INVARIANTS OF THE EQUATIONS OF WAVE MECHANICS I

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ABSTRACT: Lie's method of finding the invariants of differential equations is generalized and the generalization is applied to the time-dependent and time-independent Scroedinger equations. Invariants containing both finite and infinite order derivatives with respect to the coordinates are obtained. A physical classification of the invariants, and of their groups is outlined. The method is illustrated by applying it to several physically interesting differential equations involving one space coordinate.

1. INTRODUCTION

Recently we have developed a generalization of Lie's method[1] for finding the infinitesimal invariants and continuous groups of differential equations which makes it possible to systematically determine the continuous groups of partial differential equations[2]. In this and the following paper we use a form of the method that is particularly appropriate for linear equations to determine invariant operators Q of Schroedinger equations. We confine ourselves for the time being to systems involving one non-relativistic particle whose spin may be neglected. For such systems the most general linear operator Q can be written in the form

$$Q = q(x) + q^{k}(x)\,\partial_{k} + q^{kl}(x)\partial_{k}\partial_{l} + \ldots \tag{1.1}$$

This is possible[3] because for such systems the representation of the canonical commutation relations

$$[x^{j}, p_{k}] = i\delta^{j}_{k},\ j, k = 1, \ldots, n \tag{1.2}$$

is irreducible in $L^{2}(R^{n})$. Moreover, the set of all finite linear combinations of elements

$$\exp(ia_{j}x^{j})\ \exp(ib^{k}p_{k})$$

of this representation is strongly dense in the space of all continuous linear operators[4] in $L^{2}(R^{n})$. Since this linear combination of elements can be formally expanded in the series (1.1) this form of Q is general.

We will deal in this and the next paper with the determination of invariants ${}^{2}Q$ that contain zero, first, and second order terms, and we will also describe and use a special method for determining a class of Q operators, ${}^{\infty}Q$, whose expansion (1.1) is infinite.

It is our purpose to demonstrate that these methods allow one to systematically determine invariants that are of known physical importance, invariants which have not previously been known, and invariants that though previously known, were not found by a systematic and general method. Some of these invariants will be shown to be explicit functions of time, and some of these generate the spectrum of the Hamiltonian.

The principal limitation of the methods illustrated here is that the special method used for finding invariants ${}^{\infty}Q$ of the time-dependent Schroedinger equation requires one to first determine the spectrum of the Hamiltonian.

2. GENERAL CONSIDERATIONS

Let H be the Hamiltonian operator describing a given dynamical system in the position space representation. Then we have

$$\{H(x, \partial_x) - i\partial_t\}\Psi(x, t) = 0 \qquad (2.1)$$

for all allowed state vectors Ψ. In practice Ψ may have several components and be a function of more than one set of space-time variables, generally denoted by (x, t) i.e., equation (2.1) is in general a set of partial differential equations involving $n \geqslant 1$ unknown functions in $m \geqslant 1$ independent variables.

Now all operators Q which correspond to constants of the motion of the system satisfy the relations

$$0 = [H, Q]\Psi - i\partial_t Q\Psi = [(H - i\partial_t), Q]\Psi \qquad (2.2)$$

for all allowed state-vectors Ψ. Because $H - i\partial_t$ is self-adjoint it follows that if Q is a solution of (2.1) so is its adjoint Q^+. The operator

$$X = \tfrac{1}{2}(Q - Q^+) \qquad (2.3)$$

is then skew-adjoint, and the operator iX is self-adjoint.* Furthermore, the set of all skew-adjoint or self-adjoint operators representing constants of the motion forms a closed Lie algebra.** From the product of skew-adjoint invariants one can form further such invariants. It sometimes happens that these product elements close with the original elements and so can serve as generators of a finite Lie group larger than that generated by the simpler elements. In other cases these product elements may not close under commutation.

Thus it may happen that the set of all polynomial invariants constitutes the basis for the enveloping algebra of one or more finite-parameter Lie groups, i.e., there may exist finite sets of invariants which close under

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* Henceforth we will use the letter Q to denote any infinitesimal invariant of a differential equation, and use the letter X only when we wish to emphasize that we choose to deal with skew-adjoint invariants, and hence with real Lie algebras.

** It is necessary to allow the so-called "infinite parameter" Lie algebras.

commutation and give rise to enveloping algebras co-extensive with the set of all invariants of the differential equation. Furthermore, turning the argument around, we see that if there exists a finite Lie algebra of invariants $L(G)$, this also implies that the elements of the enveloping algebra of $L(G)$ constitute the generators of an infinite dimensional group of invariants. Hence, it is apparent that the concept of the "largest group" of a differential equation is not a particularly helpful one. Rather, what is to be sought is the smallest Lie group whose enveloping algebra contains all polynomial invariants of the equation – or perhaps all polynomial invariants with some special property of interest.

3. PHYSICAL CLASSIFICATION OF THE LIE GROUPS

There are three distinct types of continuous groups that are often discussed in the literature dealing with the symmetry properties of individual dynamical systems. These are the degeneracy, geometrical, and dynamical groups. We shall adopt definitions of these groups which limit their size. Given a system described by (2.1), a Lie group G of operators g, the associated Lie algebra $L(G)$, and a representation $\{T_g\}$ of G then

1. The degeneracy group is the smallest group G acting on an eigenspace of $h_E \in \mathcal{H}$ where $h_E = \{\Psi_E \mid H\Psi_E = E\Psi_E\}$ such that

   i) for $g \in G$, $T_g \Psi_E(x,t) \in \mathcal{H}_E$ and

   ii) the generators of any larger group are members of the enveloping algebra of $L(G)$.

2. The geometrical symmetry group is the smallest group acting on the space-time manifold such that

   i) for $g \in G$, $T_g \Psi(x,t) = \Psi(g^{-1}(x,t))$

   ii) The generators of any larger group satisfying i) are members of the enveloping algebra $L(G)$.

3. The dynamical group is the smallest group G acting on the Hilbert space of state vectors such that

   i) if $g \in G$, $T_g \Psi(x,t) = \Phi(x,t) \in \mathcal{H}$

   ii) it contains the degeneracy group.

   iii) the generators of any larger group satisfying i), ii), are members of the enveloping algebra of $L(G)$.

Note that the definition of the dynamical group of a system implies that the Hilbert space of the state vectors of the system is a carrier space for a single irreducible representation of the dynamical group.

4. PHYSICAL CLASSIFICATION OF THE OPERATORS OF LIE ALGEBRAS

If the Hamiltonian operator does not depend explicitly on the time, then $Q_t = i\partial_t$ commutes with $H - i\partial_t$ and has the same spectrum as H. This allows one to make a physically meaningful classification of the operators of a dynamical algebra analogous to the Cartan classification of the operators of a semi-simple group. If we let

$$Q = a^i Q_i \,,\; i = 1 \ldots n \,. \tag{4.1}$$

we may require the a^i to be so chosen that

$$[Q_t , Q] = \alpha Q \,, \tag{4.2}$$

where α is a number. On inserting

$$[Q_t , Q_i] = c_{ti}^{k} Q_k \tag{4.3}$$

into this equation one obtains

$$(c_{ti}^{k} - \alpha\delta_i^k)\, a_i = 0 \,. \tag{4.4}$$

If the a^i are linearly independent this equation is equivalent to a set of homogeneous linear equations that may be satisfied only for those values of α for which

$$\det (c_{ti}^{k} - \alpha\delta_i^k) = 0 \,. \tag{4.5}$$

If the secular equation gives rise to several roots with $\alpha = 0$ then the system may admit a nontrivial degeneracy group if the corresponding $Q, Q', \ldots$ do not all commute but satisfy conditions (1 - i) and (1 - ii) of Section 3. Then the degeneracy group is nonabelian and the spectrum of the system may contain degenerate levels simply because of the continuous symmetries that it admits. If the secular equation has a nonzero root α_s then the system admits a spectrum generating operator Q_s such that

$$[Q_t, Q_s] = [H, Q_s] = \alpha_s Q_s \tag{4.6}$$

so that if

$$H\Psi = E\Psi$$

then

$$H\{(Q_s)^m\Psi\} = (E + m\alpha_s)\{(Q_s)^m\Psi\} \tag{4.7}$$

In all these cases the energy E is a linear function of a set of weights, n_i :

$$E = \epsilon^i n_i + E_0 \quad .$$

Here E_0 and the ϵ^i are constants. We shall call such spectra *linear* spectra. In short, because α_s is simply a number, the operators Q_s can only generate a spectrum of evenly spaced levels. Thus, it follows that any time-independent Hamiltonian with a discrete spectrum that has a spectrum generating invariant must have a spectrum of equally spaced levels. If there are several operators $Q_{s'}, Q_{s''}, \ldots$ the spectrum will consist of equally spaced ladders or bands of equally spaced (perhaps degenerate) levels, etc.[5]

If there are non-zero degenerate roots α, then the number of linearly independent Q_α determined by (4.5) may be less than n. In this case not all linearly independent Q's may be chosen to satisfy (4.6). Some Q must then be such that

$$[Q_t, Q] = Q' \, ; \; Q' \neq Q \; . \tag{4.8}$$

If $Q' \neq Q_t$ then the effect of Q' on a wave function Ψ_E of definite energy E is to convert it to a mixture of functions of different energy

$$Q'\Psi_{E_0} = b^i \Psi_{E_i} . \tag{4.9}$$

We term such an invariant a mixing operator.

If the dynamical group contains an operator Q such that

$$[Q_t , Q] = \beta Q_t \tag{4.10}$$

then Q is the generator of a continuous spectrum because (4.10) implies that

$$H \{(\exp bQ)\Psi_0\} = i\partial_t (\exp bQ)\Psi_0 = (\exp b\beta E_0)\{(\exp bQ)\Psi_0\} \tag{4.11}$$

To sum up, when the Hamiltonian does not depend explicitly on the time, consideration of the root vectors of $i\partial_t$ leads to a useful classification of the invariants of a dynamical algebra into: a) generators of the degeneracy group; b) discrete-spectrum generating operators; c) continuous-spectrum generating operators; d) mixing generators. Of course, it does not follow that simply because the time evolution equation admits a continuous or discrete-spectrum generating operator, that the wave functions that are obtained from a properly behaved Ψ by operation with this spectrum-generating operator will themselves satisfy the boundary conditions appropriate to the system. For example, the free particle and the particle in a box, or the particle on a circle, all give rise to the same continuous group if one ignores the boundary conditions. However, when the boundary conditions are taken into account the groups are quite different. Nevertheless, it is clear that the first step in finding operators that convert eigen-solutions into eigen-solutions is to find those operators which have the necessary local properties to do so.*

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* G. Bluman has recently shown explicitly how to use local symmetry operators in the building of solutions that satisfy various boundary conditions.[6]

5. THE DETERMINING EQUATIONS

We now consider the problem of actually finding the operators Q that are invariants of a time-dependent wave equation. For simplicity of exposition we restrict ourselves in this paper to Schroedinger's equations. Let

$$K = -2(H - i\partial_t) = a^{ij}\partial_i\partial_j + b^i\partial_i + c\,;\; i, j, = 1, 2, \ldots n \qquad (5.1)$$

and let

$$\partial_i u = u_i\,,\; \partial_i\partial_j u = u_{ij}\,,\; \text{etc.},$$

so that the differential equation of interest becomes

$$Ku = a^{ij}u_{ij} + b^i u_i + cu = 0\ . \qquad (5.2)$$

Here it is assumed that the $a^{ij} (= a^{ji})$, and b^i, c, and u are functions of

$$x = x_1, x_2, \ldots x_n\,,\ \text{with}\ t = x_n\ . \qquad (5.3)$$

An operator Q will be the generator of a one-parameter continuous group that leaves equation (5.1) invariant if

$$(\exp \beta Q)\, u = 0 \qquad (5.4)$$

for all values of the parameter β. On setting β infinitesimal in the usual way, one finds that this is equivalent to the requirement

$$KQu = 0\ . \qquad (5.5)$$

As we may suppose that Q is of the form

$$Q = q(x) + q^i(x)\,\partial_i + q^{ij}(x)\,\partial_i\partial_j + \ldots \qquad (5.6)$$

we must then have, if Q is an invariant,

$$(a^{ij}\partial_i\partial_j + b^i\partial_i + c)(q + q^k\partial_k + q^{kl}\partial_k\partial_l + \ldots.)u = 0 \ . \tag{5.7}$$

Equation (5.7) is solved by the classical method of expanding it and collecting terms that multiply the functions u, u_i, u_{ij}, etc. These functions would all be linearly independent were it not for the identities contained in the original differential equation (5.2). On substituting these identities in the expanded form of (5.7) and again collecting terms that multiply a linearly independent set of functions, one obtains a set of simultaneous differential equations determining the components q, q^i, q^{ij}, etc., of Q. The process may be neatly formulated with the aid of Lagrange's undetermined multipliers. However, in the actual calculations of interest to us here the use of undetermined multipliers only complicates the computations, so we will not make use of them in this paper.

On integrating the set of differential equations, arbitrary constants of integration appear. Because of the interdependence of the equations, the same constants may occur in the final expressions for q, and the q^i, q^{ij}, etc. We shall denote these constants of integration $\gamma^0, \gamma^1 \ldots$, and write

$$q = \gamma^m q_m \ , \tag{5.8}$$

$$q^i = \gamma^m q^i_m \ , \tag{5.9}$$

$$q^{ij} = \gamma^m q^{ij}_m \ , \tag{5.10}$$

and so forth.

The general result of solving equation (5.7) for any system is thus of the form

$$Q = \gamma^m (q_m + q^i_m\partial_i + q^{ij}_m\partial_i\partial_j + \ldots). \tag{5.11}$$

Because the γ^m are linearly independent one defines the operators

$$Q_m = q_m + q^i_m\partial_i + q^{ij}_m\partial_i\partial_j + \ldots \tag{5.12}$$

so that

$$Q = \gamma^m Q_m \tag{5.13}$$

In short, the search for a particular Q that satisfies (5.7) yields a set of such Q's, the Q_m .

6. AN EXAMPLE: SOLUTION OF THE DETERMINING EQUATIONS FOR ^{1}Q WHEN THE POTENTIAL IS A TIME-INDEPENDENT FUNCTION OF ONE VARIABLE.

If we let

$$Q = q + q^j \partial_j \tag{6.1}$$

and the wave equation is the time-dependent Schroedinger equation

$$(\partial_1 \partial_1 + 2i\partial_2 - 2v)\, u = 0 \ , \tag{6.2a}$$

which we also write as

$$u_{11} + 2iu_2 - 2vu = 0 \ , \tag{6.2b}$$

then we must require

$$(\partial_1 \partial_1 + 2i\partial_2 - 2v)(q + q^j \partial_j)\ u = 0 ; \quad i = 1, 2 . \tag{6.3}$$

This becomes on expanding

$$\begin{aligned} 0 = {} & q_{11} u + 2q_1 u_1 + 2iq_2 u + q^j_{11} u_j + 2q^j_1 u_{1j} + 2iq^j_2 u_j \\ & + q^j\,(u_{11j} + 2iu_{j2} - 2vu_j) \ . \end{aligned} \tag{6.4}$$

The u, u_i, u_{kj}, are subject to the condition

$$0 = u_{11} + 2iu_2 - 2vu \tag{6.5}$$

and the two further conditions obtained by differentiating this equation

$$0 = u_{111} + 2iu_{21} - 2v_1 u - 2vu_1 \tag{6.6}$$

$$0 = u_{112} + 2iu_{22} - 2vu_2 = 0 \ . \tag{6.7}$$

Collecting terms in (6.4) gives

$$\begin{aligned} 0 &= u(q_{11} + 2iq_2) + u_1(2q_1 + q^1_{11} + 2iq^1_2 - 2vq^1) \\ &+ u_2(q^2_2 - 2vq^2) + u_{22}(2iq^2) + u_{12}(2q^2_1 + 2iq^1) \\ &+ u_{11}(2q^1_1) + u_{111}(q^1) + u_{112}(q^2) \ . \end{aligned} \tag{6.8}$$

Choosing the independent functions to be $u, u_1, u_2, u_{12}, u_{22}$, the dependent functions are u_{11}, u_{111}, and u_{112}. Eliminating these last three by means of equations (6.5), (6.6), and (6.7), we find that the last three terms in (6.8) become

$$2q^1_1(2vu - 2iu_2) = u(4vq_1) + u_2(-4iq_1) \tag{6.9}$$

$$q^1(2vu_1 + 2v_1 u - 2iu_{12}) = u(2v_1 q^1) + u_1(2vq^1) + u_{12}(-2iq^1) \tag{6.10}$$

$$q^2(2vu_2 - 2iu_{22}) = u_2(2uq^2) + u_{22}(-2iq^2) \ . \tag{6.11}$$

Substituting these relations into (6.8) and collecting terms gives

$$u(q_{11} + 2iq_2 + 4vq_1 + 2v_1q^1) = 0 \tag{6.12}$$

$$u_1(2q_1 + q^1_{11} + 2iq^1_2) = 0 \tag{6.13}$$

$$u_2(q^2_{11} + 2iq^2_2 - 4iq_1) = 0 \tag{6.14}$$

$$u_{12}(2q^2_1) = 0 \tag{6.15}$$

These equations are identical to those obtained by Osvjannikov's formulation of Lie's method[7], if one identifies $-q$ with his σ.

From equation (6.15) it follows that q^2 is a function of x^2 only. Thus (6.14) becomes

$$q^2_2 = 2q^1_1$$

or

$$q^1 = \tfrac{1}{2}(x^1q^2_2 + a),$$

with

$$a = a(x^2) .$$

Hence

$$q^1_{11} = 0 . \tag{6.16}$$

Therefore (6.13) becomes

$$2q_1 + i(x^1q^2_{22} + a_2) = 0 \tag{6.17}$$

or

$$q_1 = -\tfrac{1}{2}(x^1q^2_{22} + a_2) . \tag{6.18}$$

Thus

$$q_{11} = -\tfrac{1}{2}\, i q^2_{22} \, , \tag{6.19}$$

so

$$q = -\tfrac{1}{2}\, i \{\tfrac{1}{2}(x^1)^2 q^2_{22} + x^1 a_2 + b\} \tag{6.20}$$

with

$$b = b(x^2) \, .$$

Hence

$$q_2 = -\tfrac{1}{2} i \{\tfrac{1}{2}(x^1)^2 q^2_{222} + x^1 a_{22} + b_2\} \, . \tag{6.21}$$

This implies that (6.12) can be written as

$$-\tfrac{1}{2} i (q^2_{22}) = \tfrac{1}{2}(x^1)^2 q^2_{222} + (x^1)\, a_{22} + b_2 + 2v q^2_2 + v_1 (x^1 q^2_2 + a) = 0 \, . \tag{6.22}$$

To sum up then to this point we have

$$q^1 = \tfrac{1}{2}(x^1 q^2_2 + a); \quad q^2 = q^2(x^2), \; a = a(x^2) \tag{6.23,a, b, c}$$

$$q = -\tfrac{1}{2} i \{\tfrac{1}{2}(x^1)^2 q^2_{22} + x^1 a_2 + b\}; \; b = b(x^2) \tag{6.24,a, b}$$

$$\tfrac{1}{2} i q^2_{22} - b_2 = q^2_2 (2v + v_1 x^1) + a v_1 + \tfrac{1}{2} q^2_{222} (x^1)^2 + a_{22} x^1 \, . \tag{6.25}$$

These relations are only compatible if in the last equation the coefficients involving the same powers of x^1 are separately constants. Equation (6.25) thus gives rise to several further equations whose form will be quite different for different potentials. We first treat the system with $v = d(x^1)^{-2}$ as an example. In this case the coefficient q^2_2 becomes zero, while that of a is $-2(x^1)^{-3}$. Separating equation (6.25) then gives

$$a = 0 \tag{6.26}$$

$$\tfrac{1}{2} i q^2_{22} - b_2 = 0 \tag{6.27}$$

$$q^2_{222} = 0 \ . \tag{6.28}$$

Thus

$$q^2_{22} = \gamma^0 \ . \tag{6.29}$$

$$q^2_2 = \gamma^0 x^2 + \gamma^1 \tag{6.30}$$

$$q^2 = \tfrac{1}{2}\gamma^0 (x^2)^2 + \gamma^1 x^2 + \gamma_2 \ . \tag{6.31}$$

Also

$$b_2 = \tfrac{1}{2} i \gamma^0 \tag{6.32}$$

$$b = \tfrac{1}{2} i \gamma^0 x^2 + \gamma^3 \tag{6.33}$$

Inserting these results into equations (6.23 and 6.24) one obtains

$$q^1 = \tfrac{1}{2}(\gamma^0 x^1 x^2 + \gamma^1 x^1) \tag{6.34}$$

and

$$q = -\tfrac{1}{4} i \gamma^0 (x^1)^2 + \tfrac{1}{4}\gamma^0 x^2 - \tfrac{1}{2} i \gamma^3 \ . \tag{6.35}$$

On collecting terms as indicated in equation (5.12) one obtains the invariants

$$Q_0 = -\tfrac{1}{4} i (x^1)^2 + \tfrac{1}{4} x^2 + \tfrac{1}{2} x^1 x^2 \partial_1 + \tfrac{1}{2}(x^2)^2 \partial_2 \tag{6.36}$$

$$Q_1 = \tfrac{1}{2}x^1\partial_1 + x^2\partial_2 \tag{6.37}$$

$$Q_2 = \partial_2 \tag{6.38}$$

$$Q_3 = -\tfrac{1}{2}i \ . \tag{6.39}$$

Now let us return to equation (6.25) and view it not as an equation determining q^2, a and b, but rather as a differential equation determining the potential $v(x^1)$ that corresponds to a given set of functions q^2, a, b. We, therefore, rewrite it as

$$v_1(x^1q^2_2 + a) + v(2q^2_2) + \tfrac{1}{2}(x^1)^2q^2_{222} + x^1a_{22} + b_2 - \tfrac{1}{2}\,iq^2_{22} = 0 \tag{6.40}$$

Integrating gives

$$v = \{C_0 + C_1x^1 + C_2(x^1)^2 + C_3(x^1)^3 + C_4(x^1)^4\}(x^1 + C_5)^{-2} \ . \tag{6.41}$$

The constants are related to q^2, a, b as follows:

$$C_0 = c_0/q^2_2 \ , \quad c_0 \text{ an arbitrary constant.} \tag{6.42}$$

$$C_1 = (a/q^2_2)\cdot(b_2 - \tfrac{1}{2}iq^2_{22})/q^2_2 \tag{6.43}$$

$$2C_2 = (a/q^2_2)\cdot(a_{22}/q^2_2) + (b_2 - iq^2_{22})/q^2_2 \tag{6.44}$$

$$3C_3 = \tfrac{1}{2}(a/q^2_2)\cdot(q^2_{222}/q^2_2) + a_{22}/q^2_2 \tag{6.45}$$

$$4C_4 = \tfrac{1}{2}\,q^2_{222}/q^2_2 \tag{6.46}$$

$$C_5 = a/q^2_2 \tag{6.47}$$

TABLE I

First-Order Invariants ${}^{1}Q$, of the Equation $(\partial_x \partial_x + 2i\partial_t - 2v)u = 0$

Q_i	$v = 0$	$v = cx$	$v = \frac{1}{2}cx^{-2}$	$v = \frac{1}{2}kx^2$	$v = \frac{1}{2}(cx^{-2} + kx^2)$
Q_1	1	1	1	1	1
Q_2	$t\partial_x - ix$	$t\partial_x - ix + \frac{1}{2}ict^2$	-	$(\partial_x + \sqrt{k}x)\exp[i\sqrt{k}t]$	$\frac{1}{2}\exp[2i\sqrt{k}t]\{\sqrt{k}x^2 + \frac{1}{2} + x\partial - ik^{-\frac{1}{2}}\partial_t x\}$
Q_3	∂_x	$\partial_x + ict$	-	$(\partial_x - \sqrt{k}x)\exp[-\sqrt{k}t]$	$\frac{1}{2}\exp[-2i\sqrt{k}t]\{\sqrt{k}x^2 + \frac{1}{2} + x\partial_x + ik^{-\frac{1}{2}}\partial_t\}$
Q_4	$\partial_t = \frac{1}{2}i(Q_3)^2$	$\partial_t = \frac{1}{2}i(Q_3)^2 + cQ_2$	∂_t	∂_t	∂_t
Q_5	$\frac{1}{4}(Q_2)^2$	$\frac{1}{4}(Q_2)^2$	$\frac{1}{2}(xt\partial_x + t^2\partial_t + \frac{1}{2}t - \frac{1}{2}ix^2)$	-	-
Q_6	$\frac{1}{4}Q_3Q_2 - \frac{1}{2}Q_1$	$\frac{1}{4}Q_3Q_2 - \frac{1}{2}Q_1$	$t\partial_t + \frac{1}{2}x\partial_x$	-	-

TABLE 2

Basic Commutation Relations of the Invariants of Table 1

a)* $v = 0$:	$[Q_2, Q_3] = iQ_1$
b)* $v = cx$:	$[Q_2, Q_3] = iQ_1$
c) $v = \frac{1}{2}cx^{-2}$:	$[Q_4, Q_5] = \frac{1}{4} Q_1 + Q_6$
	$[Q_4, Q_6] = Q_4$
	$[Q_5, Q_6] = Q_5$
d) $v = \frac{1}{2}kx^2$:	$[Q_2, Q_3] = 2\sqrt{k}\, Q_1$
	$[Q_2, Q_4] = i\sqrt{k}\, Q_2$
	$[Q_3, Q_4] = -i\sqrt{k}\, Q_3$
e) $v = \frac{1}{2}(cx^{-2} + kx^2)$:	$[Q_2, Q_3] = ik^{-\frac{1}{2}} Q_4$
	$[Q_2, Q_4] = 2i\sqrt{k}\, Q_2$
	$[Q_3, Q_4] = -2i\sqrt{k}\, Q_3$

* In these cases the commutation relations of the remaining invariants in the corresponding columns of Table 1 may be obtained using the relations between invariants given therein. In both cases all invariants close under commutation.

The compatible solutions of these equations determine the potentials $v(x^1)$ that give rise to Schroedinger equations with invariants of the form (6.1). The invariants of several such systems are listed in Table 1. Their commutation relations will be found in Table 2. The allowed class of potentials clearly does not include many systems of physical interest. We must conclude that many quantum mechanical constants of the motions are not of the form (6.1).

7. TIME DILATION TRANSFORMATION FOR LINEARIZING NON-LINEAR SPECTRA

The generalization which allows constants of the motion to contain second order terms $q^{ij}\partial_i\partial_j$ has its simplest interesting applications where the wave equations are those of systems with several spatial degrees of freedom. These are discussed in paper II of this series.* The determining equations and the processes involved in solving them are there shown to be analogous to those just illustrated. Here we shall proceed directly to the discussion of a method for determining the constants of the motion of time-dependent Schroedinger equations which allows one to find invariants whose expansion (1.7) involves an infinite series of derivatives.

Consider the class of eigenvalue problems which are the form

$$(H - \lambda(n))\,\Psi_r(x) = 0 \qquad (7.1)$$

where $x = (x^1, x^2, \ldots, x^n)$ represents n independent variables,

$$H = H\,(x^i, \partial_i\,, \partial_i\partial_j\,, \ldots)$$

Here $\lambda(n) \to n$ (in is one-to-one, and the set

$$\{\Psi_{\lambda(n)} \in L^2\,(\mu(x))\}$$

constitutes a complete set for all square integrable functions $L^2(\mu(x))$ with

* See the following paper[8]

respect to the measure $\mu(x)$.

This class can be thought of as arising from the Schroedinger-type equation.

$$(H - i\partial_t)\ \exp\left[-i\lambda(n)\, t\right] \tag{7.2}$$

where

$$\Psi_n(x) \in L^2(\mu(x))\,.$$

Here t may or may not correspond to the physical time, e.g., in the case H is a Hamiltonian, t corresponds to the physical time, whereas, e.g., in the case of ***the classical orthogonal polynomials,*** t has no direct interpretation and, as will be indicated later, if desired, it can be eliminated from any result.

The method, we shall describe in this and the next two subsequent sections yields spectrum generating Lie algebras for the equation.

$$(H - i\partial_t)\Psi(x, t) = 0 \tag{7.3}$$

where

$$\Psi(x, t) = \Sigma c_n\ \exp\left[-i\lambda(n)\, t\right] \Psi_n(x) \in L^2_{\lambda(n)}(\mu(x)) \tag{7.4}$$

under the assumption that the spectrum $\lambda(n)$ is known. Because these algebras and their enveloping algebras transform the set $\{\exp\left[-i\lambda(n)\,t\right]\Psi_n(x)\}$ amongst itself they can be used to construct the dynamical group for (7.3).

If $\lambda(n)$ is a sufficiently simple function of several integers i.e., $n = (n^1, n^2, \ldots)$ e.g., in a relativistic wave equation, the technique we shall describe is applicable by considering each integer one at a time. This, of course, means that some, if not all, the corresponding t's will be auxiliary variables, and that the dynamical group's Lie algebra lies in the direct sum of spectrum generating algebras for all the integers n_i. The technique is also directly applicable to any set of partial differential equations which e.g., might arise in the separation of variables approach, and the spectrum generating algebras for the separate equations can be multiplied to give the dynamical group for the original unseparated equation.

In general, there are two types of dilation operations involved: one dilating t, and the other dilating some subset of the $\boldsymbol{x}$. The key operation is the dilation of t, which transforms to a space where the spectrum of $i\partial_t$ is linear in n. The dilation of $\boldsymbol{x}$ transforms to yet another space, and it is necessary in some cases in order to avoid extending infinitesimal methods to include other than nonnegative integral powers of the partial derivatives. Because of this aspect of the method, we shall present the method in two steps – first the dilation of t and then, in Section 9, the dilation of the $\boldsymbol{x}$. We shall proceed on the assumption that all operators are well defined i.e., we shall not examine here questions concerning the domains and ranges of our mappings.

We subject the operation $(H - i\partial_t)$ and the space $L^2_{\lambda(n)}(\mu(\boldsymbol{x}))$ to the time dilation

$$D_t = \exp\left[\ln\left(\frac{n(H)}{\lambda(n(H))}\right) t\partial_t\right] \tag{7.5}$$

where $n(H)$ is the operator solution of (7.1) for n in terms of H, i.e.,

$$(n(H) - n)\Psi_n(\boldsymbol{x}) = 0 \tag{7.6}$$

Therefore, we pass from the space $L^2_{\lambda(n)}(\mu(\boldsymbol{x}))$ and equation (7.3) to $L^2_n(\mu(\boldsymbol{x}))$

$$\left[H - \left\{\frac{\lambda(n(H))}{n(H)}\right\} i\partial_t\right]\Psi(\boldsymbol{x}, t) = 0 \tag{7.7}$$

where

$$\Psi(\boldsymbol{x}, t) = \sum a_n \exp[-int]\,\Psi_n(\boldsymbol{x}) \in L^2_n(\mu(\boldsymbol{x})) .$$

In the space $L^2_n(\mu(\boldsymbol{x}))$, the following equation is equivalent to (7.7):

$$\{H - \lambda(n(i\partial_t))\}\Psi(\boldsymbol{x}, t) = 0 \tag{7.8}$$

This is possible because in $L_n^2(\mu(x))$, we have*

$$(n(H) - i\partial_t)\,\Psi(x,t) = 0 \tag{7.9}$$

8. AN EXAMPLE OF TIME-DILATION. THE GEGENBAUER EQUATION

As an example illustrating the use of time dilations as an aid in finding spectrum-generating invariants we consider Gegenbauer's differential equation

$$\{(1 - x^2)\,\partial_x\partial_x - (2\nu+1)\,x\partial_x + n\,(n + 2\nu)\}\,g_n(x) = 0 \tag{8.1}$$

which arises by separating the equation

$$\{(1 - x^2)\,\partial_x\partial_x - (2\nu + 1)\,x\partial_x + i\partial_t\}\sum_n c_n \exp\left[-in\,(n + 2\nu)\,t\right] g_n(x) = 0\,. \tag{8.2}$$

To get a linear spectrum for ∂_t we transform equation (8.1) with the dilator

$$D = \exp\left[t\partial_t \log \frac{\tilde{n}}{\tilde{n}\,(\tilde{n} + 2\nu)}\right] \tag{8.3}$$

where

$$\tilde{n} = -\nu + \{\nu^2 - (1 - x^2)\,\partial_x\partial_x + (2\nu + 1)\,x\partial_x\}^{\frac{1}{2}}\,.$$

The transformed equation is

$$\{(1 - x^2)\,\partial_x\,\partial_x - (2\nu + 1)\,x\partial_x + (\tilde{n} + 2\nu)\,i\partial_t\}\,\sum c_n \exp\left[-int\right] g_n(x) = 0\,.$$

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* Because of the identity expressed by (7.9), we can substitute for all or any subset of the $n(H)$ in $\lambda(n(H))/n(H)$; hence we see that there are, in general, classes of "time-dependent" partial differential equations which from the point-of-view of square-integrable solutions, are indeed equivalence classes in the strict sense of possessing "the same general solution".

Within the basis set $\exp[-int]\, g_n(x)$, we have the operator identity

$$i\partial_t = \tilde{n} \ .$$

Using this identity we get

$$\{(1-x^2)\,\partial_x\partial_x - (2\nu+1)\,x\partial_x + i\partial_t\,(i\partial_t + 2\nu)\}\,f(x,t) = 0 \tag{8.4}$$

where $f(x,t) = \sum_n \exp[-int]\, g_n(x)$. From (8.4), we have

$$(1-x^2)\,f_{xx} - (2\nu+1)\,xf_x - f_{tt} + 2i\nu f_t = 0 \ , \tag{8.4'}$$

and differentiating this by x and t, we get

$$(1-x^2)\,f_{xxx} - (2\nu+1)\,xf_{xx} - f_{xtt} + 2i\nu f_{xt} = 2xf_{xx} + (2\nu+1)\,f_x \tag{8.5}$$

and

$$(1-x^2)\,f_{xxt} - (2\nu+1)\,xf_{xt} - f_{ttt} + 2i\nu f_{tt} = 0 \ . \tag{8.6}$$

From (8.4'), (8.5) and (8.6), we can choose

$$f,\, f_t,\, f_x,\, f_{xt},\, f_{tt} \tag{8.7}$$

for the independent functions. We seek an operator Q which satisfies the equation

$$\{(1-x^2)\,\partial_x\,\partial_x - (2\nu+1)\,x\partial_x - \partial_t\,\partial_t + 2i\nu\partial_t\,\}\,Qf(x,t) = 0 \tag{8.8}$$

of the form

$$Q = q^x\partial_x + q^t\,\partial_t + q^0 \ . \tag{8.9}$$

Substituting Q into (8.8), we get

$$\{(1-x^2)\,\partial_x\partial_x - (2\nu+1)\,x\partial_x - \partial_t\,\partial_t + 2i\partial_t\,\}\,(q^x f_x + q^t f_t + q^0 f) = 0 \tag{8.10}$$

Expanding it, using (8.4'), (8.5), (8.6), and collecting terms multiplying the functions (8.7), we get the determining equations

$$q^x_x + x(1-x^2)^{-1} q^x - q^t_t = 0$$

$$(1-x^2)\, q^t_x - q^t_x = 0$$

$$(1-x^2)\, q^t_{xx} - (2\nu+1)\, x q^t_x - q^t_{tt} + 2i\nu\, q^t_t - 2q^0_t - 4i\nu q^x_x - 4i\nu x(1-x^2)^{-1} q^x = 0$$

$$(1-x^2)\, q^x_{xx} + (2\nu+1)\, x q^x_x - q^x_{tt} + 2i\nu q^x_t +$$

$$+ (2\nu+1)(1+x^2)(1-x^2)^{-1} q^x + 2(1-x^2)\, q^0_x = 0$$

$$(1-x^2)\, q^0_{xx} - (2\nu+1)\, x q^0_x - q^0_{tt} + 2i\nu q^0_t = 0\ . \qquad (8.11\text{a-e})$$

Solving these simultaneous equations, we have, for $\nu \neq 0, 1$,

$$q^x = a(1-x^2)\, e^{it} + b(1-x^2)\, e^{-it}$$

$$q^t = aixe^{it} - bixe^{-it} + c \qquad (8.12\text{ a-c})$$

$$q^0 = -2b\nu xe^{-it} + d$$

where a, b, c and d are integration constants. Substituting these into (8.9), we get

$$Q = a \cdot e^{it}\{(1-x^2)\,\partial_x + ix\partial_t\} + b \cdot e^{-it}\{(1-x^2)\,\partial_x - ix\partial_t - 2\nu x\} + c \cdot \partial_t + d\ .$$

(8.13)

As a, b, c and d are arbitrary*, we have four independent operators which satisfy the equation (8.8);

$$Q_+ = e^{it}\{(1-x^2)\partial_x + ix\partial_t\},\ Q_- = e^{-it}\{(1-x^2)\partial_x - ix\partial_t - 2\nu x\}$$

$$Q_1 = \partial_t,\ Q_2 = 1\ . \qquad (8.14a\text{-}c)$$

On putting $Q_0 = Q_1 - i\nu Q_2$, we obtain the commutation relations

$$[Q_+, Q_-] = -2iQ_0,\ [Q_0, Q_1] = iQ_+,\ [Q_0, Q_-] = -iQ_-\ . \qquad (8.15)$$

These show that Q_+ or Q_- shifts the eigenvalue of Q_0, $i(n-\nu)$, by unit amount

$$Q_\pm e^{int} g_n(x) = C_\pm(n)\, e^{i(n\pm1)t} g_{n\pm1}(x)\ . \qquad (8.16)$$

As the Casimir operator $\frac{1}{2}(Q_+Q_- + Q_-Q_+) - Q_0^2$ has the eigenvalue $\nu(\nu-1)$, ν specifies the UIR of the group.

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*For $\nu = 0$ or 1, we have the following solutions for q^x, q^t, q^0:

$$\nu = 0:\ q^x = \sum_{\alpha=-\infty}^{\infty} \xi^\alpha e^{i\alpha t},\ q^t = \sum_{\alpha\neq0} \frac{1}{i\alpha}\{\xi^\alpha_x + x(1-x^2)^{-1}\xi^\alpha\}e^{i\alpha t} + a,\ q^0 = b,$$

$$\nu = 0:\ q^x = \sum_{\alpha=-\infty}^{\infty} \xi^\alpha e^{i\alpha t},\ q^t = \sum_{\alpha\neq0} \frac{1}{i\alpha}\{\xi^\alpha_x + x(1-x^2)^{-1}\xi^\alpha\}e^{i\alpha t} + a,$$

$$q^0 = -\sum_{\alpha\neq0} \frac{1}{\alpha}\{\xi^\alpha_x + (\alpha+1)x(1-x^2)^{-1}\xi^\alpha\}e^{i\alpha t} - x(1-x^2)\xi^0 + b$$

where α, a, b are arbitrary and ξ^α are the solutions of the equation

$$\{(1-x^2)\partial_x\partial_x + x\partial_x + (1+x^2)(1-x^2)^{-1} + \alpha^2\}\xi^\alpha = 0\ .$$

This eigenvalue vanishes when $\nu = 0$ or 1. In these cases a subset of the invariants of the Gegenbauer equation still generate an $O(2, 1)$ subalgebra, but as indicated in the footnote, the complete set of generators of the algebra is infinite.

Now, we shall turn to the problem of eliminating the auxiliary variable t, if desired, in this discussion of the Lie algebraic properties of the Gegenbauer polynomials. This is accomplished as follows: define $\{x_\alpha\}$ where $\alpha = +, -, 1, 2$, such that

$$x_\alpha g_n(x) = \lim_{t \to 0} Q_\alpha \exp[-int]\, g_n(x) \tag{8.17}$$

then

$$\begin{aligned} x_+ &= (1 - x^2)\,\partial_x + xn \\ x_- &= (1 - x^2)\,\partial_x - xn - 2\nu x \\ x_1 &= -in \\ x_2 &= 1\,. \end{aligned} \tag{8.18 a-d}$$

We see here clearly that the disadvantage of eliminating the auxiliary variable t is that the generators are n dependent, e.g., the x_+'s involved, in say $x_+ g_{n+1}$, are not identical in their n dependence to (8.18), but this can be overcome by substituting $\tilde{n}$ for n (eq. (8.3)).

We note that this method of taking $\lim_{t \to 0}$ is completely general.

9. SPACE DILATION TRANSFORMATIONS

In many cases, e.g., the rigid rotator or the classical orthogonal polynomials, the dilation of t is sufficient, i.e., equation (7.8) has invariants of the form discussed in Section 6. But, in other cases, e.g., the hydrogen atom, the direct application of infinitesimal methods would require the extension of the methods to other than nonnegative integer powers of the partial derivatives. In order to avoid this possibility, we shall now discuss the dilation of some or all of the x variables. For clarity and simplicity, we shall discuss in detail the procedure for dilating one variable and then indicate the generalization for dilating more than one variable. Therefore, in the following assume x represents one variable.

We introduce a set of "special functions" such that by definition

$$\Psi_n(x) = \Phi_n(g(n)\,x) \tag{9.1}$$

where g is arbitrary for the present and will be chosen later for convenience. Now, we introduce the dilation operator D_x where

$$D_x = \exp\{x\partial_x \ln [g(n(i\partial_t))]^{-1}\} \tag{9.2}$$

takes us from $L_n^2(\mu(x))$ to a new space, which we shall denote as S, i.e.,

$$D_x \exp[-int]\,\Psi_n(x) = \exp[-int]\,\Phi_n(x) \in S \tag{9.3}$$

and

$$D_x(H(x, \partial_x, \partial_x^2, \ldots \partial_x^m) - i\lambda(n(i\partial_t))\, D_x^{-1}$$

$$= H(g^{-1}x, g\partial_x, \ldots, g^n\partial_x^m) - i\lambda(n(i\partial_t)) \tag{9.4}$$

where $g = g(n(i\partial_t))$.

Now, we use the allowed freedom in choosing $g(\cdot)$ such that e.g., equation (9.4) does not contain any negative powers of ∂_t. (An example of just such a case is the hydrogen atom, c.f. paper II of this series[8].)

The generalization to the dilation of k x's then follows directly and the dilation operator in that case is given by

$$D_{x^1 \ldots x^k} = \prod_{i=1}^{k} \exp\{x^i\partial_i \ln g_i(n(i\partial_t))^{-1}\} \tag{9.5}$$

where the g_i's are defined such that

$$\Psi_n(x^1, \ldots, x^k, \ldots, x^n) = \Phi_n(g_1 x^1, , \ldots, g_k x^{k+1}, \ldots, x^n) . \tag{9.6}$$

An extremely important aspect of these dilation transformations is that the algebraic structure of the dynamical algebra for the transformed equation is the same as the original one, even though when x dilation is employed adjointness properties are changed. In other words if

$$\{ Q_a , Q_b \} = c^d_{ab} Q_d$$

represents the Lie algebra of the dynamical group for the transformed equation, then the algebraic structure for the dynamical group of the original equation is given by

$$\{ \tilde{Q}_a , \tilde{Q}_b \} = c^d_{ab} \tilde{Q}_d$$

where

$$\tilde{Q}_a = D^{-1} Q_a D .$$

10. AN EXAMPLE OF TIME AND SPACE DILATION: ONE-DIMENSIONAL KEPLER SYSTEMS

To illustrate the combined use of time and space dilations as an aid in finding invariants, we consider the one-dimensional Kepler problem.

(i) Linearization of spectrum

The Schroedinger equation for the one dimensional atom is given by

$$(-\tfrac{1}{2}\partial_r \partial_r - Z^{-1} - i\partial_t) \sum_n C_n e^{-iE_n t} \Psi_n (2\sqrt{-2E_n} r) . \; (r \geq 0) \tag{10.1}$$

where

$$E_n = -\tfrac{1}{2} Z^2 n^{-2} .$$

To linearize the spectrum of $i\partial_t$ we perform the time dilation

$$D_t(-\tfrac{1}{2}\partial_r\partial_r - Zr^{-1} - i\partial_t)D_t^{-1} \cdot D_t \sum_n C_n e^{-iE_n t}\Psi_n(2\sqrt{-2E_n}\,r)$$

where

$$D_t = \exp\{t\partial_t \ln 2Z\sqrt{-2H}^{-3}\} \qquad (10.2)$$

with $H = -\tfrac{1}{2}\partial_r\partial_r - Zr^{-1}$. The transformed equation is then

$$\{-\tfrac{1}{2}\partial_r\partial_r - Zr^{-1} - (2Z)^{-1}\sqrt{-2H}^{\,3}\,i\partial_t\}\sum_n C_n e^{int}\Psi_n(2\sqrt{-2E_n}\,r) = 0\,. \qquad (10.3)$$

In this basis set we have an operator identity

$$(-2H)^{\frac{1}{2}} = iZ(\partial_t)^{-1}$$

which allows one to rewrite (10.3) as

$$\{-\tfrac{1}{2}\partial_r\partial_r - Zr^{-1} - \tfrac{1}{2}Z^2(\partial_t)^{-2}\}\cdot\sum_n C_n e^{int}\Psi_n(2\sqrt{-2E_n}\,r) = 0\,.$$

Now to eliminate* the negative power of ∂_t we perform the space dilation

$$D_r = \exp\{r\partial_r \log(ai\partial_t)\} \qquad (10.4)$$

where a is an arbitrary constant.

The transformed equation will be

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* Although we can eliminate negative powers by multiplying through $(\partial_t)^2$, we get a fourth order differential equation.

$$- \tfrac{1}{2}\{(ai\partial_t)^{-2}\partial_r\partial_r + 2Z(ai\partial_t)^{-1}r^{-1} + Z^2(\partial_t)^{-2}\}\sum_n C_n e^{int}\Psi_n(-2aZr) = 0 ,$$

and can be rewritten as

$$(\partial_r\partial_r + 2Zar^{-1}i\partial_t - Z^2a^2)\cdot\sum_n C_n e^{int}\Psi_n(-2aZr) = 0 .$$

By choosing $-(2Z)^{-1}$ for a, we put the equation in the standard form

$$(\partial_r\partial_r - ir^{-1}\partial_t - \tfrac{1}{4})f(r,t) = 0 , \tag{10.5}$$

whre

$$f(r,t) = \sum_n C_n e^{int}\Psi_n(r) .$$

To determine the independent functions, we differentiate (10.5) with respect to r and t to get

$$f_{rrr} - ir^{-1}f_{rt} - \tfrac{1}{4}f_r = -ir^{-2}f_t \tag{10.6}$$

$$f_{rrt} - ir^{-1}f_{tt} - \tfrac{1}{4}f_t = 0 . \tag{10.7}$$

From (10.5), (10.6), and (10.7), we can choose f, f_r, f_t, f_{rt} and f_{tt} for the independent functions.

We seek an operator Q of the form

$$Q = q^r\partial_r + q^t\partial_t + q^0 \tag{10.8}$$

which satisfies the equation

$$(\partial_r\partial_r - ir^{-1}\partial_t - \tfrac{1}{4})Q\cdot f(r,t) = 0 . \tag{10.9}$$

Using (10.5), (10.6), (10.7) and (10.8), (10.9) becomes

$$0 = (\partial_r \partial_r - ir^{-1}\partial_t - \tfrac{1}{4})(q^r f_r + q^t f_t + q^0 f$$

$$= (q^0_{rr} - ir^{-1}q^0_t + \tfrac{1}{2}q^r_r) f + (q^t_{rr} - ir^{-1}q^t_t - ir^{-2}q^r +$$

$$2ir^{-1}q^r_r) f_t + (q^r_{rr} - ir^{-1}q^r_t + 2q^0_r) f_r + 2q^t_r \cdot f_{rt} \,. \qquad (10.10)$$

Using the linear independence of f, f_t, f_r and f_{rt}, we have

$$q^0_{rr} - ir^{-1}q^0_t + \tfrac{1}{2}q^r_r = 0$$

$$q^t_{rr} - ir^{-1}q^t_t - ir^{-2}q^r + air^{-1}q^r_r = 0$$

$$q^r_{rr} - ir^{-1}q^r_t + 2q^0_t = 0 \qquad (10.11\,\text{a-d})$$

$$q^t_r = 0$$

Solving these equations, we have

$$q^r = iare^{it} - ibre^{-it}$$

$$q^t = ae^{it} + be^{-it} + c \qquad (10.12\,\text{a-c})$$

$$q^0 = -\tfrac{1}{2}are^{it} - \tfrac{1}{2}ibre^{-it} + \tfrac{1}{2}d$$

where a, b, c, and d are arbitrary constants. Putting these solutions into (10.8), and collecting terms with the same constant coefficient, we get

$$Q = aie^{it}(r\partial_r - i\partial_t - \tfrac{1}{2}r)$$

$$- bie^{-it}(r\partial_r + i\partial_t + \tfrac{1}{2}r) + c\partial_t + \tfrac{1}{2}d \qquad (10.13)$$

As a, b, c, and d are arbitrary, the operators

$$Q_1 = ie^{it}(r\partial_r - i\partial_t - \tfrac{1}{2}r)$$

$$Q_2 - ie^{-it}(r\partial_r + i\partial_t + \tfrac{1}{2}r)$$

$$Q_3 = \partial_t \qquad (10.14\,\text{a-d})$$

$$Q_4 = 1$$

will satisfy the condition (10.9) independently.

These invariants satisfy the commutation relations

$$[Q_1, Q_2] = -2iQ_3\,,\ [Q_3, Q_1] = iQ_1\,,\ [Q_3, Q_2] = -iQ_2\,. \qquad (10.15)$$

From these it is clear that Q_1 and Q_2 shift the eigenvalue n by one unit;

$$Q_j \cdot e^{int}\Psi_n(r) = C_j \cdot e^{i(n\pm 1)t}\Psi_{n\pm 1}(r)\,,\ j = 1, 2\,. \qquad (10.16)$$

To obtain the shift operators for the eigenfunctions of the original equation (1), we must perform the inverse transformation :

$$D^{-1} = D_r^{-1}D_t^{-1} = \exp\{r\partial_r \log 2Z(-i\partial_t)^{-1}\} \cdot \exp\{t\partial_t \log (2Z)^{-1}\sqrt{-2H}^{\,3}\}\,. \qquad (10.17)$$

Then the corresponding operators $\tilde{Q}_1$, $\tilde{Q}_2$ and $\tilde{Q}_3$ will be

$$\tilde{Q}_j = D^{-1}Q_jD\,,\ j = 1, 2, 3\,. \qquad (10.18)$$

In this and higher dimensional Kepler problems the energy shift operators are best left in the form (10.18) as very complicated expressions are obtained on explicitly carrying out the indicated transformations.

11. CONCLUSION

We have seen that the partial differential equations of quantum mechanics can be invariant under continuous groups of transformations whose generators contain derivatives of arbitrarily high order. The currently accepted methods of finding the infinitesimal invariants of differential equations, which are due primarily to the works of Lie[1] on ordinary differential equations of Osvjannikov[7] on partial differential equations, do not allow for the existence of group generators involving derivatives of order greater than those contained in the differential equation itself. It turns out that this is of no consequence for ordinary differential equations, but for partial differential equations the accepted view is, in general, much too restrictive. We have therefore in reference (2) generalized the concept of form-invariance of differential equations by generalizing the concept of a point transformation upon which Lie founded his group theoretic treatment of differential equations. The reader is referred to this reference (2) for a more detailed discussion of the relation of the work here to the works of Lie and Osvjannikov.

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RESUMEN

Se generaliza el método de Lie para encontrar las invariantes de ecuaciones diferenciales; la generalización se aplica a la ecuación de Schröedinger independiente y dependiente del tiempo. Se obtienen invariantes que contienen derivadas con respecto a las coordenadas tanto de orden finito como infinito. Se indica una clasificación física de los invariantes y de los grupos que forman. El método se ilustra aplicándolo a varias ecuaciones diferenciales interesantes en Física y que contienen una coordenada espacial.