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MONTE CARLO STUDIES OF A CLASS OF REAL SYMMETRIC MATRICES*

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ABSTRACT: A class of ensembles of real symmetric matrices H is defined by $H = A_1^T A_1 + \xi^2 A_2^T A_2$ where the A_i are real asymmetric matrices whose elements are independent Gaussian random variables. By means of Monte Carlo calculations we study the density and spacing distributions of the eigenvalues. For all values of ξ^2 considered, except for a statistically insignificant fraction, the eigenvalues all lie within a finite interval, near the ends of which the behaviour of the level density disagrees with what is observed for nuclear levels. On the other hand, the nearest-neighbour spacing distribution corresponds to the Wigner surmise. The results for higher-order spacings are also discussed.

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I. INTRODUCTION

For the study of the spectra of complex quantum-mechanical systems an ensemble of Hermitian random matrices is introduced, with the hope that such an ensemble can represent the statistical properties of the Hamiltonian[2]. Averaging over the ensemble, one obtains the level density – a global property – and the spacing distributions – local properties.

Many different ensembles of random matrices have been proposed[2]. So far, however, none fits the experimental data more than partially, nor is there an adequate theoretical basis for choosing among them. In this paper we analyze numerically the properties of a further ensemble recently proposed by Wigner[3].

A weakness common to most of the ensembles suggested up to now is that they do not take into account the fact that a quantum-mechanical Hamiltonian has a lower bound to its eigenvalues. Wigner's proposal remedies this defect, by defining the ensemble of matrices

$$H = \mathrm{Re}\, A^{+} A \qquad (1)$$

where A is an asymmetric complex matrix, the real and imaginary parts of whose elements are independent random variables, taken from a normal distribution of width one*. If A_1 is the real and A_2 the imaginary part of A, H can also be written as

$$H = A_1^T A_1 + A_2^T A_2 \qquad (2)$$

where A_i^T is the transpose of A_i.

In the next section we give the results of a Monte Carlo calculation for the ensemble (2). We discuss both the level density (which we compare with the theoretical results**) and the spacing distributions.

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* In the SIAM Review, Wigner[4] had proposed a simpler form: $H = A^2$ where A belongs to the Gaussian orthogonal ensemble. However, the density follows the law $(E^{-1} - 1)^{\frac{1}{2}}$, which is not equal to the quarter-of-circle law and does not agree with the experimental data.

** See the following paper by F. J. Dyson[5].

In section III we consider a generalization of this ensemble, where

$$H = A_1^T A_1 + \xi^2 A_2'^T A_2' \quad . \tag{3}$$

This corresponds to selecting the matrix elements of A_1 from a normal distribution of standard deviation 1, and those of $A_2 = \xi A_2'$ from one of standard deviation ξ.

II. MONTE CARLO CALCULATIONS FOR $\xi = 1$

We first generate a number of matrices according to the prescription (2) and then diagonalize them by the Householder–Givens method.

To determine a suitable value for the dimensionality N, the density distributions for various choices of N were fitted to the theoretical curve[5] for $N \to \infty$; the probabilities of the corresponding values of χ^2 are shown in Fig. 1. Since Dyson's theory predicts abrupt limits to the region where the

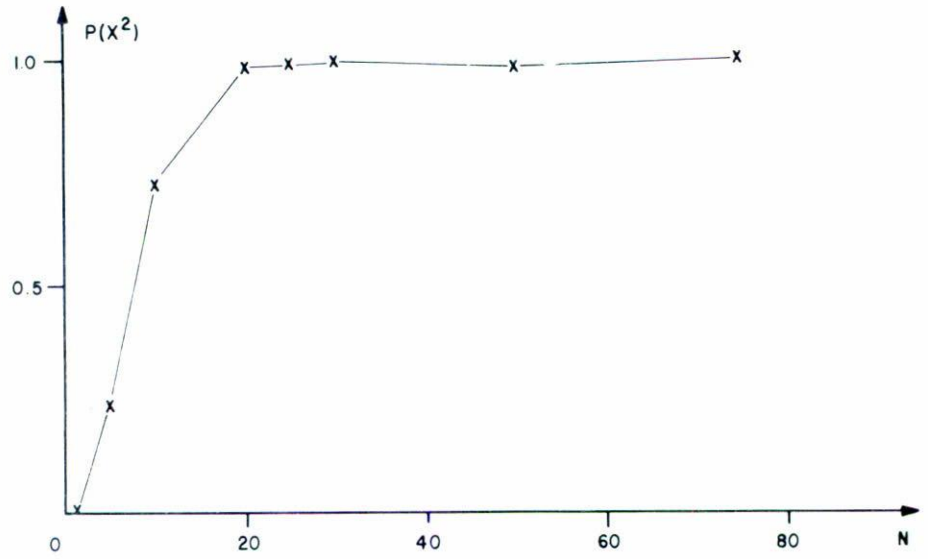

Fig. 1. $P(\chi^2)$ from the χ^2 fit of Dyson's theoretical distribution to the level-density histograms for different dimensionalities N.

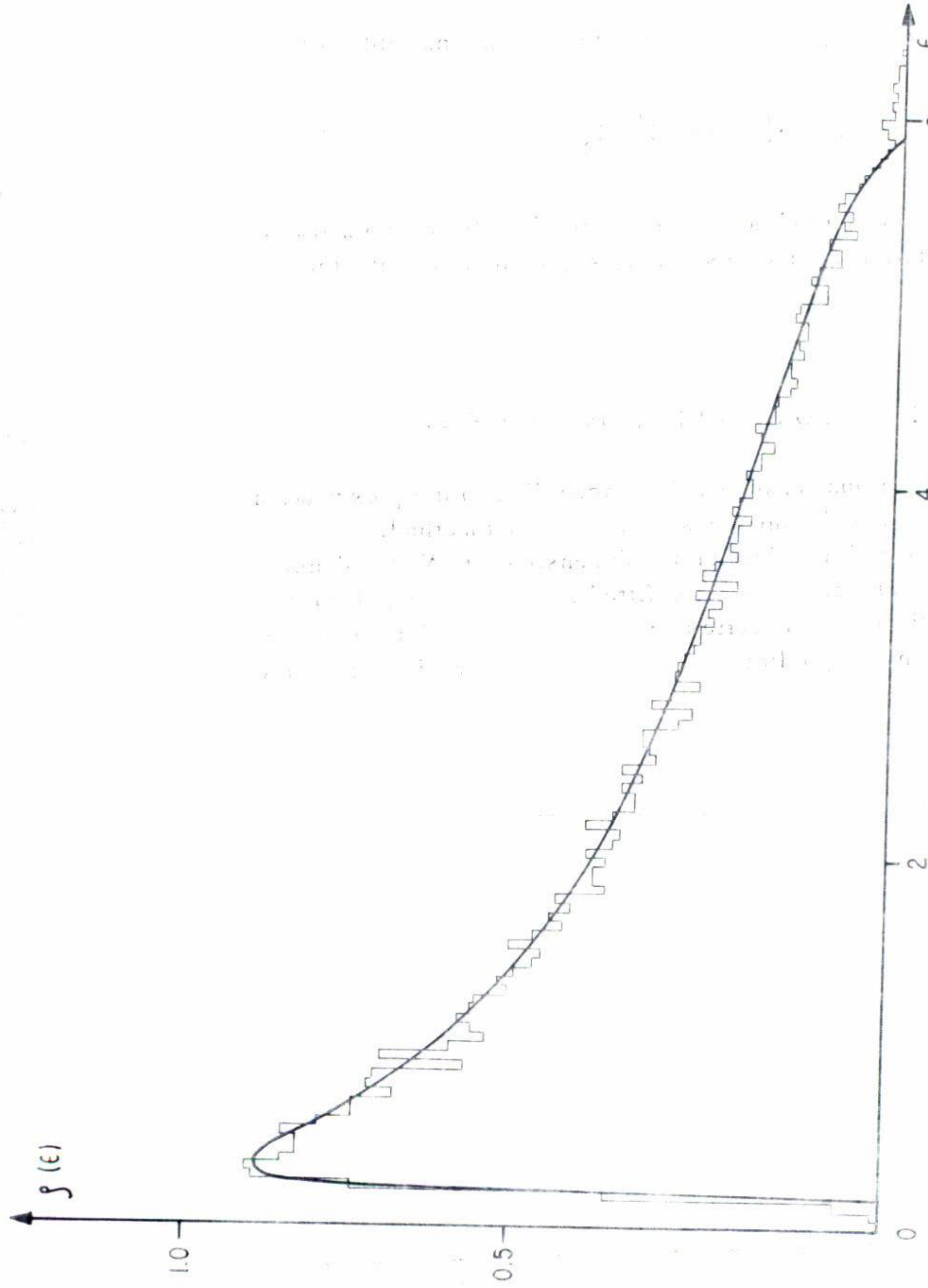

Fig. 2. Level density for a set of 800 matrices with $N = 50$ and $\hbar = 1$. The units used are $\epsilon = E/(2N)$.

theoretical frequency differs from zero, we have used only the eigenvalues falling within this region for the calculation of χ^2. The two tails falling outside the allowed region

$$3 - 2\sqrt{2} \lesssim \epsilon \lesssim 3 + 2\sqrt{2} \tag{4}$$

correspond for all $N \gtrsim 5$ to approximately 0.12 eigenvalues per matrix; this number is independent of N. This, as asserted by Dyson[5], is in qualitative agreement with the result found by Bronk[6] for the semi-circle distribution of the Gaussian orthogonal ensemble, except that the probability is in our case twice as high. It is this fact which enables us to ignore the tails and to obtain the data of fig. 1, from which we may conclude that $N = 30$ is already and excellent approximation to the limit $N \to \infty$. The Kolmogorov-Smirnov[7] test applied to the data confirms this conclusion. We have therefore used this value $N = 30$ for more extensive calculation.

We show in fig. 2 the histogram for the level density from a set of 800 matrices with $N = 30$. The units used were $\epsilon = E/(2N)$. We can see that the Monte Carlo data follow the theoretical curve rather well.

We now turn out attention to the k-th neighbour-spacing distributions $p\,(k\,;\,s)$ for the same data as above. The histograms for $p\,(k\,;\,s)$ were obtained numerically by selecting only one k-th neighbour spacing from each matrix of the set of 800.

For $k = 0$ three different cases were considered. In the first one, we constructed $p\,(0\,;\,s)$, with s being the difference in energy between the ground and first excited states of each matrix. The result is shown in fig. 3, where the histogram is compared to the Wigner distribution

$$p\,(0\,;\,s) = \frac{\pi}{2}\, s \exp\left[-\frac{1}{4}\,\pi s^2\right], \tag{5}$$

which is obeyed very closely by the Gaussian and circular ensembles[8]. For our data, the fit is such that $P\,(\chi^2) = 0.15$. (Most of the discrepancy arises from the first two classes; if they are joined, $P\,(\chi^2) = 0.82$.) As a second case we consider the spacing between the first and second excited states, which yield a similar result (see fig. 4).

As a third case we have evaluated $p\,(0\,;\,s)$ in the region near $\epsilon = 1/6$, which corresponds to the maximum in the level density. The distance considered was the one between the two eigenvalues immediately above $\epsilon = 1/6$ for each matrix. Again the Wigner distribution is followed rather closely, as shown in fig. 5. Here $P(\chi^2) = 0.64$. Over the 800 matrices considered, the spacing is that between the (3.05 ± 0.65) th state and the next one.

For the highest-density region we have also analyzed $p(k; s)$ for $k = 1, 2, 3$ and 4. Larger values of k were not considered since then the density varies appreciably. The resulting histograms are given in fig. 6, where they are compared with the theoretical distributions for the Gaussian orthogonal ensemble. For all k, the spacings were adjusted by a parameter λ_k so that their average value is equal to $k + 1$, as it should be were the density constant. The values of λ_k are also given in fig. 6.

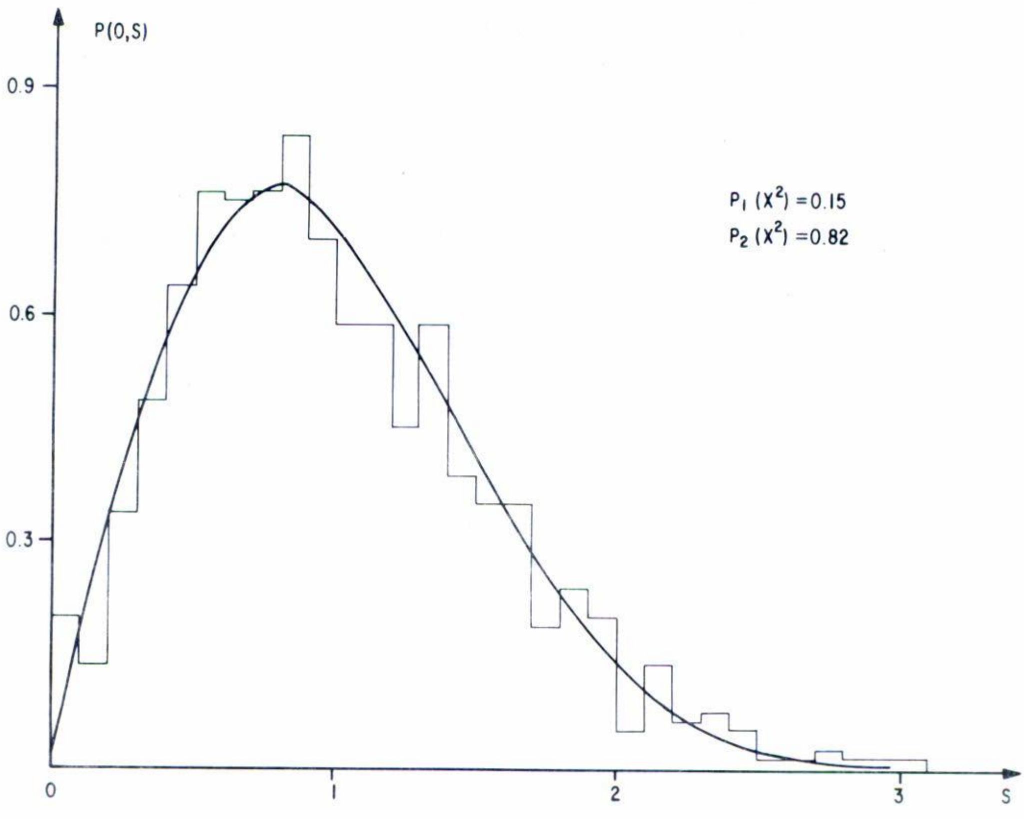

Fig. 3. Nearest-neighbour spacing distribution $p(0; s)$ for 800 matrices with $N = 30$ and s equal to the difference between the ground and first excited states in units of the mean spacing. The smooth curve is the Wigner distribution (5).

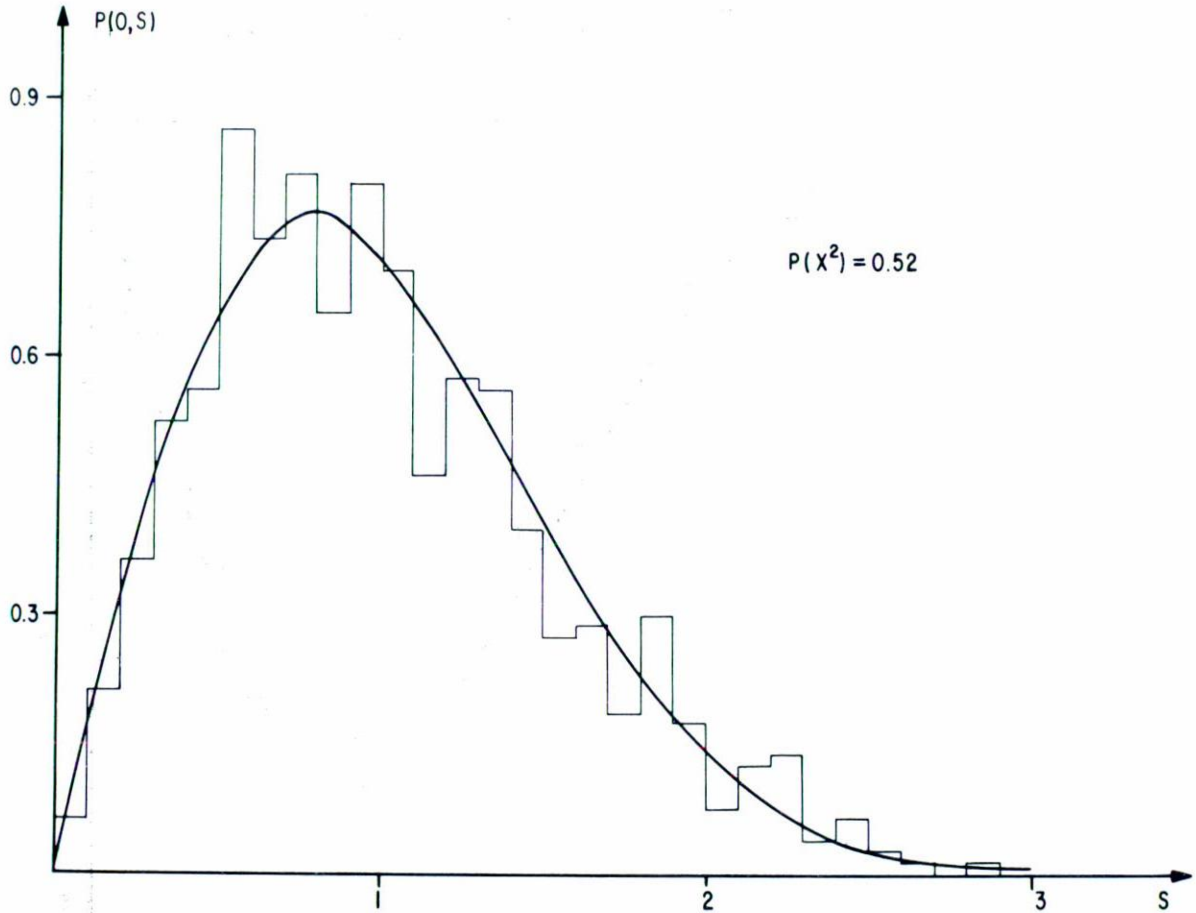

Fig. 4. The distribution $p(0\,;\,s)$ for the same set of matrices as in fig. 3 but with s equal to the difference between the first and second excited states. The smooth curve is the Wigner distribution (5).

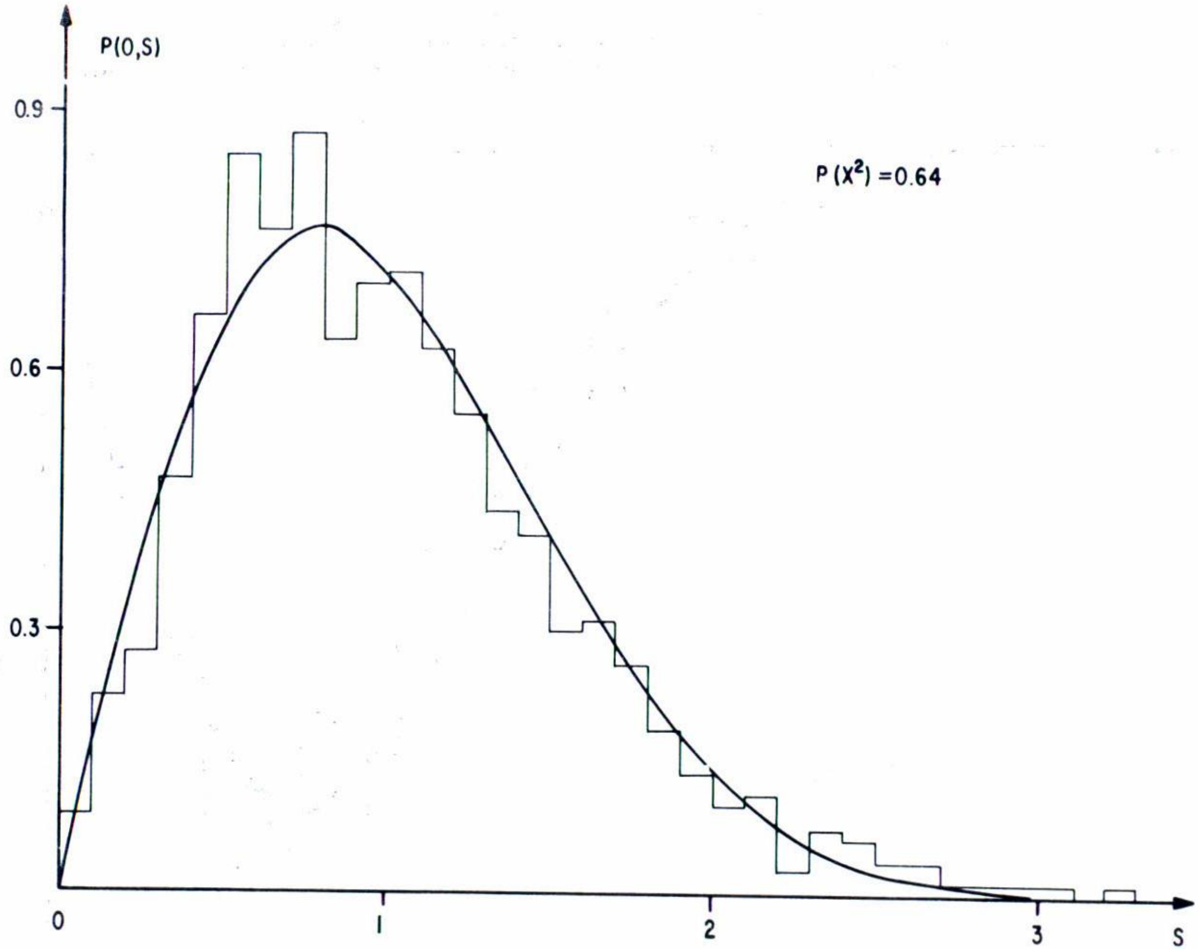

Fig. 5. The distribution $p(0\,;\,s)$ for the same set of matrices as in fig. 3. The smooth curve is the Wigner distribution (5). Here $p(0\,;\,s)$ is evaluated in the highest-density region, as defined in the text.

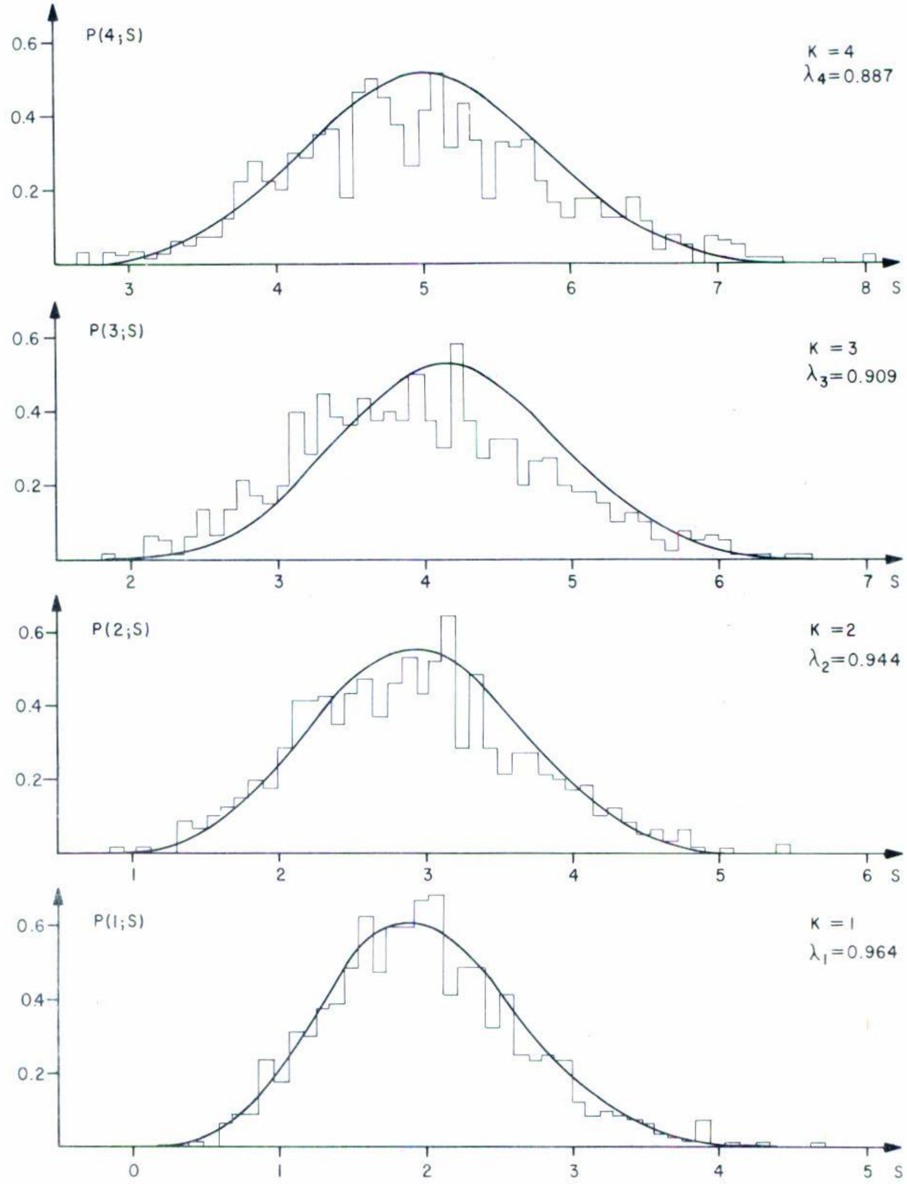

Fig. 6. The distribution $p(k;s)$, $k = 1, 2, 3, 4$, for the same set of 800 matrices as in fig. 3; the theoretical distributions correspond to the Gaussian orthogonal ensemble. Here $p(k;s)$ is evaluated in the highest-density region and λ_k is a parameter whose value is adjusted to correct for variations in the density.

III. MONTE CARLO CALCULATIONS FOR $\xi = 1$

Two cases, $\xi = 2$ and $\xi = 5$, are considered. For both a set of 400 matrices with $N = 30$ was analyzed. In figs. 7 and 8 we show the eigenvalue density for $\xi = 2$ and $\xi = 5$, respectively. The density is shown as a function of ϵ, where

$$\epsilon = \frac{E}{N(1+\xi^2)} , \qquad (6)$$

which was found to give the centroid at $\epsilon = 1$. We can see that the general behaviour of the density is like that for $\xi = 1$, except that the maximum of the density increases and shifts towards smaller values of ϵ, as ξ increases.

Values of $\xi < 1$ have not been considered, since the results for ξ are equivalent to those for $1/\xi$ except for a change of scale in the abscissa. Hence the conclusions obtained from $\xi = 2$ and $\xi = 5$ hold also for $\xi = 0.5$ and $\xi = 0.2$, respectively. Furthermore, as $\xi \to \infty$ the density tends to the same limit as for $\xi \to 0$, a case for which theoretical results have been derived[5].

We have also analyzed the nearest-neighbour spacing distribution for the two values of ξ, for the region around the second eigenvalue. The results are shown in figs. 9 and 10. As we can see, the Wigner distribution is followed quite well.

IV. CONCLUSION

As can be seen from the results we have given here, the ensemble of matrices (3), for all values of ξ, does not reproduce the nuclear global density of levels. The local properties, on the other hand, are similar to those predicted by the Gaussian orthogonal ensemble which, it seems, are in reasonable agreement with the experimental data[9].

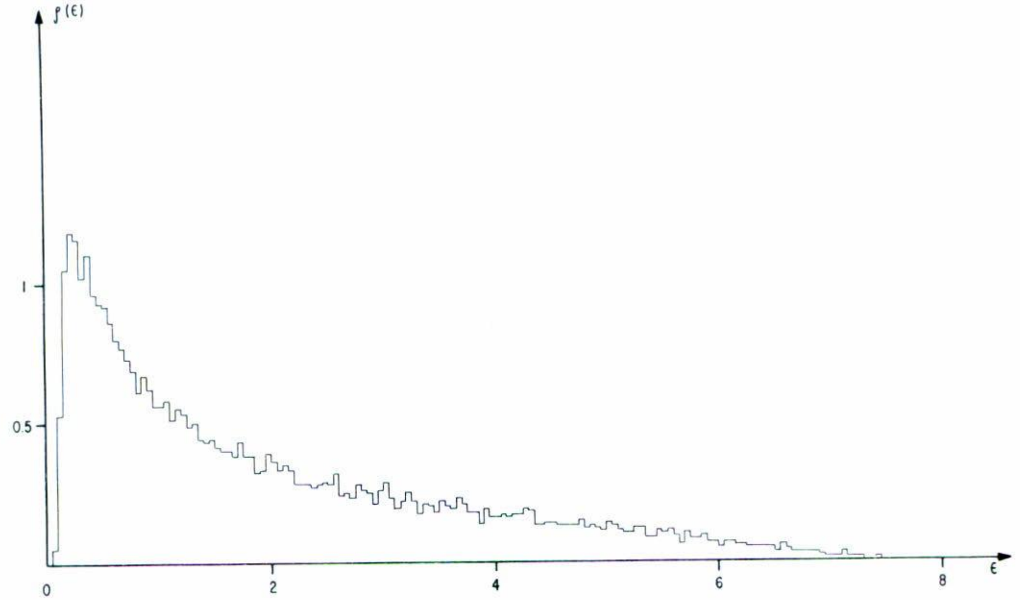

Fig. 7. Level density for a set of 400 matrices with $N = 30$ and $\xi = 2$.

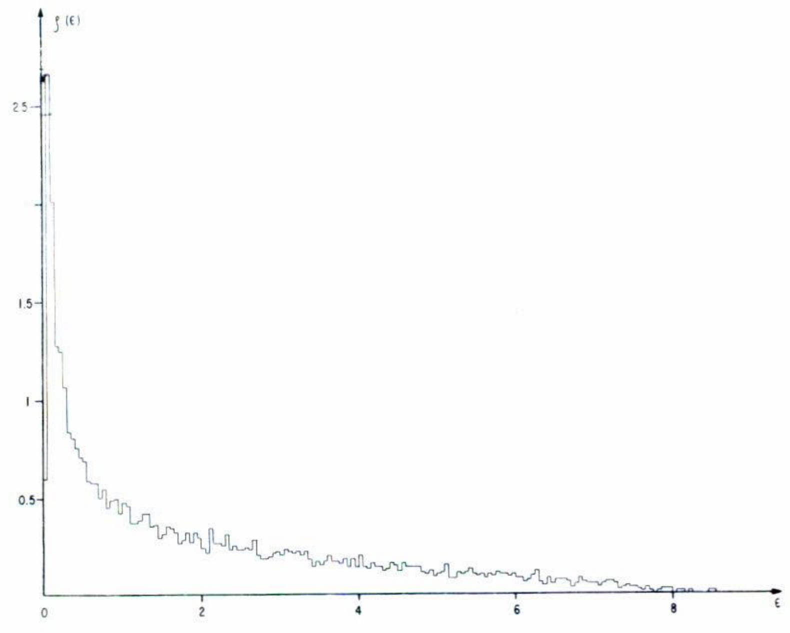

Fig. 8. Level density for a set of 400 matrices with $N = 30$ and $\xi = 5$.

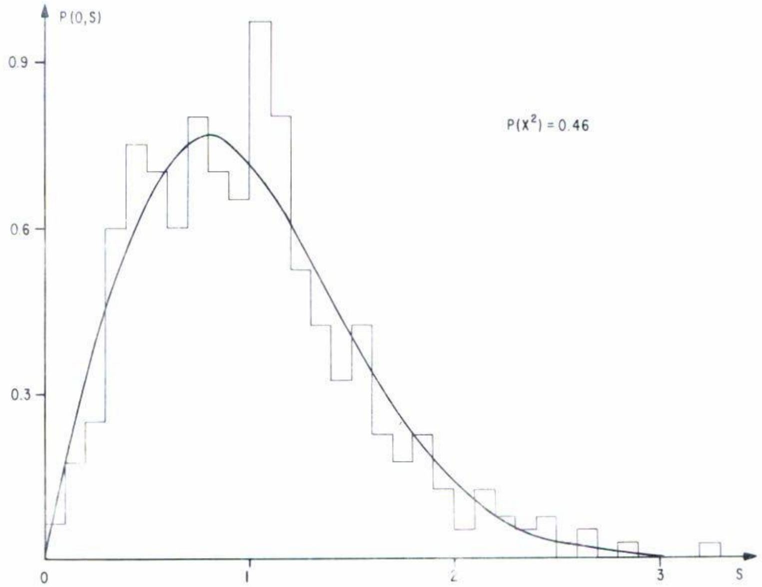

Fig. 9. Nearest-neighbour spacing distribution $p(0;s)$ for a set of 400 matrices with $N = 30$ and $\xi = 2$, compared with the Wigner distribution

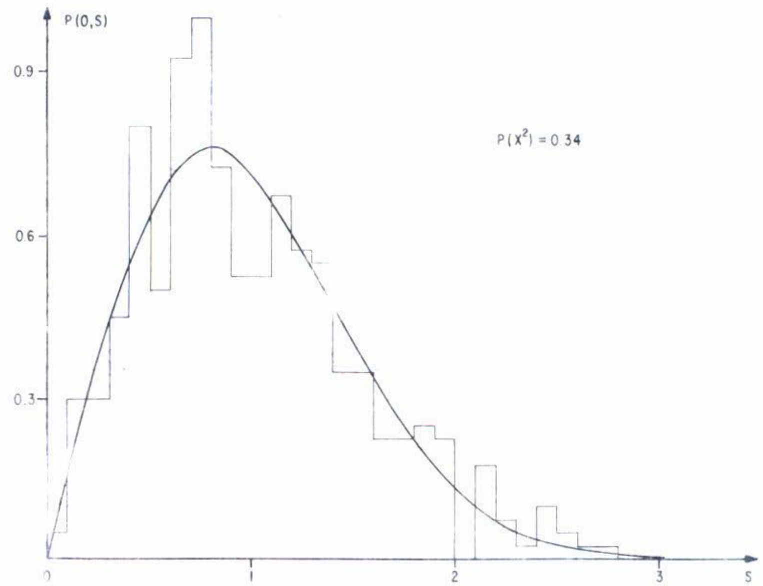

Fig. 10. Histogram for $p(0;s)$ obtained for a set of 400 matrices with $N = 30$ and $\xi = 5$, compared with the Wigner distribution.

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