10-12] as a generalization of disorder operators in statistical mechanics [15]. To interpolate between sectors of different topological charge soliton operators should apply to the field variables the relevant topological behavior of the soliton solutions. For two dimensional models these ideas allow to recover Mandelstam's operator [16] and the standard results abelian [17] and non-abelian bosonization [18–20]. They may also be applied to non-abelian gauge fields [21] and to fermionic currents [22]. In 3D, soliton operators of abelian gauge theories have been investigated along this lines [13, 14, 23, 24]. Some applications of the CP_1 model Skyrmions are discussed in Ref. [25, 26]. Here we use the canonical mapping to construct the $\sigma - O(3)$ disorder operator from the CP_1 operator.

The topological properties of the CP_1 Skyrmion are encoded in the behavior in space, like infinity and at its center given by [3],

$$\mathbf{z}(\vec{x})_{\overrightarrow{\rho\to\infty}} \begin{pmatrix} e^{-i\arg(\vec{x})/2} \\ 0 \end{pmatrix} \qquad \mathbf{z}(\vec{x})_{\overrightarrow{\rho\to 0}} \begin{pmatrix} 0 \\ e^{i\arg(\vec{x})/2} \end{pmatrix},$$
$$A_i(\vec{x})_{\overrightarrow{\rho\to\infty}} \frac{1}{2}\partial_i[\arg(\vec{x})] \qquad A_i(\vec{x})_{\overrightarrow{\rho\to 0}} -\frac{1}{2}\partial_i[\arg(\vec{x})],$$

where ρ is the radial variable and $\theta(\vec{x}) = \arctan(x_1/x_2) \equiv \arg(\vec{x})$. To apply the asymptotic behavior to the fields, the disorder operator $\mu(x)$ should satisfy the order disorder algebra

$$\mu(x;c) Z_1(y) = \begin{cases} e^{-\frac{1}{2}i\arg(\vec{y}-\vec{x})} Z_1(y)\mu(x;c) & \vec{y}-\vec{x} \notin T(c) \\ Z_1(y)\mu(x;c) & \vec{y}-\vec{x} \in T(c) \end{cases},$$
$$\mu(x;c) Z_2(y) = \begin{cases} Z_1(y)\mu(x;c) & \vec{y}-\vec{x} \notin T(c) \\ e^{\frac{1}{2}i\arg(\vec{y}-\vec{x})} Z_1(y)\mu(x;c) & \vec{y}-\vec{x} \in T(c) \end{cases},$$
$$\mu(x;c) A_i(y) = \begin{cases} \left[A_i(y) + \frac{1}{2}\partial_i \arg(\vec{y}-\vec{x})\right]\mu(x;c) & \vec{y}-\vec{x} \notin T(c) \\ \left[A_i(y) - \frac{1}{2}\partial_i \arg(\vec{y}-\vec{x})\right]\mu(x;c) & \vec{y}-\vec{x} \in T(c) \end{cases},$$

were $T(c)$ is a spatial region centered in $\vec{x}$ whose boundary is a plane curve c. $\mu(x,c)$ is identified to be

$$\mu(\vec{x},t;c) = \exp\left\{\frac{1}{2}\int_{\mathbf{R}^2-T_{\vec{x}}}\left[Z_1(\vec{w},t)\Pi_1(\vec{w},t) - \Pi_1^\dagger(\vec{w},t)Z_1^\dagger(\vec{w},t)\right]\arg(\vec{w}-\vec{x})d^2\vec{w} - \frac{1}{2}\int_{T_{\vec{x}}}\left[Z_2(\vec{w},t)\Pi_2(\vec{w},t) - \Pi_2^\dagger(\vec{w},t)Z_2^\dagger(\vec{w},t)\right]\arg(\vec{w}-\vec{x})d^2\vec{w}\right\}. \quad (34)$$

For the $O(3)-\sigma$ model, the direct construction of the disorder variable is more complicated to pursue since the topological properties of the solution depend on the whole space time configuration. This is overcome by using the canonical map. In components the map ($\vec{n} = \mathbf{z}^\dagger\vec{\sigma}\mathbf{z}$) is

$$n_1 = 2\Re(z_1^* z_2) \quad n_2 = 2\mathrm{Im}(z_1^* z_2)$$
$$n_3 = |z_1|^2 - |z_2|^2. \quad (35)$$

Using also that $\mu_\sigma^\dagger = \mu_\sigma^{-1}$, it is shown that $[\mu(\vec{x}), N_3(\vec{y}] = 0$ and the order disorder algebra (34) which is non trivial only for N_1 and N_2 is written as,

$$\mu_\sigma(\vec{x})N_1(\vec{y}) = \cos(\arg(\vec{y}-\vec{x}))N_1(\vec{y})\mu_\sigma(\vec{x}) \ ,$$
$$\mu_\sigma(\vec{x})N_2(\vec{y}) = \sin(\arg(\vec{y}-\vec{x}))N_2(\vec{y})\mu_\sigma(\vec{x}) \ . \quad (36)$$

It does not depend on $T(c)$. The inverse of the change variables (31) is

$$|z_1| = \sqrt{\frac{1+n_3}{2}} \ , \qquad |z_2| = \sqrt{\frac{1-n_3}{2}} \ ,$$
$$e^{i\phi} = \frac{n_1 + i n_2}{\sqrt{1-n_3^2}} \ , \qquad z_1 = |z_1|e^{i\varphi_1},$$
$$z_2 = |z_2|e^{i\varphi_2}, \qquad \phi = \varphi_2 - \varphi_1,$$
$$\pi_1^{CP} = \frac{1}{z_1}\left[\pi_3^\sigma + \frac{i}{2}(n_1\pi_2^\sigma - n_2\pi_1^\sigma)\right],$$
$$\pi_2^{CP} = \frac{1}{z_2}\left[-\pi_3^\sigma - \frac{i}{2}(n_1\pi_2^\sigma - n_2\pi_1^\sigma)\right]. \quad (37)$$

Substituting this in the classical expression of (34) and taking into account the ordering issues for the quantum operators already discussed we end up with

$$\mu_\sigma(\vec{x},t) = \exp\left\{i\int_{\mathbf{R}^2}\left[N_1(\vec{y},t)\Pi_2(\vec{y},t) - \Pi_1(\vec{y},t)N_2(\vec{y},t)\right]\arg(\vec{y}-\vec{x})d^2y\right\}, \quad (38)$$

which again does not depend on $T(c)$. This operator satisfies the order disorder algebra (36).

5. Conclusion

In this paper we use the complete canonical map between the Hamiltonians descriptions which results from applying Dirac's method [8], of the $O(3)-\sigma$ model and the CP_1 model in $3D$ and show that the quantum theory is free of ordering ambiguities. We demonstrate, by exhibiting the explicit functional change of variables for the path integral and computing the Jacobian determinant, that the phase space partition functions of the models computed using the complete Senjanovic's construction, are identical, as expected. Finally, we apply the results of the canonical equivalence to construct the $O(3)-\sigma$ soliton disorder operator starting from the corresponding operator of the CP_1 model and verify that it satisfies the defining order disorder algebra.

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