THE ANOMALOUS MAGNETIC MOMENT OF THE ELECTRON IN STOCHASTIC QUANTUM MECHANICS*

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ABSTRACT: In this note we introduce the first-order terms of the electromagnetic potentials for an extended electron into the stochastic scheme of quantum mechanics; thus we derive in a simple and straightforward form the corresponding observable corrections, as, for example, the anomalous magnetic moment of the electron. The final results are expressed in terms of some structure parameters, whose value may be only estimated at present; also, effects arising from the quantization of the radiation field are excluded a priori by our use of classical electrodynamics.

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I. INTRODUCTION

In a recent paper[1], we have introduced the radiation reaction force into a stochastic formulation of non-relativistic quantum mechanics[2] with the aim of studying its effect on the energy levels of a quantum system. From the results of the calculation explicitly performed for the lower levels of the hydrogen atom, we are led to conclude that this self-action effect constitutes a major contribution to the Lamb shift. Although such a nonrelativistic spinless treatment cannot be expected to predict exact numerical results, it has the advantage of offering a clear and consistent physical picture throughout.

Making use of this advantage once more, we propose in this paper to introduce the first-order self-action terms for an extended electron with spin, into the stochastic formulation generalized adequately to include the treatment of spin on the basis of the rigid-body model[3]. In this way, we expect to obtain the nonrelativistic equivalent of the first-order radiative corrections of quantum electrodynamics, which account for such measurable effects as the anomalous magnetic moment of the electron; effects arising from the quantized field, such as vacuum polarization, are a priori excluded by the use of classical electrodynamics.

II. INTRODUCTION OF SELF-ACTION TERMS

In this section we introduce the first-order self-action term of the electron into the set of fundamental equations of the stochastic theory[2]:

$$m\,(\mathcal{D}_c \mathbf{v} - \mathcal{D}_s \mathbf{u}) = \mathbf{f}^{(+)} \tag{1a}$$

$$m\,(\mathcal{D}_s \mathbf{v} + \mathcal{D}_c \mathbf{u}) = \mathbf{f}^{(-)} \tag{1b}$$

Here, as in the previous papers, $\mathbf{v}$ and $\mathbf{u}$ are the systematic and stochastic components, respectively, of the total velocity $\mathbf{c} = \mathbf{v} + \mathbf{u}$; $\mathbf{f}^{(+)}$ and $\mathbf{f}^{(-)}$ are the components of the external force, the + and − signs referring to their parity under time reversal, and $\mathcal{D}_c$, $\mathcal{D}_s$ are derivative operators which in the markoffian approximation, i. e., the approximation leading to quantum mechanics, take on the form

$$\mathcal{D}_c = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \tag{2a}$$

$$\mathcal{D}_s = \mathbf{u} \cdot \nabla + D\nabla^2 \tag{2b}$$

Eqs. (1) may be combined to yield

$$m \mathcal{D}_q \mathbf{v}_q = \mathbf{f}_q \ , \tag{3}$$

where the following definitions have been used:

$$\mathcal{D}_q = \mathcal{D}_c - i\mathcal{D}_s$$

$$\mathbf{v}_q = \mathbf{v} - i\mathbf{u}$$

$$\mathbf{f}_q = \mathbf{f}^{(+)} - i\mathbf{f}^{(-)} \ .$$

In the electromagnetic case, the force $\mathbf{f}_q$ has been shown to be[4, 5]

$$\mathbf{f}_q = e\left[\mathbf{E} + \frac{1}{c}\mathbf{v}_q \times \mathbf{H} + \frac{iD}{c}\nabla^2\mathbf{A}\right] \tag{4}$$

In the present paper we propose to add to this external force, the self-action forces which way be derived from the first-order terms of the Liénard-Wiechert series development of the electromagnetic potentials[6]:

$$\phi' = -\frac{e}{2c^2}\frac{\mathbf{R}\cdot\mathbf{w}}{R} \tag{5a}$$

$$\mathbf{A}' = \frac{e}{c}\frac{\mathbf{v}}{R} \tag{5b}$$

Here $\mathbf{v}$ and $\mathbf{w}$ are the velocity and acceleration of the charge, respectively, and $R = |\mathbf{r} - \mathbf{r}'|$ is the distance between the charge, located at $\mathbf{r}'$, and the field point $\mathbf{r}$. Hence, the field at $\mathbf{r}$ due to the entire charge distribution is given by:

$$\phi' = -\frac{1}{2c^2}\int \frac{\mathbf{R}\cdot\mathbf{w}}{R}\,\sigma'\,d\tau' \tag{6a}$$

$$\mathbf{A}' = \frac{1}{c}\int \frac{\mathbf{v}}{R}\,\sigma'\,d\tau' \tag{6b}$$

where $\sigma' = \sigma(\mathbf{r}')$ is the charge density.

The self-force acting on the electron is thus obtained by substituting the stochastic generalizations of Eqs. (6) into Eq. (4) and integrating once more over the change distribution, i. e.,

$$\mathbf{f}'_q = \int [\,\mathbf{E}'_q + \frac{1}{c}\,\mathbf{v}^t_q \times \mathbf{H}'_q + \frac{iD}{c}\nabla^2 \mathbf{A}'_q\,]\,\sigma\, d\tau\,; \tag{7}$$

$$\phi' = -\frac{1}{2c^2}\int \frac{\mathbf{R}\cdot\mathbf{w}_q}{R}\,\sigma'\,d\tau'\,, \tag{8a}$$

$$\mathbf{A}' = \frac{1}{c}\int \frac{\mathbf{v}_q}{R}\,\sigma'\,d\tau' \tag{8b}$$

where $\mathbf{v}^t_q = \mathbf{v}_q + \Omega_q \times \rho$. Here Ω_q and ρ have the same meaning as in Ref. (3), i. e., ρ is the distance of the point charge to the center of mass of the particle, and $\Omega_q = \omega - i\eta$ represents the complex angular velocity, ω and η being its systematic and stochastic components, respectively. For further details, see the references.

Since in Eq. (7) the derivations are with respect to ρ, it is convenient to write the functions to be derived in terms of this coordinate. Omitting higher-order terms in Ω_q, we may write

$$\mathbf{v}_q(\rho') = \mathbf{v}_q(\rho) - \Omega_q \times \mathbf{R}$$

$$\mathbf{w}_q(\rho') = \mathbf{w}_q(\rho)\,.$$

Upon deriving with respect to ρ and further making a Taylor series development around the center of mass, we arrive at the following results:

$$\nabla \times \frac{\mathbf{v}_q(\rho')}{R} = \frac{\nabla \times \mathbf{v}_q}{R} + \frac{\rho \cdot \nabla(\nabla \times \mathbf{v}_q)}{R} - \frac{\mathbf{R} \times \mathbf{v}_q}{R^3} - \frac{\mathbf{R} \times (\rho \cdot \nabla \mathbf{v}_q)}{R^3} -$$

$$- \frac{\nabla \times (\Omega_q \times \mathbf{R})}{R} + \frac{\mathbf{R} \times (\Omega_q \times \mathbf{R})}{R^3} \tag{9a}$$

$$\nabla^2 \frac{\mathbf{v}_q(\rho')}{R} = \frac{\nabla^2 \mathbf{v}_q}{R} + \frac{\rho \cdot \nabla(\nabla^2 \mathbf{v}_q)}{R} - \frac{2\mathbf{R} \cdot \nabla \mathbf{v}_q}{R} -$$

$$- \frac{2R_i \rho_j \partial_i \partial_j \mathbf{v}_q}{R^3} - \frac{\nabla^2 (\Omega_q \times \mathbf{R})}{R^3} + \frac{2\mathbf{R} \cdot \nabla (\Omega_q \times \mathbf{R})}{R^3} + \mathbf{v}_q \nabla^2 \frac{1}{R} \tag{9b}$$

$$\nabla \frac{\mathbf{R} \cdot \mathbf{w}_q(\rho')}{R} = \frac{\mathbf{w}_q}{R} + \frac{\rho \cdot \nabla \mathbf{w}_q}{R} + \frac{\mathbf{R} \cdot \nabla \mathbf{w}_q}{R} + \frac{R_i \rho_j \partial_i \partial_j \mathbf{w}_q}{R} +$$

$$+ \frac{\mathbf{R} \times (\nabla \times \mathbf{w}_q)}{R} + \frac{\mathbf{R} \times [\rho \cdot \nabla (\nabla \times \mathbf{w}_q)]}{R} - \frac{(\mathbf{R} \cdot \mathbf{w}_q)\mathbf{R}}{R^3} - \frac{\mathbf{R} \cdot (\rho \cdot \nabla \mathbf{w}_q)\, \mathbf{R}}{R^3} \tag{9c}$$

which, together with Eqs. (8), allow us to write the different contributions to Eq. (7) as double integrals over σ and σ'. The expressions obtained are simplified considerably by assuming a spherically symmetric charge distribution. If further we introduce the following definitions:

$$\frac{1}{e^2} \iint \frac{\sigma \sigma' \, d\tau \, d\tau'}{R} = \frac{1}{a_0} = \frac{\epsilon_0}{\lambda} \tag{10a}$$

$$\frac{1}{e^2} \iint R \sigma \sigma' \, d\tau d\tau' = a_1 = \epsilon_1 \lambda \tag{10b}$$

$$\frac{\iint - \frac{\rho^2}{R} \sigma \sigma' \, d\tau d\tau'}{\iint \rho^2 \sigma \sigma' \, d\tau d\tau'} = \frac{1}{a_2} = \frac{\epsilon_2}{\lambda} \tag{10c}$$

the self-force acting on the electron takes on the final form:

$$f'_q = \alpha\{-3/2\,\epsilon_0 f_q + \frac{\epsilon_0}{6}[5/2\nabla(m\mathbf{v}_q^2) - iD\nabla^2(m\mathbf{v}_q)] +$$

$$+ 1/2\,\epsilon_0 m\mathbf{v}_q \times \Omega_q + \frac{\epsilon_1}{12}\lambda\!\!\!^{-2}\nabla\nabla\cdot f_q - 1/2\,\epsilon_2 g\nabla[\mathbf{s}_q\cdot(\nabla\times\mathbf{v}_q)]\} \quad (11)$$

where the gyromagnetic ratio is defined through[3]

$$3/2\,\frac{ge^2I}{m} = \iint \rho^2\sigma\sigma'\,d\tau d\tau' . \quad (12)$$

In writing Eq. (11), we have omitted the last term of Eq. (9b) whose integral would represent the action of a point charge over itself, and therefore would not be associated to effects due to the extended structure of the particle.

In Eq. (11), f_q way be considered equal to the external force, since we are working in a first-order approximation. Hence, Eq. (3) takes on the form:

$$m\mathcal{D}_q\mathbf{v}_q = (1 - 3/2\,\alpha\epsilon_0)\,f_q + \alpha\{\frac{\epsilon_0}{6}[5/2\nabla(m\mathbf{v}_q^2) - iD\nabla^2(m\mathbf{v}_q)] +$$

$$+ 1/2\,\epsilon_0 m\mathbf{v}_q\times\Omega_q + \frac{\epsilon_1}{12}\lambda\!\!\!^{-2}\nabla\nabla\cdot f_q - 1/2\,\epsilon_2 g\nabla[\mathbf{s}_q\cdot(\nabla\times\mathbf{v}_q)]\} . \quad (13)$$

The factor which appears multiplying the external force way be absorbed by a redefinition of the mass. In fact, defining the mass to first order by

$$m_r = (1 + 3/2\,\alpha\epsilon_0)\,m = m + \delta m , \quad (14)$$

we obtain from Eq. (13) an equation of motion for a particle with the new mass m_r. For the sake of simplicity, we shall omit the index r and write m for the renormalized mass in what follows.

III. INTEGRATION OF THE EQUATION OF MOTION

We proceed to study the effects due to the remaining self-action terms by considering the particular problem of an electron in a hydrogenlike atom subject to an external magnetic field. The external force acting on the extended electron is given by[3]

$$f_q = e\left(\mathbf{E} + \frac{1}{c}\mathbf{v}_q \times \mathbf{H} + \frac{iD}{c}\nabla^2\mathbf{A}\right) + \frac{ge}{2mc}\nabla(\mathbf{s}_q \cdot \mathbf{H}) \qquad (15)$$

if higher-order terms due to the structure of the particle are not taken into account. The change of variable leading to Schrödinger's equation with minimal electromagnetic coupling is[5]

$$\mathbf{v}_q = -2iD\nabla w - \frac{e}{mc}\mathbf{A}; \ \nabla \cdot \mathbf{A} = 0 .$$

In the present case, we propose to integrate Eq. (15) upon the change of variable

$$\mathbf{v}_q = -2iD\nabla w - \frac{e}{mc}(\mathbf{A} + \alpha\epsilon_0\mathbf{B}_q) \qquad (16a)$$

where $\mathbf{B}_q$ is such that

$$\frac{2e}{mc}\nabla \times \mathbf{B}_q = \Omega_q ; \quad \nabla \cdot \mathbf{B}_q = 0 . \qquad (16b)$$

In deriving Eqs. (9), we have assumed that Ω_q does not depend on the center-of-mass coordinates, which is equivalent to assuming that the spinning motion of the particle is essentially independent of its translational motion. In this case, Ω_q can indeed be written as a rotational function, as in Eq. (16b), and hence we may treat $\mathbf{B}_q$ as a correction to $\mathbf{A}$, as is shown in Eq. (16a).

Introduction of Eq. (15) into Eq. (13) modified by the mass renormalization, and further integration upon the change of variable indicated by Eq. (16a), yields the equation

$$-i\hbar \frac{\partial w}{\partial t} + \frac{1}{2m} \left[-i\hbar \nabla w - \frac{e}{c} (\mathbf{A} + \alpha\epsilon_0 \mathbf{B}_q) \right]^2 - \frac{\hbar^2}{2m} \left(1 - \frac{\alpha\epsilon_0}{6}\right) \nabla^2 w +$$

$$+ V + \frac{\alpha\epsilon_1}{12} \lambda\!\!\!^{-2} \nabla^2 V - \frac{5}{12} \alpha\epsilon_0 m \mathbf{v}_q^2 - \frac{ge}{2mc} (1 + \alpha\epsilon_2) \mathbf{s}_q \cdot \mathbf{H} = 0$$

which upon the new change of variable $\psi = e^w$ takes on the form:

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{1}{2m} \left[-i\hbar \nabla - \frac{e}{c} (\mathbf{A} + \alpha\epsilon_0 \mathbf{B}_q) \right]^2 \psi + (V + \delta V - \mu \cdot \mathbf{H}) \psi \tag{17a}$$

where

$$\delta V = \frac{\alpha\epsilon_1}{12} \lambda\!\!\!^{-2} \nabla^2 V + \alpha\epsilon_0 \frac{\hbar^2}{2m} \nabla^2 w - \frac{5}{12} \alpha\epsilon_0 m \mathbf{v}_q^2 \tag{17b}$$

and

$$\mu = (1 + \alpha\epsilon_2) \mu_0 \ ; \quad \mu_0 = \frac{ge}{2mc} \mathbf{s}_q \ . \tag{17c}$$

According to Eqs. (14) and (17), the first-order self-action effects on an electron endowed with structure are the following:

a) A correction to the theoretical mass parameter, whose value is given by Eq. (14), i. e., of order $\alpha\epsilon_0$.

b) An anomalous magnetic moment which, according to (17c), is given by $\mu_a/\mu_0 = \alpha\epsilon_2$.

c) A correction to the potential energy, whose value is given by Eq. (17b):

$$\delta E = \frac{\alpha\epsilon_1}{12} \lambda\!\!\!^{-2} < \nabla^2 V > \ . \tag{18}$$

d) Additional corrections to the kinetic energy, given by the remaining terms in Eq. (17b), and a correction to the vector potential **A**, due to the spin of the electron (see Eq. 17a).

IV. SOME COMMENTS ABOUT THE PARAMETERS

Since, as is shown by Eqs. (10), the parameters ϵ_0, ϵ_1, ϵ_2 depend strongly on the charge distribution, we are unable to predict their values as long as we do not have any knowledge about the electromagnetic structure of the electron. We may, however, make some general statements, assuming that $\sigma(\rho)$ is a decent function. Let us define, on the one hand, a mean mechanical radius a in terms of the moment of inertia of a sphere with uniform mass distribution, $I = (2/5)\, ma^2$; from Eq. (12) we then obtain

$$3/5\, ge^2 a^2 = \iint \rho^2 \sigma\sigma' \, d\tau d\tau' \ . \tag{19}$$

If, on the other hand, we define a mean electrical radius b by

$$e^2 b^2 = \iint \rho^2 \sigma\sigma' \, d\tau d\tau' \ , \tag{20}$$

we see that this is of the some order as a if g is around unity. We may therefore expect the parameters a_0, a_1, a_2 introduced in Eqs. (10) to be of a similar order of magnitude, although their numerical values and even their signs depend on the specific form of the function $\sigma(\rho)$.

It has been repeatedly shown[3, 7, 8] that the rigid-body model for the spinning electron may yield a satisfactory nonrelativistic description, but then the radius must be assumed to be not smaller than the Compton wavelength for the electron. If, in fact, we assume $a \sim \lambdabar$, and furthermore take $g = 2$, we may conclude from the above arguments that the three adimensional parameters introduced in Eqs. (10) do not depart considerably from unity. On the other side, when the electron is considered as a point particle, ϵ_1 goes to zero, while ϵ_0 and ϵ_1 go to infinity, thus yielding, in particular, an infinite value for the mass correction. We may therefore state, in analogy with quantum electrodynamics, that mass renormalization is equivalent to assigning a finite value to δm, i. e., to ϵ_0, according to Eq. (14), which means endowing the particle with structure. From the above considerations it seems sensible to assume that $\epsilon_0 \sim 1$, and hence, $\epsilon_1 \sim \epsilon_2 \sim 1$. In other words, if δm is of order α, then the anomalous magnetic moment is of the same order, and the correction given by Eq. (18) is

$$\delta E \sim \frac{\alpha \lambdabar^2}{12} < \nabla^2 V > \ , \tag{21}$$

thus representing a structure dependent contribution to the Lamb shift, which is to be added to the main term obtained in Ref. (1).

We thus see that, although this nonrelativistic treatment cannot provide exact numerical results, it yields, nevertheless, consistent values for the three corrections it predicts: mass renormalization, anomalous magnetic moment and structural component of the Lamb shift; it shows, furthermore, the dependence of these effects upon the electrical structure of the electron, in the nonrelativistic approximation.

In concluding, we wish to call attention to Eq. (10c), according to which we are assuming $\sigma(\rho)$ to have special properties; in particular, for ϵ_2 to be positive, σ should have at least one change of sign. Restrictions such as this might furnish some general information about the charge distribution of the electron.

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RESUMEN

En esta nota se introducen dentro del esquema de la mecánica cuántica los términos de primer orden en los potenciales electromagnéticos para un electrón extenso; se derivan las correcciones observables como por ejemplo, el momento magnético anómalo del electrón en forma simple y directa. Los resultados finales se expresan en términos de algunos parámetros de estructura cuyos valores, en este momento sólo se pueden estimar; también, de-

bido al uso de la electrodinámica clásica, se excluyen los efectos debido a a la cuantización del campo de radiación.