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DISTRIBUTION OF EIGENVALUES FOR A CLASS OF REAL SYMMETRIC MATRICES

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ABSTRACT: An ensemble of real symmetric matrices H is defined by $H = \mathrm{Re}(M^{+}M)$, where M is an asymmetric complex matrix whose elements are independent Gaussian random variables. Using the theory of Brownian motion, the probability-distribution of the eigenvalues of H is determined exactly. When the order of the matrices tends to infinity, the density of eigenvalues in the neighbourhood of E tends to $x^{-1}(6x - 1 - x^2)^{\frac{1}{2}}$, where $x = \frac{1}{2}(E/E)$ and E is the mean of the eigenvalues. The density is zero in the interval $0 < E < (3/2 - \sqrt{2})\, E$.

I. STATEMENT OF RESULTS

Bohigas et al (see the preceeding paper[1] have calculated numerically the eigenvalues of matrices chosen by a Monte Carlo technique from an ensemble suggested by Wigner[2] as a possible statistical model for a nuclear Hamiltonian. We here analyze Wigner's ensemble by constructing a Brownian motion model for it, following the method of Dyson[3]. The Brownian model

leads to a simple and exact determination of the distribution of eigenvalues. The results agree in detail with the calculations of Bohigas et al.

Wigner's ensemble consists of matrices H of the form

$$H = \mathrm{Re}(M^{\dagger}M) = A^{T}A + B^{T}B \ , \tag{1.1}$$

where

$$M = A + iB \ , \tag{1.2}$$

and A, B are real asymmetric matrices whose elements are independent Gaussian random variables with equal variance v. The ensemble (1.1) is the case $k = 2$ of the ensemble W_k defined by

$$H = \sum_{r=1}^{k} (A_r{}^{T} A_r) \ , \tag{1.3}$$

where k is a positive integer and the A_r are k independent real asymmetric Gaussian random matrices. Let N be the order of the matrices A_r and H. Our results are as follows. The joint probability distribution of the eigenvalues $(E_1, \ldots, E_N)$ of a matrix H in W_k is

$$p(E_1, \ldots, E_N) = c_{kN} \prod_{j=1}^{N} \{ E_j^{\alpha} \exp(-E_j/2v) \} \prod_{i<j} |E_i - E_j| \ ,$$

where

$$\alpha = \frac{1}{2}[(k-1)N + 1] \tag{1.4}$$

and c_{kN} is a normalization constant. The limiting form as $N \to \infty$ of the single single-eigenvalue density distribution for W_k is

$$\rho(E) = (2\pi v x)^{-1} (2(k+1)x - (k-1)^2 - x^2)^{1/2} \ , \quad a < x < b \ ,$$

$$\rho(E) = 0 \ , \qquad x < a \quad \text{or} \quad x > b \ , \tag{1.6}$$

where

$$x = (E/N\nu) \ , \tag{1.8}$$

$$a = (\sqrt{k} - 1)^2 \ , \tag{1.9}$$

$$b = (\sqrt{k} + 1)^2 \ . \tag{1.10}$$

The distribution (1.6) has its centroid at

$$\overline{E} = kN\nu \tag{1.11}$$

and its peak at

$$E_{max} = (k-1)^2 \, \overline{E}/(k(k+1)) \ . \tag{1.12}$$

For $k = 1$ the density decreases monotonically from infinity at $E = 0$ to zero at $E = 4\overline{E}$. In this case, which was also briefly discussed by Wigner[4], the eigenvalue distribution is identical to that found by Bronk[5] for the ensemble of complex Hermitian matrices

$$H = M^{\dagger}M \ . \tag{1.13}$$

When expressed in terms of the variable $E^{\frac{1}{2}}$ instead of E, the distribution becomes a quadrant of a circle. For $k = 2$ (the case suggested by Wigner[2] the distribution is still extremely unsymmetrical with its peak at $E_{Max} = 1/6 \ \overline{E}$. When k is large the distribution tends to a semi-circle with center at $E = (k+1)N\nu$ and radius $2k^{\frac{1}{2}}N\nu$.

According to Bronk[6], the probability that a single eigenvalue lies far outside the limits of the allowed interval (1.6) tends to zero extremely rapidly as $N \to \infty$. Bronk proved this for the usual Gaussian matrix ensemble for which the distribution of eigenvalues is a semicircle. His argument extends with only minor modification to the cases considered here. Hence the ratio of the largest to the smallest eigenvalue of the matrix (1.3) tends with probability 1 to the definite limit

$$(b/a) = (\sqrt{k} + 1)^2 (\sqrt{k} - 1)^{-2} \tag{1.14}$$

as $N \to \infty$. In particular, for the Wigner ensemble (1.1) this ratio is

$$(b/a) = (\sqrt{2} + 1)^4 = 33.97 . \tag{1.15}$$

II. PROOFS

The matrix elements of H are given by (1.3) as quadratic forms

$$H_{ij} = \sum_{r=1}^{k} \sum_{m=1}^{N} A_{rmi} A_{rmj} \tag{2.1}$$

in the random variables A_{rmn}. The ensemble W_k is the unique stationary state of the Brownian process in which each A_{rmn} is independently fluctuating according to the rules

$$< \delta A_{rmn} > = - v^{-1} A_{rmn} \delta t , \tag{2.2}$$

$$< \delta A_{rij} \delta A_{smn} > = 2 \delta_{rs} \delta_{im} \delta_{jn} \delta t . \tag{2.3}$$

The fluctuation of the eigenvalue E_j of H is given by second-order perturbation theory,

$$\delta E_j = \delta H_{jj} + \sum_{l \neq j} (E_j - E_l)^{-1} \delta H_{ji} \delta H_{lj} , \tag{2.4}$$

in the representation in which H is diagonal. We substitute (2.1) into (2.4) and calculate the ensemble averages to first order in δt using (2.2) and (2.3). In this way we find

$$< \delta E_j > = 2 \delta t \left[kN - v^{-1} E_j + \sum_{l \neq j} (E_j - E_l)^{-1} (E_j + E_l) \right] \tag{2.5}$$

$$< \delta E_i \, \delta E_j > = 8 \delta t \, E_j \, \delta_{ij} \tag{2.6}$$

The Brownian process defined by (2.5) and (2.6) can equivalently be described by the Ornstein-Uhlenbeck equation

$$\frac{1}{2} \frac{\partial p}{\partial t} = \sum_j \frac{\partial}{\partial E_j} \Big[2E_j \frac{\partial p}{\partial E_j} - \big((k-1)N + 1 - v^{-1} E_j + 2E_j \sum_{l \neq j} (E_j - E_l)^{-1} \big) p \Big] \tag{2.7}$$

for the time-dependent joint distribution function $p(E_1, \ldots, E_N, t)$ of the eigenvalues of H. The stationary solution of (2.7) must satisfy

$$(\partial p / \partial E_j) = \big(\alpha E_j^{-1} - \frac{1}{2} v^{-1} + \sum_{l \neq j} (E_j - E_l)^{-1} \big) p \, , \tag{2.8}$$

with α given by (1.5). The distribution (1.4) is uniquely determined by (2.8). This proof of (1.4) succeeds because the ensemble averages (2.5) and (2.6) involve only the eigenvalues of H and not the individual matrices A_r. If for example the various A_r had occurred in (2.1) multiplied by unequal coefficients, we could not have eliminated the A_r from (2.5), and this method of deducing the eigenvalue distribution would have failed.

The distribution (1.4) is the canonical equilibrium state at temperature $T = 1$ of a classical Coulomb gas of N point charges, with the potential energy

$$w = \sum_{j=1}^{N} ((E_j / 2v) - \alpha \log E_j) - \sum\sum_{i<j} \log | E_i - E_j | \, , \tag{2.9}$$

free to move on the semi-infinite line $0 < E_j < \infty$. As $N \to \infty$ the Coulomb gas can be approximated by a continuous charge-distribution $\rho(E)$ with energy

$$w = \frac{1}{2}\int (v^{-1}E - (k-1)N \log E)\, \rho(E)\, dE$$

$$- \frac{1}{2}\iint \log |E - E'|\, \rho(E)\, \rho(E')\, dE\, dE' \quad . \tag{2.10}$$

The single-eigenvalue distribution-function $\rho(E)$ is the unique non-negative function which minimizes (2.10) subject to the constraint

$$\int \rho(E)\, dE = N \quad . \tag{2.11}$$

Suppose that $\rho(E)$ is non-zero in the interval

$$Nva < E < Nvb \quad . \tag{2.12}$$

In this interval it must satisfy the condition obtained by minimizing (2.10), namely

$$\int (E - z)^{-1} z\rho(z)\, dz = \frac{1}{2}v^{-1}E - \frac{1}{2}(k+1)N \quad . \tag{2.13}$$

But (2.13) is precisely the equation which defines the semi-circle distribution for the function $E\rho(E)$, regarded as a function of the variable $(E - (k+1)Nv)$, (see Wigner[7]). Thus

$$\varphi(y) = \int (y - z)^{-1} z\rho(z)\, dz$$

$$= \frac{1}{2}v^{-1}\left[y - (y - Nvb)^{1/2}(y - Nva)^{1/2}\right] - \frac{1}{2}(k+1)N \tag{2.14}$$

is an analytic function of the complex variable y in the plane cut along the interval (2.12). Its imaginary part on the cut is

$$Im\, \varphi(E \pm i\epsilon) = \mp \pi E\rho(E) \quad . \tag{2.15}$$

Letting $y \to \infty$ in (2.14) we find

$$a + b = 2(k + 1) . \tag{2.16}$$

Letting $y = 0$ and using (2.11) we find

$$(ab)^{1/2} = k - 1 . \tag{2.17}$$

Eq. (2.16) and (2.17) imply (1.9) and (1.10). From (2.14) and (2.15) it follows that $\rho(E)$ has the form (1.6). This completes the proof of the statements made in Section I and in the bastract of this paper.

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