INVARIANTS OF THE EQUATIONS OF WAVE MECHANICS II ONE-PARTICLE SCHROEDINGER EQUATIONS*

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ABSTRACT: Employing the systematic methods of the previous paper (I), we derive the invariants and dynamical groups of the time-dependent Schrödinger equation for the two-dimensional harmonic oscillator, and the two and three-dimensional hydrogenlike atom.

1. INTRODUCTION

The application of the systematic methods of the previous paper* to some common Schroedinger equations is illustrated here.

Time-independent and time-dependent invariants are sought for the two dimensional harmonic oscillator. Using both time and space dilatations

* From the M. Sc. Thesis of Sukeyuki Kumei, Department of Physics, University of the Pacific, 1971.

* See the preceding paper.

we then determine a set of invariants of the two-dimensional Kepler problem. A subset of these generates the dynamical group which is shown to be $O(3,2)$. An interesting feature of the result is that all the $O(3,2)$ generators may be constructed from further invariants that shift the radial and angular quantum numbers by one-half unit[2].

Calculations similar to those performed on the two-dimensional system yield the generators of the dynamical group of the three-dimensional Kepler systems. We find this to be $O(4,2)$ in agreement with the analyses of Malkin and Manko[3], Barut and Kleinert[4] and Fronsdal[5]. The calculations on Kepler systems presented here obtain for the first time their spectrum generating invariants as explicit functions of the time.

Finally, in the appendix we determine the spectrum generating groups of the radial equations of two and three-dimensional Kepler systems.

2. THE TWO-DIMENSIONAL HARMONIC OSCILLATOR

The Schroedinger equation is

$$-\tfrac{1}{2}(\partial_r\partial_r + r^{-1}\partial_r + r^{-2}\partial_\phi\partial_\phi - kr^2 + 2i\partial_t)\cdot f(r,\phi,t) = 0 \,. \quad (2.1)$$

We assume a Q operator of the form

$$Q = Q^{\phi\phi}\partial_\phi\partial_\phi + Q^{r\phi}\partial_r\partial_\phi + Q^r\partial_r + Q^\phi\partial_\phi + Q^t\partial_t + Q^0 \quad (2.2)$$

and choose the independent functions to be

$$f,\, f_r,\, f_\phi,\, f_t,\, f_{\phi\phi},\, f_{rt},\, f_{r\phi},\, f_{\phi t},\, f_{tt},\, f_{r\phi\phi},\, f_{\phi\phi\phi},\, f_{\phi\phi t}\,.$$

The determining equations derived from

$$(\partial_r\partial_r + r^{-1}\partial_r + r^{-2}\partial_\phi\partial_\phi - kr^2 + 2i\partial_t)\,Qf = 0$$

are

$$Q^t_r = 0\ ,\ r^2 Q^{\phi\phi}_r + Q^{r\phi}_\phi = ,\ \ rQ^{\phi\phi}_\phi - rQ^{r\phi}_\phi + Q^{r\phi} = 0\ ,$$

$$Q^t_\phi - 2ir^2 Q^{r\phi}_r = 0\ ,\ Q^t_{rr} + r^{-1}Q^t_r + r^{-2}Q^t_{\phi\phi} + 2iQ^t_t - 4iQ^r_r = 0$$

$$Q^{r\phi}_{rr} - r^{-1}Q^{r\phi}_r + r^{-2}Q^{r\phi}_{\phi\phi} + 2iQ^{r\phi}_t + r^{-2}Q^{r\phi} + 2r^{-2}Q^r_\phi + 2Q^\phi_r = 0$$

$$Q^{\phi\phi}_{rr} + r^{-1}Q^{\phi\phi}_r + r^{-2}Q^{\phi\phi}_{\phi\phi} + 2iQ^{\phi\phi}_t - 2r^{-2}Q^r_r + 2r^{-3}Q^r + 2r^{-2}Q^\phi_\phi = 0$$

$$Q^r_{rr} - r^{-1}Q^r_r + r^{-2}Q^r_{\phi\phi} + 2iQ^r_t + r^{-2}Q^r + 2Q^0_r = 0$$

$$Q^0_{rr} + r^{-1}Q^0_r + r^{-2}Q^0_{\phi\phi} + 2iQ^0_t + 2kr^2Q^r_r + 2krQ^r = 0$$

$$Q^\phi_{rr} + r^{-1}Q^\phi_r + r^{-2}Q^\phi_{\phi\phi} + 2iQ^\phi_t + 2kr^2Q^{r\phi}_r + 2krQ^{r\phi} + 2r^{-2}Q^0_\phi = 0\ .$$

Their general solutions give

$$Q = \sum_{i=1}^{16} a^i Q_i$$

where a^i's are the integration constants and Q_i's are

$$Q_{\substack{1\\2}} = \exp\left[\pm 2i\phi\right]\left(\pm ir^{-2}\cdot\partial_\phi\partial_\phi + r^{-1}\partial_r\partial_\phi \pm ir^{-1}\partial_r - r^{-2}\partial_\phi \mp \partial_t\right) = \mp\frac{i}{2}\, Q_{\substack{11\\13}}\cdot Q_{\substack{12\\14}}$$

$$Q_3 = \partial_\phi = \tfrac{1}{4}i\,(k)^{-\frac{1}{2}}\left(Q_{11}Q_{14} - Q_{13}Q_{12}\right)$$

$$Q_{\substack{4\\5}} = \exp\left[\mp 2i\sqrt{kt}\right]\left(\mp i\sqrt{k}r\partial_r + \partial_t + ikr^2 \mp i\sqrt{k}\right) = \frac{i}{2}\, Q_{\substack{12\\11}}\, Q_{\substack{14\\13}}$$

$$Q_6 = \partial_t = \tfrac{1}{4}i\left(Q_{11}Q_{14} + Q_{12}Q_{13}\right)$$

$$Q_{\substack{7\\8\\9\\10}} = \exp\left\{ \substack{\pm\\\pm} i2\sqrt{k}t \substack{+\\+\\-\\-} 2i\phi \right\}\left\{ \substack{+\\+\\-\\-} ir^{-2}\partial_\phi\partial_\phi + i\left(\substack{+\\+\\-\\-} r^{-1} \substack{-\\+\\+\\-} \sqrt{k}r\right)\partial_r - \left(r^{-2} \substack{-\\+\\-\\+} \sqrt{k}\right)\partial_\phi \right.$$

$$\left. + r^{-1}\partial_r\partial_\phi \substack{-\\-\\+\\+} \partial_t \substack{-\\-\\+\\+} ikr^2 \right\} = \substack{-\\-\\+\\+} \frac{i}{2}\left(Q_{\substack{11\\12\\13\\14}}\right)^2$$

$$Q_{\substack{11\\12\\13\\14}} = \exp\left\{ \substack{+\\-\\+\\-} i\sqrt{k}t \substack{+\\+\\-\\-} i\phi \right\}\left(\partial_r \substack{+\\+\\-\\-} ir^{-1}\partial_\phi \substack{+\\-\\+\\-} \sqrt{k}r\right), \; Q_{15} = \partial_\phi\partial_\phi , \; Q_{16} = 1$$

Here Q_i, $i = 1, 2, \ldots, 10$, are expressed in terms of Q_{11}, Q_{12}, Q_{13} and Q_{14} as listed and Q is equal to the operator defined in (2.2)

3. THE TWO DIMENSIONAL HYDROGENLIKE ATOM

The two dimensional Kepler problem has several interesting features[2]. The Schroedinger equation is

$$-\frac{i}{2}\left(\partial_r\partial_r + r^{-1}\partial_r + r^{-2}\partial_\phi\partial_\phi + 2Zr^{-1} + 2i\partial_t\right)\sum_{nm} C_{nm}\exp\left[-iE_n t\right]\Psi_{nm}(r,\phi) = 0 ,$$

where E_n is given by $-\frac{1}{2}Z^2\left(n - \frac{1}{2}\right)^{-2}$. The transformation operator D leading to the linear spectrum can be chosen to be

$$D = D_r \cdot D_t = \exp\left\{ r\partial_r \log\left(-\frac{i}{2Z}\partial_t\right)\right\} \cdot \exp\left\{ t\partial_t \log\left[2Z/\sqrt{-2H}^{\,3}\right]\right\} \tag{3.2}$$

and the transformed equation is then

$$-\frac{1}{2}\left(\partial_r\partial_r + r^{-1}\partial_r + r^{-2}\partial_\phi\partial_\phi - ir^{-1}\partial_t - \frac{1}{4}\right)f(r,\phi,t) = 0 \qquad (3.3)$$

where

$$f(r,\phi,t) = \sum_{nm} C_{nm} \exp\left[i(n-\tfrac{1}{2})t\right]\Psi_{nm}\left(\frac{n}{2Z}r,\phi\right).$$

We choose as independent functions the set

$$f, f_r, f_\phi, f_t, f_{\phi\phi}, f_{rt}, f_{\phi t}, f_{r\phi}, f_{r\phi\phi}, f_{\phi\phi\phi},$$

and let the Q operator be

$$Q = Q^{\phi\phi}\partial_\phi\partial_\phi + Q^{r\phi}\partial_r\partial_\phi + Q^r\partial_r + Q^\phi\partial_\phi + Q^t\partial_t + Q^0 . \qquad (3.4)$$

Then the determining equations for Q derived from the equation

$$\left(\partial_r\partial_r + r^{-1}\partial_r + r^{-2}\partial_\phi\partial_\phi - ir^{-1}\partial - \frac{1}{4}\right)Q\,f(r,\phi) = 0$$

are

$$Q^t_r = 0\,,\;\; Q^{\phi\phi}_\phi - Q^{r\phi}_r + r^{-1}Q^{r\phi} = 0\,,$$

$$Q^{\phi\phi}_r + r^{-2}Q^{r\phi}_\phi = 0\,,\;\; Q^t_\phi + irQ^{r\phi}_r - \frac{i}{2}Q^{r\phi} = 0\,,$$

$$Q^t_{rr} + r^{-1}Q^t_r + r^{-2}Q^t_{\phi\phi} - ir^{-1}Q^t_t + 2ir^{-1}Q^r_r - ir^{-2}Q^r = 0$$

$$Q^{r\phi}_{rr} - r^{-1}Q^{r\phi}_r + r^{-2}Q^{r\phi}_{\phi\phi} - ir^{-1}Q^{r\phi}_t + r^{-2}Q^{r\phi} + 2r^{-2}Q^r_\phi + 2Q^\phi_r = 0\,,$$

$$Q^{\phi\phi}_{rr} + r^{-1}Q^{\phi\phi}_r + r^{-2}Q^{\phi\phi}_{\phi\phi} - ir^{-1}Q^{\phi\phi}_t + 2r^{-3}Q^r + 2r^{-2}Q^\phi_\phi - 2r^{-2}Q^r_r = 0\,,$$

$$Q^r_{rr} - r^{-1}Q^r_r + r^{-2}Q^r_{\phi\phi} - ir^{-1}Q^r_t + r^{-2}Q^r + 2Q^0_r = 0\,,$$

$$Q^0_{rr} + r^{-1}Q^0_r + r^{-2}Q^0_{\phi\phi} - ir^{-1}Q^0_t + \frac{1}{2}Q^r = 0\,,$$

$$Q^\phi_{rr} + r^{-1}Q^\phi_r + r^{-2}Q^\phi_{\phi\phi} - ir^{-1}Q^\phi_t + 2r^{-2}Q^0_\phi + \frac{1}{2}Q^{r\phi}_r = 0\,.$$

Solving these equations, one obtains a 16 parameter generator

$$Q = \sum_{i=1}^{16} a^i Q_i$$

where a^i are the integration constants, and the Q_i are

$$Q_{\substack{1\\2}} = \exp\,[\pm i\phi]\,(\pm ir^{-1}\partial_\phi\partial_\phi + \partial_r\partial_\phi \pm \frac{i}{2}\partial_r - \frac{1}{2}r^{-1}\partial_\phi \pm \frac{1}{2}\partial_t) = \mp\frac{i}{2}\,Q_{\substack{11\\13}}\cdot Q_{\substack{12\\14}}$$

$$Q_3 = \partial_\phi = \frac{i}{2}(Q_{13}\cdot Q_{12} - Q_{11}\cdot Q_{14})\,,$$

$$Q_{\substack{4\\5}} = \exp\,[\pm it]\,(\pm ir\partial_r + \partial_t - \frac{i}{2}r \pm \frac{i}{2}) = -i\,Q_{\substack{11\\12}}\cdot Q_{\substack{13\\14}}\,,$$

$$Q_6 = \partial_t = -\frac{i}{2}(Q_{11}\cdot Q_{14} + Q_{12}\cdot Q_{13})\,,$$

$$Q_{\substack{7\\8\\9\\10}} = \exp\left\{\substack{+\\-\\+\\-}\,it\;\substack{+\\+\\-\\-}\,i\phi\right\}\times$$

$$\times\left\{\substack{+\\+\\-\\-}\,r^{-1}\partial_\phi\partial_\phi + \partial_r\partial_\phi\;\substack{+\\+\\-\\-}\,(1\;\substack{+\\-\\+\\-}\,r)\,\partial_r - \frac{1}{2}(r^{-1}\;\substack{+\\-\\+\\-}\,1)\,\partial_\phi\;\substack{+\\+\\-\\-}\,\frac{1}{2}\partial_t\;\substack{-\\-\\+\\+}\,\frac{i}{4}r\right\}$$

$$= \substack{-\\-\\+\\+}\,\frac{i}{2}\,(Q_{\substack{11\\12\\13\\14}})^2$$

$$Q_{\substack{11\\12\\13\\14}} = \exp\left\{ \substack{+\\-\\+\\-} \tfrac{1}{2} it \substack{+\\+\\-\\-} \tfrac{1}{2} i\phi \right\} \left(r^{\frac{1}{2}} \partial_r \substack{+\\+\\-\\-} i r^{-\frac{1}{2}} \partial_\phi \substack{-\\+\\-\\+} \frac{1}{2} r^{\frac{1}{2}} \right), \quad Q_{15} = \partial_\phi \partial_\phi , \quad Q_{16} = 1 .$$

Here Q is equal to the operator defined in (3.4).

It is interesting to notice that all the operators listed above can be transformed into those of the harmonic oscillator in section two by the transformation $r \to r^2$, $\phi \to 2\phi$, $t \to 2t$. This is because under this transformation the equation (3.1) becomes exactly the same as equation (2.1). As a result, we can express the Q_i $(i = 1, 2, \ldots, 10)$ in terms of Q_{11}, Q_{12}, Q_{13} and Q_{14} as listed* above as in the case of the harmonic oscillator.

As the Q_i listed above were obtained by using the transformed equation (3.3), the corresponding operators $\tilde{Q}_i$ for the original equation (3.1) are given by

$$\tilde{Q}_i = D^{-1} Q_i D \ .$$

Now we analyze these operators. The commutation relations of $\tilde{Q}_{11}$, $\tilde{Q}_{12}$, $\tilde{Q}_{13}$ and $\tilde{Q}_{14}$ with $\tilde{Q}_3$ and $\tilde{Q}_6$ show that these operators raise or lower the eigenvalues of $\tilde{Q}_3$ and $\tilde{Q}_6$ by $i/2$. The commutation relations among $\tilde{Q}_{11}$, $\tilde{Q}_{12}$, $\tilde{Q}_{13}$ and $\tilde{Q}_{14}$ are given by

$$[\tilde{Q}_{11}, \tilde{Q}_{12}] = 0, \quad [\tilde{Q}_{11}, \tilde{Q}_{13}] = 0, \quad [\tilde{Q}_{11}, \tilde{Q}_{14}] = 1,$$

$$[\tilde{Q}_{12}, \tilde{Q}_{13}] = -1, \quad [\tilde{Q}_{12}, \tilde{Q}_{14}] = 0, \quad [\tilde{Q}_{13}, \tilde{Q}_{14}] = 0 .$$

From this and from the fact that $\tilde{Q}_i$ $i = 1, 2, \ldots, 10$, can be expressed in terms of $\tilde{Q}_{11}$, $\tilde{Q}_{12}$, $\tilde{Q}_{13}$ and $\tilde{Q}_{14}$, one can see that the set $\{\tilde{Q}_i\}$ $i = 1, 2, \ldots, 10$, forms a closed Lie algebra.

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*To derive the expression for the Q_i one must use the operator identity $\partial_r \partial_r = -r^{-1}\partial_r - r^{-2}\partial_\phi\partial_\phi + i r^{-1}\partial_t + \tfrac{1}{4}$, which holds in the space $\{\exp[int]\, \Psi_{nm}(\frac{n}{2Z} r, \phi)\}$.

This set contains two subalgebras $\{\tilde{Q}_1, \tilde{Q}_2, \tilde{Q}_3\}$ and $\{\tilde{Q}_4, \tilde{Q}_5, \tilde{Q}_6\}$, for which the commutation relations are given by

$$[\tilde{Q}_1, \tilde{Q}_2] = -\frac{i}{2}\tilde{Q}_3, \quad [\tilde{Q}_3, \tilde{Q}_1] = i\tilde{Q}_1, \quad [\tilde{Q}_3, \tilde{Q}_2] = -i\tilde{Q}_2,$$

$$[\tilde{Q}_4, \tilde{Q}_5] = -2i\tilde{Q}_6, \quad [\tilde{Q}_6, \tilde{Q}_4] = i\tilde{Q}_4, \quad [\tilde{Q}_6, \tilde{Q}_5] = -i\tilde{Q}_5,$$

$$[\tilde{Q}_1, \tilde{Q}_4] = i\tilde{Q}_7, \quad [\tilde{Q}_1, \tilde{Q}_5] = -i\tilde{Q}_8, \quad [\tilde{Q}_2, \tilde{Q}_4] = i\tilde{Q}_9,$$

$$[\tilde{Q}_2, \tilde{Q}_5] = -i\tilde{Q}_{10}, \quad [\tilde{Q}_i, \tilde{Q}_6] = 0 \ (i = 1, 2, 3), \quad [\tilde{Q}_i, \tilde{Q}_3] = 0 \ (i = 4, 5, 6).$$

These imply that $\tilde{Q}_1$ and $\tilde{Q}_2$ shift the eigenvalue of $\tilde{Q}_3$, that is im, by one unit, and also $\tilde{Q}_4$ and $\tilde{Q}_5$ shift the eigenvalue of $\tilde{Q}_6$, in, by unit amount. As $\tilde{Q}_1$, $\tilde{Q}_2$ and $\tilde{Q}_3$ commute with $\tilde{Q}_6$, which is a labeling operator of the energy, they comprise the Lie algebra of the degeneracy group, and the operators $-i(\tilde{Q}_1 - \tilde{Q}_2)$ and $-(\tilde{Q}_1 + \tilde{Q}_2)$ are identified* as A_x and A_y where $\mathbf{A}$ is the two dimensional analogue of the Runge-Lenz vector defined by

$$\mathbf{A} = (A_x, A_y, 0) = (-2H)^{-\frac{1}{2}}\{\frac{1}{2}(\mathbf{L} \times \mathbf{P} - \mathbf{P} \times \mathbf{L}) + Z\frac{\mathbf{r}}{r}\}.$$

Here we have defined the cross product by, for example,

$$\mathbf{L} \times \mathbf{P} = \begin{vmatrix} i & j & k \\ 0 & 0 & L_z \\ P_x & P_y & 0 \end{vmatrix}$$

The raising and lowering operators of n, $\tilde{Q}_4$ and $\tilde{Q}_5$, satisfy the equations

$$\tilde{Q}_4 \exp[-iE_n t]\Psi_{nm} = i\left(\frac{n+\frac{1}{2}}{n-\frac{1}{2}}\right)^{3/2}\{(n+l)(n-l)\}^{\frac{1}{2}} \exp[-iE_{n+1m} t]\Psi_{n+1m}$$

* See appendix I for the analogous calculation in three dimensions.

$$\tilde{Q}_5 \exp\left[-iE_n t\right] \Psi_{nm} = i\left(\frac{n-3/2}{n-\frac{1}{2}}\right) \left\{(n-1+l)(n-1-l)\right\}^{\frac{1}{2}} \exp\left[-iE_{n-1} t\right] \Psi_{n-1m}$$

$$\tilde{Q}_6 \exp\left[-iE_n t\right] \Psi_{nm} = i\,n \exp\left[-iE_n t\right] \Psi_{nm}$$

where[6]

$$\Psi_{nm} = \frac{4}{(2|m|)!} \left\{ \frac{(n+|m|-1)!}{2\pi(2n-1)^3 (n-|m|-1)!} \right\}^{\frac{1}{2}} F(-n+|m|+1 \,|\, 2|m|+1 \,|\, \alpha r)$$

$$\times\, e^{-\frac{1}{2}\alpha r} (\alpha r)^l e^{im\phi}$$

and

$$\alpha = 2\sqrt{-2E_n}\,, \quad E_n = -\tfrac{1}{2} Z^2 (n - \tfrac{1}{2})^{-2} .$$

Now to obtain the skew-adjoint operators under the scalar product

$$(f, g) = \int_0^{2\pi}\int_0^{\infty} f^* \cdot g \; r dr d\phi$$

We define the new operators $\bar{Q}_i$ by

$$\bar{Q}_i = (\tilde{Q}_6)^{-3/2}\, \tilde{Q}_i\, (\tilde{Q}_6)^{3/2} .$$

The $\bar{Q}_4$ and $\bar{Q}_5$ then satisfy the relations

$$\bar{Q}_4 \exp\left[-iE_n t\right] \Psi_{nm} = i\left\{(n+l)(n-l)\right\}^{\frac{1}{2}} \exp\left[-iE_{n+1} t\right] \Psi_{n+1m}$$

$$\bar{Q}_5 \exp\left[-iE_n t\right] \Psi_{nm} = i\left\{(n-1+l)(n-1-l)\right\}^{\frac{1}{2}} \exp\left[-iE_{n-1} t\right] \Psi_{n-1m}$$

and the operators

$$J_{23} = -i(\bar{Q}_1 + \bar{Q}_2), \qquad J_{31} = -(\bar{Q}_1 - \bar{Q}_2), \quad J_{12} = \bar{Q}_3$$

$$J_{53} = -\tfrac{1}{2}(\bar{Q}_4 + \bar{Q}_5), \qquad J_{34} = -\frac{i}{2}(\bar{Q}_4 - \bar{Q}_5), \quad J_{45} = \bar{Q}_6$$

$$J_{24} = -\frac{i}{2}(\bar{Q}_7 + \bar{Q}_8 + \bar{Q}_9 + \bar{Q}_{10}), \quad J_{25} = \tfrac{1}{2}(\bar{Q}_7 - \bar{Q}_8 + \bar{Q}_9 - \bar{Q}_{10})$$

$$J_{14} = \tfrac{1}{2}(\bar{Q}_7 + \bar{Q}_8 - \bar{Q}_9 - \bar{Q}_{10}), \qquad J_{15} = \frac{i}{2}(\bar{Q}_7 - \bar{Q}_8 - \bar{Q}_9 + \bar{Q}_{10})$$

are skew adjoint.

Calculating the commutation relations directly, we find that J_{ij}'s generate an $O(3,2)$ algebra

$$[J_{ab}, J_{cd}] = -g_{bc}J_{ad} + g_{ac}J_{bd} - g_{ad}J_{bc} + g_{bd}J_{ac}$$

with

$$g_{11} = g_{22} = g_{33} = -g_{44} = -g_{55} = -1, \; (J_{ab})^{\dagger} = -J_{ab}$$

Therefore the set $\{\exp[-iE_n t]\Psi_{nm}\}$ where n and m are both integers or both half odd integers form the basis for a UIR of $O(3,2)$, and m or n specify the UIR of an $O(1,2)$ or $O(3)$ subgroup of $O(3,2)$.

If we allow both integral and half integral n and m, then the set $\{\exp[-iE_n t]\Psi_{nm}\}$ will comprise the basis of a UIR of the group generated by the set of generators $\{Q_1, Q_2, Q_3, Q_6, Q_{11}, Q_{12}, Q_{13}, Q_{14}, Q_{16}\}$. For this case the scalar product must be defined by

$$(f, g) = \int_0^{4\pi}\int_0^{\infty} f^* \cdot g \; r dr d\phi \; .$$

4. HYDROGENLIKE ATOM

We shall determine the invariants of the time-depdendent Schroedinger equation of the hydrogenlike atom

$$-\tfrac{1}{2}(\partial_r\partial_r + 2r^{-1}\partial_r - r^{-2}L^2 + 2Zr^{-1} + 2\partial_t)\cdot$$

$$\sum_{nlm} C_{nlm}\cdot\exp[-iE_n t]\,\Psi_{nlm}(r,\theta,\phi) = 0\,, \qquad (4.1)$$

where the C_{nlm} are arbitrary constants, and

$$L^2 = -y^2\partial_x\partial_x + 2x\partial_x - y^{-2}\partial_\phi\partial_\phi$$

with $x = \cos\theta$, $y = \sin\theta$.

The transformation operator D leading to the linear spectrum can be chosen to be

$$D = D_r\cdot D_t = \exp\{r\partial_r\cdot\log(-\frac{i}{2Z}\partial_t)\}\cdot\exp\{t\partial_t\cdot\log(\frac{2Z}{\sqrt{-2H}^3})\} \qquad (4.2)$$

where

$$H = -\frac{1}{2}\partial_r\partial_r - r^{-1}\partial_r + \frac{1}{2}r^{-2}\cdot L^2 - Zr^{-1}\,.$$

The transformed equation is then

$$(\partial_r\partial_r + 2r^{-1}\partial_r - r^{-2}\cdot L^2 - r^{-1}\partial_t - \frac{1}{4})\cdot f(r,\theta,\phi) = 0 \qquad (4.3)$$

with $f(r,\theta,\phi) = \sum_{nlm} C_{nlm}\exp[int]\,\Psi_{nlm}(\frac{n}{2Z}r,\theta,\phi)$.

We choose as independent functions, the set

$$f,\ f_r,\ f_x,\ f_\phi,\ f_t,\ f_{rx},\ f_{r\phi},\ f_{rt},\ f_{xx},\ f_{x\phi},\ f_{xt},\ f_{\phi\phi},$$

$$f_{\phi t},\ f_{rxx},\ f_{rx\phi},\ f_{r\phi\phi},\ f_{xxx},\ f_{xx\phi},\ f_{x\phi\phi},\ f_{\phi\phi\phi},$$

where $x = \cos\theta$, and let the Q operator be

$$Q = Q^{rx}\partial_x\partial_r + Q^{r\phi}\partial_r\partial_\phi + Q^{xx}\partial_x\partial_x + Q^{x\phi}\partial_x\partial_\phi + Q^{\phi\phi}\partial_\phi\partial_\phi$$

$$+ Q^r\partial_r + Q^x\partial_x + Q^\phi\partial_\phi + Q^t\partial_t + Q^0 \qquad (4.4)$$

The determining equations derived from the equation

$$(\partial_r\partial_r + 2r^{-2}\partial_r - r^{-2}L^2 - r^{-1}\partial_t - \frac{1}{4})\, Q \cdot f = 0$$

are then found to be

$$A \cdot Q^0 + \frac{1}{2}Q^r_r = 0\,,\; A \cdot Q^r + 2r^{-2}Q^r - 4r^{-1}Q^r_r + 2Q^0_r = 0\,,$$

$$A \cdot Q^x + 2r^{-2}Q^x + 4r^{-2}xQ^r_r - 4r^{-3}xQ^r +$$

$$+ (4r^{-2} + \frac{1}{2})\, Q^{rx}_r - 4r^{-3}Q^{rx} + 2r^{-2}y^2Q^0_x = 0\,,$$

$$A \cdot Q^\phi + 2r^{-2}y^{-2}Q^0_\phi + \frac{1}{2}Q^{r\phi}_r = 0\,,\; A \cdot Q^t + 2ir^{-1}Q^r_r - ir^{-2}Q^r = 0\,,$$

$$A \cdot Q^{rx} - 4r^{-1}Q^{rx}_r + 4r^{-2}Q^{rx} + 2r^{-2}y^2Q^r_x + 2Q^x_r = 0\,,$$

$$A \cdot Q^{r\phi} - 4r^{-1}Q^{r\phi}_r + 2r^{-2}Q^{r\phi} + 2r^{-2}y^{-2}Q^r_\phi + 2Q^\phi_r = 0\,,$$

$$A \cdot Q^{xx} + 6r^{-2}Q^{xx} + 8r^{-2}xQ^{rx}_r - 8r^{-3}xQ^{rx} - 2r^{-2}y^2Q^r_r + 2r^{-3}y^2Q^r$$

$$+ 2r^{-2}y^2Q^x_x + 2r^{-2}xQ^x = 0\,,$$

$$A \cdot Q^{x\phi} + 2r^{-2}Q^{x\phi} + 4r^{-2}Q^{r\phi}_r - 4r^{-3}xQ^{r\phi} + 2r^{-2}y^{-2}Q^x_\phi + 2r^{-2}y^2Q^\phi_x = 0\,,$$

$$A \cdot Q^{\phi\phi} - 2r^{-2}y^{-2}Q^{r}_{r} + 2r^{-3}y^{-2}Q^{r} - 4r^{-2}xy^{-4}Q^{rx}_{r} + 4r^{-3}xy^{-4}Q^{rx}$$

$$- 2r^{-2}y^{-6}(1+3x^{2})Q^{xx} + 2r^{-2}y^{-2}Q^{\phi}_{\phi} - 2r^{-2}xy^{-4}Q^{x} = 0\,,$$

$$2ir^{-1}Q^{rx}_{r} - ir^{-2}Q^{rx} + 2r^{-2}y^{2}Q^{t}_{x} = 0\,,\ 2ir^{-1}Q^{r\phi}_{r} - ir^{-2}Q^{r\phi} + 2r^{-2}y^{-2}Q^{t}_{\phi} = 0\,,$$

$$r^{-2}y^{2}Q^{rx}_{x} + r^{-2}xQ^{rx} + Q^{xx}_{r} = 0\,,\ \ r^{-2}y^{2}Q^{r\phi}_{x} + Q^{x\phi}_{r} + r^{-2}y^{-2}Q^{rx}_{\phi} = 0\,,$$

$$Q^{\phi\phi} + r^{-2}y^{-2}Q^{r\phi}_{\phi} - r^{-2}xy^{-4}Q^{rx} = 0\,,\ -y^{2}Q^{rx}_{r} + r^{-1}y^{2}Q^{rx} + y^{2}Q^{xx}_{x} + 2xQ^{xx} = 0\,,$$

$$- y^{2}Q^{r\phi}_{r} + r^{-1}y^{2}Q^{r\phi} + y^{2}Q^{x\phi}_{x} + xQ^{x} + y^{-2}Q^{xx}_{\phi} = 0\,,$$

$$- y^{-2}Q^{rx}_{r} + r^{-1}y^{-2}Q^{rx} + y^{-2}Q^{x\phi}_{\phi} + y^{2}Q^{\phi\phi}_{x} - 2xy^{-4}Q^{xx} = 0\,,$$

$$- y^{-2}Q^{r\phi}_{r} + r^{-1}y^{-2}Q^{r\phi} + y^{-2}Q^{\phi\phi}_{\phi} - xy^{-4}Q^{x\phi} = 0\,,\quad Q^{t}_{r} = 0\,,$$

where

$$A = \partial_r\partial_r + 2r^{-1}\partial_r + r^{-2}y^{2}\partial_x\partial_x - 2r^{-2}x\partial_x + r^{-2}y^{-2}\partial_\phi\partial_\phi - r^{-1}\partial_t$$

Solving these equations, one obtains a 22 parameter generator.

$$Q = \sum_{i=1}^{22} a^{i} Q_{i}$$

where the a^{i}'s are integration constants and the Q_i's are

$$Q_{\substack{1\\2}} = \exp\left[\pm i\phi\right](y\,\partial_x \mp ixy^{-1}\partial_\phi)\,,\quad Q_3 = \partial_\phi$$

$$Q_{\substack{4\\5}} = 2i \exp[\pm i\phi] \left(xy\partial_r\partial_x + r^{-1}y^3\partial_x\partial_x \mp iy^{-1}\partial_r\partial_\phi + r^{-1}y^{-1}\partial_\phi\partial_\phi + y\partial_r - 2r^{-1}xy\partial_x - \frac{i}{2}y\partial_t\right)$$

$$Q_6 = 2i\left(-y^2\partial_r\partial_x + r^{-1}xy^2\partial_x\partial_x + r^{-1}xy^{-2}\partial_\phi\partial_\phi + x\partial_r + - 2r^{-1}x^2\partial_x - \frac{i}{2}x\partial_t\right),$$

$$Q_{\substack{7\\8}} = \exp[\pm it]\left(\pm ir\partial_r + \partial_t - \frac{i}{2}r \pm i\right), \quad Q_9 = \partial_t ,$$

$$Q_{\substack{10\\11\\12\\13}} = 2i\exp\left\{\substack{+\\-\\+\\-}\,it\ \substack{+\\+\\-\\-}\,i\phi\right\}\left\{xy\partial_r\partial_x + r^{-1}y^3\partial_x\partial_x \substack{-\\-\\+\\+}\, iy^{-1}\partial_r\partial_\phi + r^{-1}y^{-1}\partial_\phi\partial_\phi + + \left(1 \substack{+\\-\\+\\-} \frac{1}{2}r\right)y\partial_r - \frac{1}{2}\left(4r^{-1} \substack{+\\-\\+\\-}\, 1\right)xy\partial_x \substack{+\\-\\-\\+}\, \frac{i}{2}y^{-1}\partial_\phi - \frac{i}{2}y\partial_t - \frac{1}{4}ry\right\}$$

$$Q_{\substack{14\\15}} = 2i\exp[\pm it]\left\{-y^2\partial_r\partial_x + r^{-1}xy^2\partial_x\partial_x + r^{-1}xy^{-2}\partial_\phi\partial_\phi + \left(1 \pm \frac{1}{2}r\right)x\partial_r + \frac{1}{2}\left(\pm y^2 - 4r^{-1}x^2\right)\partial_x + x\partial_t - \frac{1}{4}rx\right\},$$

$$Q_{\substack{16\\17}} = \exp[\pm 2i\phi]\left\{y^2\partial_x\partial_x \mp 2ix\partial_x\partial_\phi - x^2y^{-2}\partial_\phi\partial_\phi \mp iy^{-2}(1+x^2)\partial_\phi\right\} = \left(Q_{\substack{1\\2}}\right)^2,$$

$$Q_{18} = y^2\partial_x\partial_x + x^2y^{-2}\partial_\phi\partial_\phi - 2x\partial_x = \frac{1}{2}\left(Q_1Q_2 + Q_2Q_1\right),$$

$$Q_{\substack{19\\20}} = \exp[\pm i\phi]\left(y\partial_x\partial_\phi \mp ixy^{-1}\partial_\phi\partial_\phi\right) = Q_{\substack{1\\2}}Q_3 ,$$

$$Q_{21} = \partial_\phi\partial_\phi , \quad Q_{22} = 1 .$$

and Q is the operator defined in (4.4).

As these are obtained from the transformed equation (4.3) the corresponding operators, $\tilde{Q}_i$, for the original equation (4.1) are given by

$$\tilde{Q}_i = D^{-1} Q_i D \ .$$

This transformation can be carried through easily for Q_i $i = 1, 2, \ldots, 6$, as demonstrated in the appendix I.

Now we analyze those operators. First $\tilde{Q}_{16}, \tilde{Q}_{17}, \ldots, \tilde{Q}_{21}$ are expressed in terms of the operators $\tilde{Q}_1$, $\tilde{Q}_2$ and $\tilde{Q}_3$, as listed, and $\tilde{Q}_{22}$ is a unit operator. $\tilde{Q}_i$ $i = 1, 2, \ldots, 6$, commute with $\tilde{Q}_9$ or the Hamiltonian, and are identified* as

$$\tilde{Q}_1 = -L_+, \ \tilde{Q}_2 = L_-, \ \tilde{Q}_3 = L_z,$$

$$\tilde{Q}_4 = i(A_x + iA_y), \ \tilde{Q}_5 = i(A_x - iA_y), \ \tilde{Q}_6 = iA_z$$

where L_+, L_- and L_Z are the usual angular momentum operators and $\mathbf{A} = (A_x, A_y, A_z)$ is the well known Runge-Lenz vector defined by

$$\mathbf{A} = (-2H)^{-\frac{1}{2}} \left\{ \frac{1}{2} (\mathbf{L} \times \mathbf{P} - \mathbf{P} \times \mathbf{L}) + Z \frac{\mathbf{r}}{r} \right\} .$$

As is well known, $\mathbf{A}$ and $\mathbf{L}$ generate an $O(4)$ algebra.

Furthermore, $\tilde{Q}_7$, $\tilde{Q}_8$ and $\tilde{Q}_9$ comprise a closed Lie algebra**:

$$[\tilde{Q}_7, \tilde{Q}_8] = -2i\tilde{Q}_9, \ [\tilde{Q}_9, \tilde{Q}_7] = i\tilde{Q}_7, \ [\tilde{Q}_9, \tilde{Q}_8] = -i\tilde{Q}_8 \ .$$

From these it is clear that $\tilde{Q}_7$ and $\tilde{Q}_8$ shift the eigenvalue of $\tilde{Q}_9$, that is *in*, by unit amount. They satisfy the relations

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* See appendix I.

** By using a dummy variable t, Armstrong[7] obtained the identical generators in his treatment of the radial function. Our result shows that his t is in fact the physical time.

$$\tilde{Q}_7 \cdot \exp\left[-iE_n t\right] \cdot \Psi_{nlm} = i\left(1+\frac{1}{n}\right)^2 \{(n+l+1)(n-l)\}^{\frac{1}{2}} \cdot \exp\left[-iE_{n+1}t\right] \cdot \Psi_{n+1lm}$$

$$\tilde{Q}_8 \cdot \exp\left[-iE_n t\right] \cdot \Psi_{nlm} = i\left(1-\frac{1}{n}\right)^2 \{(n-l-1)(n+l)\}^{\frac{1}{2}} \cdot \exp\left[-iE_{n-1}t\right] \cdot \Psi_{n-1lm}$$

$$\tilde{Q}_9 \cdot \exp\left[-iE_n t\right] \cdot \Psi_{nlm} = n \cdot \exp\left[-iE_n t\right] \cdot \Psi_{nlm}$$

where Ψ_{nlm} is a normalized hydrogenic wave function and $E_n = -Z^2/2n^2$. Because of the factor $(1 \pm 1/n)^2$ in the above coefficients, no linear combinations of the operators $\tilde{Q}_7$ and $\tilde{Q}_8$ are skew adjoint under the usual scalar product

$$(f, g) = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} f^* \cdot g\, r^2 \sin\theta\, dr d\theta d\phi \ .$$

To remove these factors we define the new operators $\overline{Q}_i$ by*

$$\overline{Q}_i = (\tilde{Q}_9)^{-2} \cdot \tilde{Q}_i \cdot (\tilde{Q}_9)^2 \ .$$

Using the new operators, we have

$$\overline{Q}_7 \cdot \exp\left[-iE_n t\right] \cdot \Psi_{nlm} = i\{(n+l+1)(n-l)\}^{\frac{1}{2}} \cdot \exp\left[-iE_{n+1}t\right] \cdot \Psi_{n+1lm}$$

$$\overline{Q}_8 \cdot \exp\left[-iE_n t\right] \cdot \Psi_{nlm} = i\{(n-l-1)(n+l)\}^{\frac{1}{2}} \cdot \exp\left[-iE_{n-1}t\right] \cdot \Psi_{n-1lm} \ .$$

A straightforward analysis then shows that

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*It is clear that the new set $\{\overline{Q}_i\}$ satisfy the same commutation relations as the set $\{\tilde{Q}_i\}$, and as $\tilde{Q}_i$ $i = 1, 2, \ldots, 6, 9$, commute with $\tilde{Q}_9$ we have $\overline{Q}_i = \tilde{Q}_i$ for $i = 1, 2, \ldots, 6, 9$.

$$M_1 = -\frac{1}{2}(\overline{Q}_7 + \overline{Q}_8)\,,\quad M_2 = -\frac{i}{2}(\overline{Q}_7 - \overline{Q}_8)\,,\quad M_3 = \overline{Q}_9$$

are skew adjoint under the above scalar product. They satisfy the relations of the well known $O(1,2)$ algebra

$$[M_1, M_2] = -M_3\,,\quad [M_2, M_3] = M_1\,,\quad [M_3, M_1] = M_2\ .$$

The remaining operators can be identified as

$$\overline{Q}_{10} = -i\,[\overline{Q}_4, \overline{Q}_7]\,,\qquad \overline{Q}_{11} = i\,[\overline{Q}_4, \overline{Q}_8]\,,\qquad \overline{Q}_{12} = -i\,[\overline{Q}_5, \overline{Q}_7]$$

$$\overline{Q}_{13} = i\,[\overline{Q}_5, \overline{Q}_8]\,,\qquad \overline{Q}_{14} = -i\,[\overline{Q}_6, \overline{Q}_7]\,,\qquad \overline{Q}_{15} = i\,[\overline{Q}_6, \overline{Q}_7]$$

by direct calculation. It is clear then that a particular complex extention of the set $\{\overline{Q}_i\}\, i = 1, 2, \ldots, 15$, comprises the well known $O(4,2)$ algebra of the dynamical group of the hydrogenlike atom,[3,4,5] and that the hydrogenic wave functions form the basis of a UIR of $O(4.2)$.

APPENDIX I.
THE INVERSE TRANSFORMATION OF CONSTANTS OF THE MOTION TO THE ORIGINAL PHYSICAL SPACE

We illustrate the process of inverse transformation by using the operators which arise in the treatment of the hydrogenlike atom.

The inverse transformation operator D^{-1} is given by

$$D^{-1} = (D_t)^{-1}(D_r)^{-1} = \exp\{t\partial_t \log (2Z)^{-1}(-2H)^{\frac{1}{2}}\}$$

$$\exp\{r\partial_r \log\left(\frac{-i}{2Z}\partial_t\right)^{-1}\}$$

and the transformed operator $\tilde{Q}_i$ will be

$$\tilde{Q}_i = D^{-1}Q_iD = (D_t)^{-1}(D_r)^{-1}Q_iD_rD_t .$$

as Q_1, Q_2 and Q_3 commute with D^{-1}, $\tilde{Q}_1$, $\tilde{Q}_2$ and $\tilde{Q}_3$ have the same form as Q_1, Q_2 and Q_3.

We next calculate the transformation of Q_6 explicitly. (The transformation of Q_4 and Q_5 can be done in a similar manner.) Performing the transformation $(D_r)^{-1}$ on Q_6, we get

$$(D_r)^{-1}(-2iy^2\partial_r\partial_x + 2ir^{-1}xy^2\partial_x\partial_x + 2ir^{-1}xy^{-2}\partial_\phi\partial_\phi + 2ix\partial_r$$

$$-4ir^{-1}x^2\partial_x + x\partial_t)D_r = Z^{-1}\partial_t(-y^2\partial_r\partial_x + r^{-1}xy^2\partial_x\partial_x$$

$$+r^{-1}xy^{-2}\partial_\phi\partial_\phi + x\partial_r - 2r^{-1}x^2\partial_x + Zx) . \qquad (1)$$

Here we have used the relations

$$[\partial_t, Q_6] = 0 , (D_r)^{-1}\partial_rD_r = -\tfrac{1}{2}iZ^{-1}\partial_t\partial_r , \quad (D_r)^{-1}r^{-1}D_r = -\tfrac{1}{2}iZ^{-1}r^{-1}\partial_t .$$

Next we perform the time dilation $(D_t)^{-1}$ on (1) to get

$$\tilde{Q}_6 = 2(-2H)^{-3/2}\partial_t\,(-y^2\partial_r\partial_x + r^{-1}xy^2\partial_x\partial_x + r^{-1}xy^{-2}\partial_\phi\partial_\phi$$

$$+ x\partial_r - 2r^{-1}x^2\partial_x + Zx)$$

after using the relations

$$[H, (D_r)^{-1}Q_6D_r] = 0\ ,\ \ (D_t)^{-1}\partial_tD_t = 2Z(-2H)^{-3/2}\partial_t\ .$$

As we have an operator identity $H = i\partial_t$ in the original space $\{\exp[-iE_nt]\Psi_{nlm}\}$, the last equation can be rewritten as

$$\tilde{Q}_6 = i(-2H)^{-\frac{1}{2}}(-y^2\partial_r\partial_x + r^{-1}xy^2\partial_x\partial_x + r^{-1}xy^{-2}\partial_\phi\partial_\phi$$

$$+ x\partial_r - 2r^{-1}x^2\partial_x + Zx)\ .$$

This is a Z component of the wellknown Runge-Lenz vector to within the phase factor i.

APPENDIX II

Sometimes we have to make use of an operator identity implied by the original differential equation to obtain a closed Lie algebra. We illustrate the process by using the two-dimensional hydrogenlike atom, taking the commutator $[Q_1, Q_9]$ as an example.

By calculating the commutator explicitly, we get

$$[Q_1, Q_9] = e^{it}\Big\{ -\frac{1}{2}r\left(\partial_r\partial_r + r^{-1}\partial_r + r^{2}\partial_\phi\partial_\phi + \frac{1}{4}\right) + \frac{1}{2}r\partial_r + \frac{1}{4}$$

$$- 2i\left(\partial_r\partial_r + r^{-1}\partial_r + r^{-2}\partial_\phi\partial_\phi - ir^{-1}\partial_t - \frac{1}{4}\right)\partial_\phi\Big\}$$

$$= e^{it}\Big\{ -\frac{1}{2}r\left(\partial_r\partial_r + r^{-1}\partial_r + r^{-2}\partial_\phi\partial_\phi - ir^{-1}\partial_t - \frac{1}{4}\right)$$

$$- 2i\left(\partial_r\partial_r + r^{-1}\partial_r + r^{-2}\partial_\phi\partial_\phi - ir^{-1}\partial_t - \frac{1}{4}\right)\partial_\phi$$

$$- \frac{i}{2}r^{-1}\partial_t - \frac{1}{4}r + \frac{1}{2}r\partial_r + \frac{1}{4}\Big\}$$

But as we have an operator identity (see equation 3.3)

$$\partial_r\partial_r + r^{-1}\partial_r + r^{-2}\partial_\phi\partial_\phi - ir^{-1}\partial_t - \frac{1}{4} = 0$$

in the transformed space $\{\exp[i(n - \tfrac{1}{2})t]\,\Psi_{nlm}(\frac{n}{2Z}r, \phi)\}$, the above equation reduces to

$$[Q_1, Q_9] = -\frac{i}{2}e^{it}\left(ir\partial_r + \partial_t - \frac{i}{2}r + \frac{i}{2}\right) = Q_4 .$$

APPENDIX III

To illustrate the way in which one can proceed in determining the groups of equations which arise on separation of variables we consider the radial equations for the two and three-dimensional Kepler problem.

The radial equations of two and three-dimensional hydrogenlike atoms are

$$\left(\partial_r \partial_r + pr^{-1}\partial_r - qr^{-2} + 2Zr^{-1} + 2i\partial_t\right) \sum_n C_n \exp\left[-iE_n t\right] R_{nl}(r) = 0$$

where

$$p = 1,\ q = l^2,\ E_n = -\tfrac{1}{2} Z^2 (n - \tfrac{1}{2})^{-2}$$

for 2-dimensions or

$$p = 2,\ q = l(l+1),\ E_n = -\tfrac{1}{2} Z^2 n^{-2}$$

for 3-dimensions.

The appropriate compound dilation operator D is

$$D = D_r \cdot D_t = \exp\{r\partial_r \log(-\tfrac{1}{2} iZ^{-1}\partial_t)\} \cdot \exp\{t\partial_t \log(2Z/\sqrt{-2H^3})\}$$

where

$$H = -\tfrac{1}{2}(\partial_r \partial_r + pr^{-1}\partial_r - qr^{-2} + 2Zr^{-1}).$$

After the transformation by D, the radial equation becomes

$$(\partial_r \partial_r + pr^{-1}\partial_r - qr^{-2} - tr^{-1}\partial_t - \tfrac{1}{4}) f(r, t) = 0$$

where

$$f(r,t)=\sum_n C_n \exp\left[i(-2E_n)^{-\frac{1}{2}} Zt\right] R_{nl}\left(\frac{r}{2\sqrt{-2E_n}}\right) .$$

We choose the independent functions to be

$$f,\ f_t,\ f_r,\ f_{rt}$$

and let the Q operator be of the form

$$Q = Q^r \partial_r + Q^t \partial_t + Q^0 .$$

The determining equations then are

$$Q^t_r = 0\ ,\ \ Q^r_{rr} - pr^{-1}Q^r_r - ir^{-1}Q^r_t + pr^{-2}Q^r + 2Q^0_r = 0$$

$$Q^t_{rr} + pr^{-1}Q^t_r - ir^{-1}Q^t_t - ir^{-2}Q^r + 2ir^{-1}Q^r_r = 0$$

$$Q^0_{rr} + pr^{-1}Q^0_r - ir^{-1}Q^0_t - 2qr^{-3}Q^r + 2(qr^{-2} + \tfrac{1}{4})Q^r_r = 0 .$$

The solution is

$$Q = \sum_{i=1}^{4} a^i Q_i$$

where a^i's are integration constants, and

$$Q_{\substack{1\\2}} = \pm ie^{\pm it}(r\partial_r \mp i\partial_t \mp \tfrac{1}{2}r + \tfrac{1}{2}p)$$

$$Q_3 = \partial_t\ \ ,\ \ Q_4 = 1 .$$

Q_1 and Q_2 are the same as the n shift operators obtained in section 3 and 4.

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RESUMEN

Usando los métodos sistemáticos del artículo previo (I), se obtienen los invariantes y los grupos dinámicos correspondientes a la ecuación de Schröedinger dependiente del tiempo, para el oscilador armónico en dos dimensiones y para el átomo hidrogenoide en dos y en tres dimensiones.