$SU(4)$ VAN DER WAERDEN INVARIANT*

R. T. Sharp and David Lee**

Department of Physics, McGill University

Montreal, Canada

(Recibido: septiembre 4, 1971)

ABSTRACT: A solution for the $SU(4)$ external labeling problem, symmetric within a phase factor, is found by explicit construction of the general van der Waerden invariant as a product of elementary scalars. The connection between external and internal multiplicity is demonstrated by exhibiting a one-to-one correspondence between product IR's and internal states. The groups $SU(2)$ and $SU(3)$ are similarly treated for illustrative purposes.

1. INTRODUCTION

An important problem in the Racah algebra of a compact group is the construction of its Wigner coefficients (which couple the states of three IR's

---

* Supported by the National Research Council of Canada

** Present address: Physics Department, California Institute of Technology, Pasadena, California 91109

to give a scalar) or, equivalently, its Clebsch-Gordan coefficients (which couple the states of two IR's to give composite states belonging to a third). In addition to their role in coupling independent systems the coefficients are needed, in connection with the Wigner-Eckart theorem, to help evaluate matrix elements of quantities which transform as components of irreducible tensors under the action of the group. A difficulty in the definition of Clebsch-Gordan, or Wigner, coefficients is that of specifying product states unambiguously – the so-called external labeling problem; the group itself does not generally provide enough labels.

We are chiefly concerned here with the group $SU(4)$. Its interest for physicists stems mainly from its application to nuclear states as the Wigner supermultiplet scheme[1]; it has also been used in speculative classification schemes for elementary particles[2].

Moshinsky[3] has given a prescription for defining product states for $SU(n)$; in principle general Clebsch-Gordan coefficients are thereby specified. From this definition, Moshinsky and Renero[4] have derived recursion formulas for the actual calculation of the coefficients. Biedenharn, Giovanni and Louck[5] have given a solution of the external labeling problem for $SU(3)$ with the suggestion that their method can be extended to $SU(n)$.

In this paper we adopt a somewhat different approach, patterned on that used long ago by van der Waerden[6] for the group $SU(2)$; he wrote down a general invariant in the bases of three IR's in the form of a product of powers of certain "elementary scalars". Although van der Waerden's derivation is a text-book way of obtaining $SU(2)$ Wigner coefficients[7] (expand the general invariant as a sum of products of basis states of the three IR's), it has been little used for higher groups[8].

In § 3 the general van der Waerden invariant for $SU(4)$ is given as a product of elementary scalars. It turns out that Speiser's[9] connection between internal states and external couplings is a valuable heuristic aid in finding these elementary scalars. In § 2 the groups $SU(2)$ and $SU(3)$ are similarly treated to illustrate the method. The remainder of this section is a description of the general approach.

We are interested in the states of the general IR of a (compact) group and their labeling because of their connection with external couplings. According to Cartan[10] the general IR can be constructed as the stretched product of the l fundamental IR's each taken an integer number of times (l is the rank of the group); the l integers are the Cartan labels of the IR. In this spirit the basis states of the general IR can be represented by products of powers of the states of the fundamental IR's. To avoid redundancy certain combinations of fundamental IR states must be regarded as incompatible, and may not appear together for the purpose of forming states of higher IR's. The resulting basis states are essentially those of Weyl[11], but we use a different solution of the redundancy problem in order to achieve a symmetrical van der Waerden invariant (and Wigner coefficients); see also Reference 12, Case 10. The labeling makes no use of subgroups, except perhaps the $U(1)$ subgroups, l in number, associated with the l weights. The indices of the powers of the fundamental IR states are the state labels and are $\frac{1}{2}(r+l)$ in number (r is the order of the group); they provide the l IR labels and the $\frac{1}{2}(r-l)$ internal labels.

Elementary multiplets (the elementary permissible diagrams of Devi, Venkatarayudu and Moshinsky[13, 14] have been used by a number of authors to solve internal labeling problems[12, 13, 14, 15]. Now the external labeling problem is just a special case of internal labeling - that of the group subgroup chain $G \times G \supset G$. An elementary multiplet or coupling $(j_1, j_2; \overline{j_3})$ couples two low-lying IR's j_1 and j_2 to form a composite IR j_e (the IR $\overline{j_3}$) conjugate to j_3 is used for reasons of symmetry). It is elementary if it cannot be expressed as the stretched (all Cartan labels additive) product of simpler elementary multiplets. A set of elementary multiplets is complete if all couplings can be represented by stretched products of powers of them; in general relations between elementary multiplets render certain combinations of them redundant.

According to Speiser[9] the connection between IR's in the direct product $J_1 \otimes J_2$ and internal states of the IR J_1 is displayed by placing the weight diagram for J_1 with its center on the heaviest state of J_2; then each state in the weight diagram of J_1 coincides with the heaviest state of a product IR (if the states of J_1 spill into the region of non-dominant weights, corresponding to negative values of one or more Cartan labels λ_1, they must be reflected in the hyperplanes $\lambda_i = -1$ and counted as positive or negative according to whether an even or odd number of reflections is required to bring them into the region of dominant weights; points with $\lambda_i = -1$ are ignored). This suggests a procedure for setting up an explicit correspondence between internal states and external product IR's. Each elementary multiplet $(j_1, j_2; \overline{j_3})$ defines a weight determined by subtracting the heaviest weight of j_2 from the heaviest weight of j_3. This weight is that of a state belonging to the IR j_1, and the elementary multiplet is made to correspond to that state, written as a product of fundamental IR states. Any external product IR, represented by a stretched product of powers of elementary multiplets, then corresponds to the internal state represented by the corresponding product of powers of fundamental IR states. The l elementary multiplets $(0, j_2; j_2)$ where j_2 is a fundamental IR, are ignored in setting up the correspondece; their role is to center the j_1 weight diagram. (The spilling of product IR's into the region of non-dominant weights with consequent reflections and cancellations occur when the Cartan labels of the IR J_2 are too small to permit the formation of products of elementary multiplets corresponding to all internal states of J_1.) The indices of the powers of the elementary scalars are the external coupling labels and are $\frac{1}{2}(r+3l)$ in number ($3l$ IR labels and $\frac{1}{2}(r-3l)$ "missing" labels).

For symmetry we replace the elementary multiplet $(j_1, j_2; \overline{j}_3)$ by the elementary scalar (j_1, j_2, j_3) in the next two sections. Products of powers of elementary scalars then define the general van der Waerden invariant (J_1, J_2, J_3) in place of the corresponding coupling $(J_1, J_2, \overline{J}_3)$.

2. $SU(2)$ AND $SU(3)$

$SU(2)$ has a single fundamental IR whose two states may be written $\eta = |1\rangle$ and $\xi = |2\rangle$ with weights $\frac{1}{2}$ and $-\frac{1}{2}$ respectively. The general internal state is the Wigner monomial[16], a product of powers of η, ξ. The elementary scalars are $B_1 = (0,1,1) = \eta_2 \xi_3 - \xi_2 \eta_3$, $B_2 = (1,0,1)$, $B_3 = (1,1,0)$. The van der Waerden invariant is[6] a product of powers of the B's. The correspondence is $B_2 \sim \eta$, $B_3 \sim \xi$; B_1 is to be ignored in setting up the correspondence.

$SU(3)$ has two fundamental IR's; the three states of the first may be written $\eta = |1\rangle$, $\xi = |2\rangle$ $\zeta = |3\rangle$, with weights $(1/3, 1/2)$, $(1/3, -1/2)$, $(-2/3, 0)$ respectively, and those of the second may be written

$$\zeta^* = \left|\begin{smallmatrix}1\\2\end{smallmatrix}\right\rangle, \quad -\xi^* = \left|\begin{smallmatrix}1\\3\end{smallmatrix}\right\rangle, \quad \eta^* = \left|\begin{smallmatrix}2\\3\end{smallmatrix}\right\rangle,$$

with weights $(2/3, 0)$, $(-1/3, 1/2)$, $(-1/3, 1/2)$. Because $\eta\eta^* + \xi\xi^* + \zeta\zeta^*$ is a scalar, states containing powers of it are redundant. $\xi\xi^*$ can be replaced by $\eta\eta^* + \zeta\zeta^*$ for the purpose of labeling states and ξ, ξ^* regarded as incompatible. The general $SU(3)$ state is represented by a power of ξ or ξ^* multiplied by powers of the other four variables. The elementary scalars for $SU(3)$ are[17]

$$B_{12} = (10, 01, 00) = \eta_1 \eta_2^* + \xi_1 \xi_2^* + \zeta_1 \zeta_2^*, \qquad B_{23} = (00, 10, 01),$$

$$B_{31} = (01, 00, 10), \qquad B_{21} = (01, 10, 00), \qquad B_{32} = (00, 01, 10),$$

$$B_{13} = (10, 00, 01), \qquad C^* = (01, 01, 01),$$

$$C = (10, 10, 10) = \begin{vmatrix} \eta_1 & \eta_2 & \eta_3 \\ \xi_1 & \xi_2 & \xi_3 \\ \zeta_1 & \zeta_2 & \zeta_3 \end{vmatrix}, \tag{1}$$

They are not all independent: CC^* can be expressed as a linear combination of $B_{12}B_{23}B_{31}$ and $B_{21}B_{32}B_{13}$; hence C, C^* should be regarded as incompatible and products involving both discarded as redundant. The general van der Waerden invariant is represented by a power of C or C^* multiplied by arbitrary powers of the B's. The connection between elementary scalars and fundamental IR states is $B_{13} \sim \eta$, $B_{12} \sim \zeta$, $C \sim \xi$, $B_{31} \sim \zeta^*$, $B_{21} \sim \eta^*$, $C^* \sim \xi^*$.

It is apparent that for $SU(2)$ and $SU(3)$ the correspondence between the general van der Waerden invariant and the general internal state is the one implied by Speiser's theorem, at least when, on the one hand, the Speiser diagram involves no spilling and when, on the other hand, the Cartan labels of J_2 are large enough to allow information of products of elementary multiplets corresponding to all internal states of J_1 (the two conditions can be shown to be equivalent). The proof that the elementary scalars (1) continue to give a complete non-redundant set of couplings when spilling occurs is relegated to an appendix.

3. $SU(4)$

$SU(4)$ has three fundamental IR's, (100), (010) and (001). The four states of the first may be written $\eta = |1\rangle$, $\xi = |2\rangle$, $\zeta = |3\rangle$, $\theta = |4\rangle$ with weights $(1/4, 1/3, 1/2)$, $(1/4, 1/3, -1/2)$, $(1/4, -2/3, 0)$, $(-3/4, 0, 0)$ The six states of the second may be written

$$\gamma^* = \left|\begin{matrix}1\\2\end{matrix}\right\rangle, \beta^* = \left|\begin{matrix}1\\3\end{matrix}\right\rangle, \alpha^* = \left|\begin{matrix}2\\3\end{matrix}\right\rangle, \alpha = \left|\begin{matrix}1\\4\end{matrix}\right\rangle, \beta = \left|\begin{matrix}2\\4\end{matrix}\right\rangle, \gamma = \left|\begin{matrix}3\\4\end{matrix}\right\rangle;$$

the four states of the third may be written

$$\theta^* = \left|\begin{matrix}1\\2\\3\end{matrix}\right\rangle, \zeta^* = \left|\begin{matrix}1\\2\\4\end{matrix}\right\rangle, \xi^* = \left|\begin{matrix}1\\3\\4\end{matrix}\right\rangle, \eta^* = \left|\begin{matrix}2\\3\\4\end{matrix}\right\rangle.$$

The weights are additive; e.g., the weight of $\alpha = \left|\begin{matrix}1\\4\end{matrix}\right\rangle$ is the sum of the weights of $\eta = |1\rangle$ and $\theta = |4\rangle$.

The scalar $\eta\eta^* + \xi\xi^* + \zeta\zeta^* + \theta\theta^*$ allows ξ, ξ^* to be regarded as incompatible and the scalar $\alpha\alpha^* + \beta\beta^* + \gamma\gamma^*$ allows β, β^* to be considered as incompatible. There is a (001) quartet formed from (010) and (100) and a (100) quartet formed from (010) and (001); because of them the following pairs of states may be regarded as incompatible: α, ξ; α, ζ; β, ζ; β^*, ξ; α^*, ξ^*; α^*, ζ^*; β, ξ^*; β^*, ζ^*. Twelve types of state may be distinguished, each represented by a product of powers of nine ($= \frac{1}{2}(r+l)$) variables; the variables are $\eta, \eta^*, \gamma, \gamma^*, \theta, \theta^*$ together with one of the following twelve sets of three: $\alpha\xi^*\zeta^*$, $\alpha^*\xi\zeta$, $\alpha\beta^*\xi^*$, $\alpha\beta\zeta^*$, $\alpha^*\beta^*\zeta$, $\alpha^*\beta\xi$, $\beta^*\xi^*\zeta$, $\alpha\alpha^*\beta$, $\alpha\alpha^*\beta^*$, $\beta\xi\zeta^*$, $\xi\zeta\zeta^*$, $\xi^*\zeta^*\zeta$. It can be verified straightforwardly that the states thus defined are independent and have the correct number, i.e., imply the correct dimension formula for $SU(4)$.

The elementary scalars for $SU(4)$ are

$$\begin{array}{lll}
A_1 = (000, 010, 010), & A_2 = (010, 000, 010), & A_3 = (010, 010, 000), \\
B_{23} = (000, 100, 001), & B_{32} = (000, 001, 100), & B_{31} = (001, 000, 100), \\
B_{13} = (100, 000, 001), & B_{12} = (100, 001, 000), & B_{21} = (001, 100, 000), \\
C_1 = (010, 001, 001), & C_2 = (001, 010, 001), & C_3 = (001, 001, 010), \\
D_1 = (010, 100, 100), & D_2 = (100, 010, 100), & D_3 = (100, 100, 010), \\
E_1 = (101, 010, 010), & E_2 = (010, 101, 010), & E_3 = (010, 010, 101).
\end{array} \tag{2}$$

They are not independent, for C_iD_j is a linear combination of E_kB_{ij} and $A_kB_{ik}B_{kj}$; C_iE_i is a linear combination of $C_jA_jB_{ij}$ and $C_kA_kB_{ik}$; D_iE_i is a linear combination of $D_jA_jB_{ji}$ and $D_kA_kB_{ki}$; E_iE_j is a linear combination of

$A_kC_kD_k$ and $A_iA_jB_{ij}B_{ji}$. In the above ijk are 123 in any order. Accordingly we regard C_iD_j , C_iE_i , D_iE_i , E_iE_j as incompatible pairs; products containing both members of any pair are to be discarded. Fourteen types of van der Waerden invariant may be distinguished, each represented by a product of powers of twelve elementary scalars; the scalars are the three A's and six B's together with one of the following fourteen sets of three: $C_1C_2C_3$, $D_1D_2D_3$, $C_1C_2E_3$, $C_2C_3E_1$, $C_3C_1E_2$, $D_1D_2E_3$, $D_2D_3E_1$, $D_3D_1E_2$, $C_1D_1E_2$, $C_1D_1E_3$, $C_2D_2E_1$, $C_2D_2E_3$, $C_3D_3E_1$, $C_3D_3E_2$. An invariant containing fewer than three C's, D's and E's may belong to more than one type; to make the types mutually exclusive we assign such an invariant to the first type for which it qualifies in the above list.

When obvious symmetries are taken into account there are just three distinct types of van der Waerden invariant. Thus the first two types above differ only by conjugation. The third, fourth and fifth differ among themselves only by a relabeling of IR's and from the sixth, seventh and eight by conjugation. The last six differ among themselves by a relabeling of IR's. The indices of the powers of the scalars in each product are the needed twelve $(= \frac{1}{2}(r + 3l))$ labels, nine IR labels and three degeneracy labels.

The connection between elementary scalars and fundamental IR states is $A_2 \sim \gamma^*$, $A_3 \sim \gamma$, $B_{21} \sim \eta^*$, $B_{12} \sim \theta$, $B_{13} \sim \eta$, $B_{31} \sim \theta^*$, $C_2 \sim \xi^*$, $C_1 \sim \alpha$, $C_3 \sim \zeta^*$, $D_2 \sim \zeta$, $D_1 \sim \alpha^*$, $D_3 \sim \xi$, $E_2 \sim \beta$, $E_3 \sim \beta^*$; B_{23} , B_{32} and A_1 are ignored in setting up the correspondence. Then the ten types of van der Waerden invariant which do not contain the elementary scalar E_1 are seen to correspond respectively to the first ten types of internal state. The elementary scalar E_1 is bilinear in the (100) and (001) states of IR number 1 and should be made to correspond to $\zeta\zeta^*$. Since the elementary scalars D_2 and C_3 which correspond to ζ and ζ^* respectively are incompatible, E_1 is needed to provide invariants corresponding to states containing both ζ and ζ^*. Thus the invariants of type $C_3D_3E_1$ correspond to states of type $\xi\zeta\zeta^*$ in which the degree in ζ^* is greater than that in ζ; Those of type $D_2D_3E_1$ correspond to type $\xi\zeta\zeta^*$ in which the degree in ζ is greater than or equal to that in ζ^*. Similarly invariants of type $C_2D_2E_1$ correspond to states of type $\xi^*\zeta\zeta^*$ with degree in ζ greater than that in ζ^* and those of type $C_2C_3E_1$ correspond to type $\xi^*\zeta\zeta^*$ in which the degree in ζ^* is greater than or equal to that in ζ. The correspondence between product IR's and internal states is therefore complete, in agreement with Speiser's theorem, at least when there is no spilling. The demonstration that the van der Waerden invariant is complete and non-redundant even when spilling occurs is given in the appendix.

K. Ahmed and one of us (RTS) are using van der Waerden invariants as presented here to calculate $SU(4) \supset SU(2) \times SU(2)$ Wigner coefficients for certain simple couplings involving no degeneracy. An attempt is being made to derive van der Waerden invariants for other groups.

REFERENCES

1. E.P. Wigner, Phys. Rev. 51 (1937) 106; for a recent review see L.A. Radicati, 15th Solvay Meeting, Brussels (1970), to be published.
2. V. Amar, A. Esteve and P.G. Sona, Nuovo Cimento 30 (1963) 722; J.-P. Antoine, D. Speiser and R.J. Oakes, Phys. Rev. 141 (1966) 1542.
3. M. Moshinsky, J. Math. Phys. 4 (1963) 1128.
4. T.A. Brody, M. Moshinsky and I. Renero, J. Math. Phys. 6 (1965) 1540 and Rev. Mex. Fís. 15 (1966) 145.
5. L.C. Biedenharn, A. Giovanni and J.D. Louck, J. Math. Phys. 8 (1969) 691.
6. B.L. van der Waerden, *Die Gruppentheoretische Methode in der Quantenmechanik*, Verlag Julius Springer, Berlin (1932).
7. V. Heine, *Group Theory in Quantum Mechanics*, Pergamon Press, New York (1960); G.Y. Lyubarskii, *The Application of Group Theory in Physics*, Pergamon Press, New York (1960); M. Hamermesh, *Group Theory*, Addison-Wesley Publishing Co., Reading, Mass. (1962).
8. See however 2 for references to applications to $SU(3)$; the method has been applied to $O(5)$ by D.E. Tilley and R.T. Sharp (unpublished).
9. D. Speiser, *Lectures of the Istanbul Summer School of Theoretical Physics*, Vol. 1, Gordon & Breach, New York (1962).
10. E. Cartan, These Paris (1894) reprinted in E. Cartan, *Oeuvres Completes*, Gauthier-Villars, Paris, France (1952); a clear discussion is given by R.E. Behrends, J. Dreitlein, C. Fronsdal and W. Lee, Rev. Mod. Phys. 34 (1962) 1 E, Theorem IV.
11. H. Weyl, *The Classical Groups*, Princeton University Press, Princeton (1946).
12. R.T. Sharp and C.S. Lam, J. Math. Phys. 10 (1969) 2033.
13. V.S. Devi and T. Venkatarayudu, J. Math. Phys. 9 (1968) 1057.
14. V.S. Devi and M. Moshinsky, J. Math. Phys. 10 (1969) 455.
15. V. Bargmann and M. Moshinsky, Nucl. Phys. 23 (1961) 177; V.S. Devi, Nuovo Cimento 68A (1970) 1; M. Brunet and M. Resnikoff, J. Math. Phys. 11 (1970) 1474; K. Ahmed and R.T. Sharp, Ann. Phys. (to be published).
16. E.P. Wigner, *Group Theory*, Academic Press, New York (1959).

17. P.-Y. Hou, Scientia Sinica 14 (1965) 367; G. Ponzano, Nuovo Cimento 41A (1966) 142; C.-K. Chew and R. T. Sharp, Nucl. Phys. B2 (1967) 697.

RESUMEN

Se encuentra una solución para el problema de clasificación externa en $SU(4)$, que es simétrica hasta una fase, construyendo explícitamente el invariante general de van der Waerden, como un producto de escalares elementales. Se demuestra la conexión entre la multiplicidad externa y la interna exhibiendo la correspondencia biunívoca entre productos de representaciones irreducibles y los estados internos. Se tratan igualmente, a manera de ilustración, los casos de $SU(2)$ y $SU(3)$.

APPENDIX

The van der Waerden invariants given in the body of this paper for $SU(2)$, $SU(3)$ and $SU(4)$ were justified by demonstrating, with the help of Speiser's theorem, that they imply the correct Clebsch-Gordan series for $J_1 \otimes J_2$; the demonstration is valid only when the Cartan labels of J_2 are so large that the weight diagram of J_1 centered on the heaviest state of J_2, does not spill into the region of negative Cartan labels, or equivalently, that products of powers of elementary scalars can be formed corresponding to all states of J_1. The purpose of this appendix is to extend the proof to the case where spilling occurs. Because of the desirability of extending the treatment to higher groups we would like to present an elegant, general proof; since we have not found one, the following cumbersome, specific demonstration is offered.

Nothing further needs to be said about $SU(2)$, since if there is spilling for $J_1 \otimes J_2$ there is none when the roles of J_1 and J_2 are interchanged.

Speiser's reflection and cancellation rules are based on the fact that the character function, characteristic function, or dimension formula changes sign when the arguments (Cartan labels, not group transformation parameters) are reflected in a hyperplane $\lambda_i = -1$ (λ_i is a Cartan label). When summing over the character, characteristic or dimension, then, it is legitimate to reflect the lower limit of the sum in a hyperplane $\lambda_i = 0$ (only alternate integer values of λ_i appear in a line perpendicular to $\lambda_i = 0$); the terms omitted or added by reflecting the lower limit all cancel. Our proof will consist of showing that by the use of such reflections all lower limits on sums due to the smallness of the Cartan labels of J_2 may be removed and only those appropriate to J_1 retained. It then follows that our van der Waerden invariant implies a character or dimension for $J_1 \otimes J_2$ given by the same mathematical formula which holds when the labels of J_2 are large and there is no spilling; it is known of course that this formula, just the product of the characters or dimensions of J_1 and J_2, is the correct one.

Consider $SU(3)$ first. Since the elementary scalar $B_{12} = (10, 01, 00)$ is a scalar in the 3-variables and compatible with all other elementary scalars, it is apparent that the product IR's in $(p_1, q_1) \otimes (p_2, q_2)$ containing it as a factor are just the IR's in the product $(p_1 - 1, q_1) \otimes (p_2, q_2 - 1)$; a similar remark applies to B_{21} (p and q are the Cartan labels for $SU(3)$). Hence we omit the scalars B_{12} and B_{21} from our list and recognize that the product IR's that remain should be those of $(p_1, q_1) \otimes (p_2, q_2) - (p_1 - 1, q_1) \otimes (p_2, q_2 - 1) - (p_1, q_1 - 1) \otimes (p_2 - 1, q_2) + (p_1 - 1, q_1 - 1) \otimes (p_2 - 1, q_2 - 1)$; this trick simplifies the proof considerably. Those subtracted van der Waerden invariant containing the elementary scalar C to the power a yield product IR's with $(p, q) = (p_1 + p_2 - 2a,\ q_1 + q_2 + a)$, $0 \leqslant a \leqslant \min p_1, p_2$. These IR's lie on a line perpendicular to $p = 0$ and the IR with lowest p is either $(p_2 - p_1, q_1 + q_2 + p_1)$ or $(p_1 - p_2,\ q_1 + q_2 + p_2)$, which are images of each other in $p = 0$ (to reflect in $p = 0$ substitute $p \to -p,\ q \to q + p$). Similarly the subtracted van der Waerden invariants with C^* lie in a line perpendicular to $q = 0$ and the lower limits on q implied by q_1 and q_2 are images in $q = 0$. To complete the argument, one can verify that the elementary factors lead to the correct character or dimension formula for those boundary cases in which one of p_1, q_1, p_2, q_2 vanishes.

Turning to $SU(4)$ we drop the elementary scalars A_3, B_{12}, B_{21} which are scalars in the 3-variables and consider the appropriate triply subtracted character or dimension formula. For ease of reference we call the fourteen types of invariant defined in § 3 W_1 to W_{14} in the order in which they appear there.

The the product IR's $(\lambda\mu\nu)$ corresponding to W_1 and W_3 are those lying in the hexagonal face

$$3\lambda + 2\mu + \nu = 3(\lambda_1 + \lambda_2) + 2(\mu_1 + \mu_2) + \nu_1 + \nu_2$$

of the Speiser diagram. Their distribution on the face and their properties under reflection in $\mu = 0$ and $\nu = 0$ are identical to those of an $SU(3)$ Speiser diagram with μ, ν playing the role of p, q. We conclude that W_1 and W_3 taken together have the reflection properties required by Speiser's theorem. W_2 and W_4 behave like W_1 and W_3 with the roles of λ and ν interchanged.

We divide the remaining ten types of invariant into two sets of five each, the first set including W_6, W_8, W_9, W_{10}, W_{13}, the second W_5, W_7, W_{11}, W_{12}, W_{14}. Each of the ten types defines a three-dimensional sub-region of $\lambda\mu\nu$ space in which the points $\lambda\mu\nu$ form a regular lattice with unit multiplicity. The five sub-regions of each set fit together with no overlapping to form a single region of fairly simple shape. Since the two regions go into each other on interchanging the IR's 1 and 2, it is necessary to discuss in detail only the first region, R_I, comprising $W_{6,8,9,10,13}$.

R_I consists of the region of $\lambda\mu\nu$ space common to two volumes, whose lower boundaries are determined by the Cartan labels of the first and second IR's respectively and whose upper boundaries are common. The common upper boundaries are the three planes

$$\lambda + 2\mu + \nu \leqslant \lambda_1 + \lambda_2 + 2(\mu_1 + \mu_2) + \nu_1 + \nu_2 ,$$

$$\lambda + 2\mu + 3\nu \leqslant \lambda_1 + \lambda_2 + 2(\mu_1 + \mu_2) + 3(\nu_1 + \nu_2) ,$$

$$3\lambda + 2\mu + \nu \leqslant 3(\lambda_1 + \lambda_2) + 2(\mu_1 + \mu_2) + \nu_1 + \nu_2 .$$

The lower boundaries set by λ_1, μ_1, ν_1 are

$$\lambda + 2\mu + 3\nu \geqslant \lambda_1 + \lambda_2 + 2(\mu_1 + \mu_2) + 3\nu_2 - \nu_1 ,$$

$$3\lambda + 2\mu + \nu \geqslant 3\lambda_2 - \lambda_1 + 2(\mu_1 + \mu_2) + \nu_1 + \nu_2 ,$$

$$\lambda + \nu \leqslant \lambda_1 + \lambda_2 + 2\mu_1 + \nu_1 + \nu_2$$

while those set by λ_2, μ_2, ν_2 are

$$\lambda + 2\mu - \nu \leqslant \lambda_1 + \lambda_2 + 2(\mu_1 + \mu_2) + 3\nu_2 - \nu_1 ,$$

$$-\lambda + 2\mu + \nu \leqslant 3\lambda_2 - \lambda_1 + 2(\mu_1 + \mu_2) + \nu_1 + \nu_2 ,$$

$$\lambda + 2\mu + \nu \geqslant \lambda_1 + \lambda_2 + 2\mu_1 + \nu_1 + \nu_2 .$$

We want to show that either set of lower boundaries may be dropped.

First consider dropping the second set of lower boundaries, those determined by λ_2, μ_2, ν_2. Reflect the first of the three planes in $\nu = 0$ $(\lambda \to \lambda,\ \mu \to \mu + \nu,\ \nu \to -\nu)$. It goes into the first lower boundary plane of the first set; moreover the other two planes of the second set are invariant under this reflection (i. e., perpendicular to $\nu = 0$) and may be extended if necessary to the new boundary. Next reflect the second boundary of the second set in $\lambda = 0$ $(\lambda \to -\lambda,\ \mu \to \mu + \lambda,\ \nu \to \nu)$. It goes into the second boundary of the first set, while the remaining third plane is invariant and may be extended. Finally reflect the third boundary of the second set in $\mu = 0$ $(\lambda \to \lambda + \mu,\ \mu,\ \mu \to -\mu,\ \nu \to \nu + \mu)$; it goes into the third of the first set. To justify dropping of the first set (and retaining the second) the same three reflections are performed in reverse order. The second region,

R_{II} , need not be discussed separately, for it differs from the first only in the exchange of IR labels 1 and 2.

Since we are considering a subtracted character or dimension function, or Clebsch-Gordan series, it is necessary for completion of the proof to verify those boundary cases with one of λ_1, μ_1, ν_1, λ_2, μ_2, ν_2 equal to zero. This can be done by methods similar to those used above. The details are considerably simpler and are omitted here.