Integrals of the motion and Green function for dual damped oscillators and coupled harmonic oscillators

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The application for the integrals of the motion of a quantum system in deriving Green function or propagator is presented. The Green function is shown to be the eigenfunction of the integrals of the motion which described initial points of the system trajectory in the phase space. The exact expressions for the Green functions of the dual damped oscillators and the coupled harmonic oscillators are evaluated in co-ordinate representations. The relation between the integrals of the motion method and other methods such as Feynman path integral and Schwinger method are also presented.

Keywords: Integrals of the motion; Green function; dual damped oscillators.

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1. Introduction

In non-relativistic quantum mechanics the Green function or propagator gives the probability amplitude for a particle to travel from one spatial point at one time to another spatial point at a later time. The propagator can be calculated by the well-known methods of Feynman path integral [1] and Schwinger method [2-6]. In 1975, V.V. Dodonov, I.A. Malkin, and V.I. Man'ko [7] presented the connection between the integrals of the motion of a quantum system and its Green function that is the eigenfunction of the integrals of the motion describing initial points of the system trajectory in the phase space. In 1977, V.V. Dodonov *et al*. [8] constructed a new method of calculating non-equilibrium density matrices with the aid of the quantum integrals of motion. D.B. Lemeshevskiy and V.I. Man'ko applied the integrals of the motion method to the problem of the driven harmonic oscillator in 2012 [9].

The aim of this paper is to calculate the Green functions for the dual damped oscillators and the coupled harmonic oscillators by the integrals of the motion method. The organization of this paper are as follows. In Sec. 2, the Green function for the dual damped oscillators is derived. In Sec. 3, the Green function for the coupled harmonic oscillators is obtained with the aid of the integrals of the motion. Finally, the conclusion is presented in Sec. 4.

2. The Green function for a dual damped oscillators

The Bateman damped harmonic oscillator is described as an open system in which energy is dissipated by interaction with a heat bath [10]. Bateman has shown that dissipative systems can be presented as a pair of damped oscillators, the so called dual damped oscillators. This system includes a primary one expressed by q_1 variables and its time reversed image by q_2 variables. The Hamiltonian operator for a dual damped oscillators can be expressed as

$$\hat{H}(t) = \frac{\hat{p}_1\hat{p}_2}{m} + \frac{\gamma}{m}(\hat{q}_2\hat{p}_2 - \hat{q}_1\hat{p}_1) + \left(k - \frac{\gamma^2}{4m}\right)\hat{q}_1\hat{q}_2, \quad (1)$$

where k is the harmonic coefficients and γ is a damping coefficient.

The aim of this section is to derive the Green function $G(x_1, x_2, x_1', x_2', t)$ of the Schrödinger equation by the method of integrals of the motion [7-9]. The classical equation of motion for this system are

$$m\ddot{q}_1 + \gamma\dot{q}_1 + kq_1 = 0, \quad (2)$$

$$m\ddot{q}_2 - \gamma\dot{q}_2 + kq_2 = 0. \quad (3)$$

The classical paths in the phase space under the initial conditions $q_1(0) = q_{10}$, $q_2(0) = q_{20}$, $p_1(0) = p_{10}$, and $p_2(0) = p_{20}$ are given by

$$q_1(t) = q_{10}e^{-\gamma t/2m}\cos\Omega t + \frac{p_{20}}{m\Omega}e^{-\gamma t/2m}\sin\Omega t, \quad (4)$$

$$q_2(t) = q_{20}e^{\gamma t/2m}\cos\Omega t + \frac{p_{10}}{m\Omega}e^{\gamma t/2m}\sin\Omega t, \quad (5)$$

$$p_1(t) = p_{10}e^{\gamma t/2m}\cos\Omega t - q_{20}m\Omega e^{\gamma t/2m}\sin\Omega t, \quad (6)$$

$$p_2(t) = p_{20}e^{-\gamma t/2m}\cos\Omega t - q_{10}m\Omega e^{-\gamma t/2m}\sin\Omega t, \quad (7)$$

where $\Omega = \sqrt{(k/m) - (\gamma^2/4m^2)}$. Now we consider the system of Eqs. (4)-(7) as an algebraic system for unknown initial positions q_{10} and q_{20} and initial momentums p_{10} and p_{20}.

The variables q_1, q_2, p_1, p_2, and t are taken as the parameters. The solution of this system can be written as the operator in Hilbert space as

$$\hat{q}_{10}(\hat{q}_1,\hat{p}_2,t) = \hat{q}_1 e^{\gamma t/2m}\cos\Omega t - \frac{\hat{p}_2}{m\Omega}e^{\gamma t/2m}\sin\Omega t, \quad (8)$$

$$\hat{p}_{10}(\hat{q}_2,\hat{p}_1,t) = \hat{p}_1 e^{-\gamma t/2m}\cos\Omega t + \hat{q}_2 m\Omega e^{-\gamma t/2m}\sin\Omega t, \quad (9)$$

$$\hat{q}_{20}(\hat{q}_2,\hat{p}_1,t) = \hat{q}_2 e^{-\gamma t/2m}\cos\Omega t - \frac{\hat{p}_1}{m\Omega}e^{-\gamma t/2m}\sin\Omega t, \quad (10)$$

$$\hat{p}_{20}(\hat{q}_1,\hat{p}_2,t) = \hat{p}_2 e^{\gamma t/2m}\cos\Omega t + \hat{q}_1 m\Omega e^{\gamma t/2m}\sin\Omega t. \quad (11)$$

The operators $\hat{q}_{10}$, $\hat{q}_{20}$, $\hat{p}_{10}$, and $\hat{p}_{20}$ are the integrals of the motion because theirs satisfy equation of

$$\frac{d\hat{I}}{dt} = \frac{\partial \hat{I}}{\partial t} + \frac{i}{\hbar}[\hat{H},\hat{I}] = 0, \quad (12)$$

where $\hat{I}$ may be $\hat{q}_{10}$, $\hat{q}_{20}$, $\hat{p}_{10}$, and $\hat{p}_{20}$. Then these operators must satisfy equations for the Green function $G(x_1,x_2,x_1',x_2',t)$ [7-9],

$$\hat{q}_{10}(x_1)G(x_1,x_2,x_1',x_2',t) = \hat{q}_1(x_1') \times G(x_1,x_2,x_1',x_2',t), \quad (13)$$

$$\hat{p}_{10}(x_1)G(x_1,x_2,x_1',x_2',t) = -\hat{p}_1(x_1') \times G(x_1,x_2,x_1',x_2',t), \quad (14)$$

$$\hat{q}_{20}(x_2)G(x_1,x_2,x_1',x_2',t) = \hat{q}_2(x_2') \times G(x_1,x_2,x_1',x_2',t), \quad (15)$$

$$\hat{p}_{20}(x_2)G(x_1,x_2,x_1',x_2',t) = -\hat{p}_2(x_2') \times G(x_1,x_2,x_1',x_2',t), \quad (16)$$

where the operators on the left-hand sides of the equations act on variables x_1 and x_2, and on the right-hand sides, on x_1' and x_2'. Now we write Eqs. (13)-(16) explicitly,

$$\left(x_1\left(e^{\gamma t/2m}\cos\Omega t\right) + \left(\frac{i\hbar}{m\Omega}e^{\gamma t/2m}\sin\Omega t\right)\frac{\partial}{\partial x_2}\right) \times G(x_1,x_2,x_1',x_2',t) = x_1' G(x_1,x_2,x_1',x_2',t), \quad (17)$$

$$\left(\left(-i\hbar e^{-\gamma t/2m}\cos\Omega t\right)\frac{\partial}{\partial x_1} + x_2\left(m\Omega e^{-\gamma t/2m}\sin\Omega t\right)\right) \times G(x_1,x_2,x_1',x_2',t) = i\hbar\frac{\partial G(x_1,x_2,x_1',x_2',t)}{\partial x_1'}, \quad (18)$$

$$\left(x_2\left(e^{-\gamma t/2m}\cos\Omega t\right) + \left(\frac{i\hbar}{m\Omega}e^{-\gamma t/2m}\sin\Omega t\right)\frac{\partial}{\partial x_1}\right) \times G(x_1,x_2,x_1',x_2',t) = x_2' G(x_1,x_2,x_1',x_2',t), \quad (19)$$

$$\left(x_1\left(m\Omega e^{\gamma t/2m}\sin\Omega t\right) - \left(i\hbar e^{\gamma t/2m}\cos\Omega t\right)\frac{\partial}{\partial x_2}\right) \times G(x_1,x_2,x_1',x_2',t) = i\hbar\frac{\partial G(x_1,x_2,x_1',x_2',t)}{\partial x_2'}. \quad (20)$$

By modifying Eqs. (17)-(20), the system of equation for deriving the Green function $G(x_1,x_2,x_1',x_2',t)$ are

$$\frac{\partial G(x_1,x_2,x_1',x_2',t)}{\partial x_2} = -\frac{i}{\hbar}(m\Omega e^{-\gamma t/2m}\csc\Omega t x_1' - m\Omega\cot\Omega t x_1)G(x_1,x_2,x_1',x_2',t), \quad (21)$$

$$\frac{\partial G(x_1,x_2,x_1',x_2',t)}{\partial x_1} = -\frac{i}{\hbar}(m\Omega e^{\gamma t/2m}\csc\Omega t x_2' - m\Omega\cot\Omega t x_2)G(x_1,x_2,x_1',x_2',t), \quad (22)$$

$$\frac{\partial G(x_1,x_2,x_1',x_2',t)}{\partial x_1'} = \frac{i}{\hbar}(m\Omega\cot\Omega t x_2' - m\Omega e^{-\gamma t/2m}\csc\Omega t x_2)G(x_1,x_2,x_1',x_2',t), \quad (23)$$

$$\frac{\partial G(x_1,x_2,x_1',x_2',t)}{\partial x_2'} = \frac{i}{\hbar}(m\Omega\cot\Omega t x_1' - m\Omega e^{\gamma t/2m}\csc\Omega t x_1)G(x_1,x_2,x_1',x_2',t) \quad (24)$$

Now one can integrate Eq. (21) with respect to the variable x_2 to obtain

$$G(x_1,x_2,x_1',x_2',t) = C(x_1,x_1',x_2',t)\exp\left[\frac{i}{\hbar}(m\Omega \times \cot\Omega t x_1 x_2 - m\Omega e^{-\gamma t/2m}\csc\Omega t x_1' x_2)\right], \quad (25)$$

where $C(x_1,x_1',x_2',t)$ is the function of x_1, x_1', x_2', and t. Substituting Eq. (25) into Eq. (22) to obtain $C(x_1,x_1',x_2',t)$, the result is

$$C(x_1,x_1',x_2',t) = C(x_1',x_2',t) \times \exp\left[-\frac{i}{\hbar}m\Omega e^{\gamma t/2m}\csc\Omega t x_1 x_2'\right]. \quad (26)$$

So, the Green in Eq. (25) can be written as

$$G(x_1,x_2,x_1',x_2',t) = C(x_1',x_2',t)\exp\left[\frac{i}{\hbar}\left(m\Omega\cot\Omega t x_1 x_2 - m\Omega\csc\Omega t(e^{\gamma t/2m}x_1 x_2' + e^{-\gamma t/2m}x_1' x_2)\right)\right] \quad (27)$$

The next step is substituting the Green function in Eq. (27) into Eq. (23) to obtain $C(x_1',x_2',t)$ as

$$C(x_1',x_2',t) = C(x_2',t)\exp\left[\frac{im\Omega}{\hbar}\cot\Omega t x_1' x_2'\right]. \quad (28)$$

Thus, the Green function in Eq. (27) becomes

$$G(x_1,x_2,x_1',x_2',t) = C(x_2',t)\exp\left[\frac{i}{\hbar}\Big(m\Omega\cot\Omega t \times (x_1 x_2 + x_1' x_2') - m\Omega\csc\Omega t(e^{\gamma t/2m}x_1 x_2' + e^{-\gamma t/2m}x_1' x_2)\Big)\right]. \quad (29)$$

Substituting the Green function in Eq. (29) into Eq. (24) to obtain $C(x_2', t)$, the result is

$$\frac{\partial C(x_2',t)}{\partial x_2'} = 0. \tag{30}$$

So, it can be implied that $C(x_2', t) = C(t)$. The Green function in Eq. (29) can be expressed as

$$G(x_1,x_2,x_1',x_2',t) = C(t)\exp\left[\frac{i}{\hbar}\times\Big(m\Omega\cot\Omega t(x_1x_2 + x_1'x_2') - m\Omega\csc\Omega t(e^{\gamma t/2m}x_1x_2' + e^{-\gamma t/2m}x_1'x_2)\Big)\right]. \tag{31}$$

To find $C(t)$, we must substitute the Green function of Eq. (31) into the Schrodinger equation

$$i\hbar\frac{\partial G(x_1,x_2,x_1',x_2',t)}{\partial t} = -\frac{\hbar^2}{m}\frac{\partial^2 G(x_1,x_2,x_1',x_2',t)}{\partial x_1 x_2} - \frac{i\hbar\gamma}{2m}x_2\frac{\partial G(x_1,x_2,x_1',x_2',t)}{\partial x_2} + \frac{i\hbar\gamma}{2m}x_1\frac{\partial G(x_1,x_2,x_1',x_2',t)}{\partial x_1} + \left(k - \frac{\gamma^2}{4m}\right)x_1x_2G(x_1,x_2,x_1',x_2',t). \tag{32}$$

After some algebra, we obtain an equation

$$\frac{dC(t)}{dt} = -C(t)(\Omega\cot\Omega t). \tag{33}$$

Equation (33) can be simply integrated with respect to time t, and one obtains

$$C(t) = \frac{C}{\sin\Omega t}, \tag{34}$$

where C is a constant. Substituting Eq. (34) into Eq. (31) and applying the initial condition

$$\lim_{t\to 0^+} G(x_1,x_2,x_1',x_2',t) = \delta(x_1 - x_1')\delta(x_2 - x_2'), \tag{35}$$

we obtain

$$C = \frac{m\Omega}{2\pi i\hbar}. \tag{36}$$

So, the Green function for a dual damped oscillator is

$$G(x_1,x_2,x_1',x_2',t) = \frac{m\Omega}{2\pi i\hbar\sin\Omega t}\times\exp\left[\frac{i}{\hbar}\Big(m\Omega\cot\Omega t(x_1x_2 + x_1'x_2') - m\Omega\csc\Omega t(e^{\gamma t/2m}x_1x_2' + e^{-\gamma t/2m}x_1'x_2)\Big)\right], \tag{37}$$

which is the same form as the result of S. Pepore and B. Sukbot calculated by the Schwinger method [5].

3. The Green function for the coupled harmonic oscillators

Considering a system of two harmonic oscillators which are coupled together by another spring. Assuming that the masses of the oscillators and three spring constants are all the same. Let their displacements be q_1 and q_2. The Hamiltonian operator for the coupled harmonic oscillator can be written as [6]

$$\hat{H}(t) = \frac{\hat{p}_1^2}{2m} + \frac{\hat{p}_2^2}{2m} + m\omega^2(\hat{q}_1^2 - \hat{q}_1\hat{q}_2 + \hat{q}_2^2), \tag{38}$$

where ω is the constant frequency. The classical equations of motion determining the oscillator positions and momentums are

$$\ddot{q}_1 + \omega^2(2q_1 - q_2) = 0, \tag{39}$$

$$\ddot{q}_2 + \omega^2(2q_1 - q_2) = 0. \tag{40}$$

The classical paths in the phase space under the initial conditions $q_1(0) = q_{10}$, $q_2(0) = q_{20}$, $p_1(0) = p_{10}$, and $p_2(0) = p_{20}$ are

$$q_1(t) = q_{10}\left(\frac{\cos\omega t + \cos\sqrt{3}\omega t}{2}\right) + q_{20}\left(\frac{\cos\omega t + \cos\sqrt{3}\omega t}{2}\right) + p_{10}\left(\frac{3\sin\omega t + \sqrt{3}\sin\sqrt{3}\omega t}{6m\omega}\right) + p_{20}\left(\frac{3\sin\omega t - \sqrt{3}\sin\sqrt{3}\omega t}{6m\omega}\right), \tag{41}$$

$$q_2(t) = q_{10}\left(\frac{\cos\omega t - \cos\sqrt{3}\omega t}{2}\right) + q_{20}\left(\frac{\cos\omega t + \cos\sqrt{3}\omega t}{2}\right) + p_{10}\left(\frac{3\sin\omega t - \sqrt{3}\sin\sqrt{3}\omega t}{6m\omega}\right) + p_{20}\left(\frac{3\sin\omega t + \sqrt{3}\sin\sqrt{3}\omega t}{6m\omega}\right), \tag{42}$$

$$p_1(t) = p_{10}\left(\frac{\cos\omega t + \cos\sqrt{3}\omega t}{2}\right) + p_{20}\left(\frac{\cos\omega t - \cos\sqrt{3}\omega t}{2}\right) - q_{10}\left(\frac{m\omega\sin\omega t + \sqrt{3}m\omega\sin\sqrt{3}\omega t}{2}\right) - $$
$$+ q_{20}\left(\frac{m\omega\sin\omega t - \sqrt{3}m\omega\sin\sqrt{3}\omega t}{2}\right), \tag{43}$$
$$p_2(t) = p_{10}\left(\frac{\cos\omega t - \cos\sqrt{3}\omega t}{2}\right) + p_{20}\left(\frac{\cos\omega t + \cos\sqrt{3}\omega t}{2}\right) - q_{10}\left(\frac{m\omega\sin\omega t - \sqrt{3}m\omega\sin\sqrt{3}\omega t}{2}\right)$$
$$- q_{20}\left(\frac{m\omega\sin\omega t + \sqrt{3}m\omega\sin\sqrt{3}\omega t}{2}\right). \tag{44}$$

Now we consider the system of Eqs. (41)-(44) as an algebraic system for unknown initial positions q_{10} and q_{20} and initial momentums p_{10} and p_{20}. The variables q_1, q_2, p_1, p_2, and t are taken as the parameters. The solution of this system can be written as the operator in Hilbert space as

$$\hat{q}_{10}(\hat{q}_1,\hat{q}_2,\hat{p}_1,\hat{p}_2,t) = \hat{q}_1\left(\frac{\cos\omega t + \cos\sqrt{3}\omega t}{2}\right) + \hat{q}_2\left(\frac{\cos\omega t - \cos\sqrt{3}\omega t}{2}\right)$$
$$- \hat{p}_1\left(\frac{\sqrt{3}\sin\sqrt{3}\omega t + 3\sin\omega t}{6m\omega}\right) + \hat{p}_2\left(\frac{\sqrt{3}\sin\sqrt{3}\omega t - 3\sin\omega t}{6m\omega}\right), \tag{45}$$
$$\hat{q}_{20}(\hat{q}_1,\hat{q}_2,\hat{p}_1,\hat{p}_2,t) = \hat{q}_1\left(\frac{\cos\omega t - \cos\sqrt{3}\omega t}{2}\right) + \hat{q}_2\left(\frac{\cos\omega t + \cos\sqrt{3}\omega t}{2}\right)$$
$$+ \hat{p}_1\left(\frac{\sqrt{3}\sin\sqrt{3}\omega t - 3\sin\omega t}{6m\omega}\right) - \hat{p}_2\left(\frac{\sqrt{3}\sin\sqrt{3}\omega t + 3\sin\omega t}{6m\omega}\right), \tag{46}$$
$$\hat{p}_{10}(\hat{q}_1,\hat{q}_2,\hat{p}_1,\hat{p}_2,t) = \hat{q}_1\left(\frac{m\omega\sin\omega t + \sqrt{3}m\omega\sin\sqrt{3}\omega t}{2}\right) + \hat{q}_2\left(\frac{m\omega\sin\omega t - \sqrt{3}m\omega\sin\sqrt{3}\omega t}{2}\right)$$
$$+ \hat{p}_1\left(\frac{\cos\omega t + \cos\sqrt{3}\omega t}{2}\right) + \hat{p}_2\left(\frac{\cos\omega t - \cos\sqrt{3}\omega t}{2}\right), \tag{47}$$
$$\hat{p}_{20}(\hat{q}_1,\hat{q}_2,\hat{p}_1,\hat{p}_2,t) = \hat{q}_1\left(\frac{m\omega\sin\omega t - \sqrt{3}m\omega\sin\sqrt{3}\omega t}{2}\right) + \hat{q}_2\left(\frac{m\omega\sin\omega t + \sqrt{3}m\omega\sin\sqrt{3}\omega t}{2}\right)$$
$$+ \hat{p}_1\left(\frac{\cos\omega t - \cos\sqrt{3}\omega t}{2}\right) + \hat{p}_2\left(\frac{\cos\omega t + \cos\sqrt{3}\omega t}{2}\right). \tag{48}$$

The operators $\hat{q}_{10}$, $\hat{q}_{20}$, $\hat{p}_{10}$, $\hat{p}_{20}$, are the integrals of the motion because their satisfy Eq. (12). Then these operators must satisfy Eqs. (13)-(16) and can be explicitly written as

$$\left[x_1\left(\frac{\cos\omega t + \cos\sqrt{3}\omega t}{2}\right) + x_2\left(\frac{\cos\omega t - \cos\sqrt{3}\omega t}{2}\right) + i\hbar\left(\frac{\sqrt{3}\sin\sqrt{3}\omega t + 3\sin\omega t}{6m\omega}\right)\frac{\partial}{\partial x_1}\right.$$
$$\left.- i\hbar\left(\frac{\sqrt{3}\sin\sqrt{3}\omega t - 3\sin\omega t}{6m\omega}\right)\frac{\partial}{\partial x_2}\right]G(x_1,x_2,x_1',x_2',t) = x_1'G(x_1,x_2,x_1',x_2',t), \tag{49}$$
$$\left[x_1\left(\frac{m\omega\sin\omega t + \sqrt{3}m\omega\sin\sqrt{3}\omega t}{2}\right) + x_2\left(\frac{m\omega\sin\omega t - \sqrt{3}m\omega\sin\sqrt{3}\omega t}{2}\right) - i\hbar\left(\frac{\cos\omega t + \cos\sqrt{3}\omega t}{2}\right)\frac{\partial}{\partial x_1}\right.$$
$$\left.- i\hbar\left(\frac{\cos\omega t - \cos\sqrt{3}\omega t}{2}\right)\frac{\partial}{\partial x_2}\right]G(x_1,x_2,x_1',x_2',t) = i\hbar\frac{\partial G(x_1,x_2,x_1',x_2',t)}{\partial x_1'}, \tag{50}$$

$$\left[x_1\left(\frac{\cos\omega t-\cos\sqrt{3}\omega t}{2}\right)+x_2\left(\frac{\cos\omega t+\cos\sqrt{3}\omega t}{2}\right)-i\hbar\left(\frac{\sqrt{3}\sin\sqrt{3}\omega t-3\sin\omega t}{6m\omega}\right)\frac{\partial}{\partial x_1}\right.$$

$$\left.+i\hbar\left(\frac{\sqrt{3}\sin\sqrt{3}\omega t+3\sin\omega t}{6m\omega}\right)\frac{\partial}{\partial x_2}\right]G(x_1,x_2,x_1',x_2',t)=x_2'G(x_1,x_2,x_1',x_2',t), \quad (51)$$

$$\left[x_1\left(\frac{m\omega\sin\omega t-\sqrt{3}m\omega\sin\sqrt{3}\omega t}{2}\right)+x_2\left(\frac{m\omega\sin\omega t+\sqrt{3}m\omega\sin\sqrt{3}\omega t}{2}\right)-i\hbar\left(\frac{\cos\omega t-\cos\sqrt{3}\omega t}{2}\right)\frac{\partial}{\partial x_1}\right.$$

$$\left.-i\hbar\left(\frac{\cos\omega t+\cos\sqrt{3}\omega t}{2}\right)\frac{\partial}{\partial x_2}\right]G(x_1,x_2,x_1',x_2',t)=i\hbar\frac{\partial G(x_1,x_2,x_1',x_2',t)}{\partial x_2'}. \quad (52)$$

By modifying Eqs. (49)-(52), the system of equations for deriving the Green function $G(x_1,x_2,x_1',x_2',t)$ are

$$\frac{\partial G(x_1,x_2,x_1',x_2',t)}{\partial x_1}=\frac{i}{\hbar}\left[x_1\left(\frac{m\omega}{2}\cot\omega t+\frac{\sqrt{3}}{2}m\omega\cot\sqrt{3}\omega t\right)+x_2\left(\frac{m\omega}{2}\cot\omega t-\frac{\sqrt{3}}{2}m\omega\cot\sqrt{3}\omega t\right)\right.$$

$$\left.-x_1'\left(\frac{m\omega}{2}\csc\omega t+\frac{\sqrt{3}}{2}m\omega\csc\sqrt{3}\omega t\right)-x_2'\left(\frac{m\omega}{2}\csc\omega t-\frac{\sqrt{3}}{2}m\omega\csc\sqrt{3}\omega t\right)\right]G(x_1,x_2,x_1',x_2',t), \quad (53)$$

$$\frac{\partial G(x_1,x_2,x_1',x_2',t)}{\partial x_2}=\frac{i}{\hbar}\left[x_1\left(\frac{m\omega}{2}\cot\omega t-\frac{\sqrt{3}}{2}m\omega\cot\sqrt{3}\omega t\right)+x_2\left(\frac{m\omega}{2}\cot\omega t+\frac{\sqrt{3}}{2}m\omega\cot\sqrt{3}\omega t\right)\right.$$

$$\left.-x_1'\left(\frac{m\omega}{2}\csc\omega t-\frac{\sqrt{3}}{2}m\omega\csc\sqrt{3}\omega t\right)-x_2'\left(\frac{m\omega}{2}\csc\omega t+\frac{\sqrt{3}}{2}m\omega\csc\sqrt{3}\omega t\right)\right]G(x_1,x_2,x_1',x_2',t), \quad (54)$$

$$\frac{\partial G(x_1,x_2,x_1',x_2',t)}{\partial x_1'}=-\frac{i}{\hbar}\left[x_1\left(\frac{m\omega}{2}\csc\omega t+\frac{\sqrt{3}}{2}m\omega\csc\sqrt{3}\omega t\right)+x_2\left(\frac{m\omega}{2}\csc\omega t-\frac{\sqrt{3}}{2}m\omega\csc\sqrt{3}\omega t\right)\right.$$

$$\left.-x_1'\left(\frac{m\omega}{2}\cot\omega t+\frac{\sqrt{3}}{2}m\omega\cot\sqrt{3}\omega t\right)-x_2'\left(\frac{m\omega}{2}\cot\omega t-\frac{\sqrt{3}}{2}m\omega\cot\sqrt{3}\omega t\right)\right]G(x_1,x_2,x_1',x_2',t), \quad (55)$$

$$\frac{\partial G(x_1,x_2,x_1',x_2',t)}{\partial x_2'}=-\frac{i}{\hbar}\left[x_1\left(\frac{m\omega}{2}\csc\omega t-\frac{\sqrt{3}}{2}m\omega\csc\sqrt{3}\omega t\right)+x_2\left(\frac{m\omega}{2}\csc\omega t+\frac{\sqrt{3}}{2}m\omega\csc\sqrt{3}\omega t\right)\right.$$

$$\left.-x_1'\left(\frac{m\omega}{2}\cot\omega t-\frac{\sqrt{3}}{2}m\omega\cot\sqrt{3}\omega t\right)-x_2'\left(\frac{m\omega}{2}\cot\omega t+\frac{\sqrt{3}}{2}m\omega\cot\sqrt{3}\omega t\right)\right]G(x_1,x_2,x_1',x_2',t), \quad (56)$$

Now we can integrate Eq. (53) with respect to the variable x_1 to obtain

$$G(x_1,x_2,x_1',x_2',t)=C(x_1',x_2,x_2',t)\exp\left[\frac{i}{\hbar}\left(x_1^2\left(\frac{m\omega}{4}\cot\omega t+\frac{\sqrt{3}}{4}m\omega\cot\sqrt{3}\omega t\right)\right.\right.$$

$$+x_1x_2\left(\frac{m\omega}{2}\cot\omega t-\frac{\sqrt{3}}{2}m\omega\cot\sqrt{3}\omega t\right)-x_1x_1'\left(\frac{m\omega}{2}\csc\omega t+\frac{\sqrt{3}}{2}m\omega\csc\sqrt{3}\omega t\right)$$

$$\left.\left.-x_1x_2'\left(\frac{m\omega}{2}\csc\omega t-\frac{\sqrt{3}}{2}m\omega\csc\sqrt{3}\omega t\right)\right)\right], \quad (57)$$

where $C(x_1',x_2,x_2',t)$ is the function of x_1',x_2,x_2', and t. Substituting Eq. (57) into Eq. (54) to find $C(x_1',x_2,x_2',t)$, we get

$$C(x'_1,x_2,x'_2,t)=C(x'_1,x'_2,t)\exp\left[\frac{i}{\hbar}\left(x_2^2\left(\frac{m\omega}{4}\cot\omega t+\frac{\sqrt{3}}{4}m\omega\cot\sqrt{3}\omega t\right)\right.\right.$$
$$\left.\left.-x_2x'_1\left(\frac{m\omega}{2}\csc\omega t-\frac{\sqrt{3}}{2}m\omega\csc\sqrt{3}\omega t\right)-x_2x'_2\left(\frac{m\omega}{2}\csc\omega t+\frac{\sqrt{3}}{2}m\omega\csc\sqrt{3}\omega t\right)\right)\right], \tag{58}$$

So, the Green function in Eq. (57) can be written as

$$G(x_1,x_2,x'_1,x'_2,t)=C(x'_1,x'_2,t)\exp\left[\frac{i}{\hbar}\left((x_1^2+x_2^2)\left(\frac{m\omega}{4}\cot\omega t+\frac{\sqrt{3}}{4}m\omega\cot\sqrt{3}\omega t\right)\right.\right.$$
$$+x_1x_2\left(\frac{m\omega}{2}\cot\omega t-\frac{\sqrt{3}}{2}m\omega\cot\sqrt{3}\omega t\right)-(x_1x'_1+x_2x'_2)\left(\frac{m\omega}{2}\csc\omega t+\frac{\sqrt{3}}{2}m\omega\csc\sqrt{3}\omega t\right)$$
$$\left.\left.-(x_1x'_2+x_2x'_1)\left(\frac{m\omega}{2}\csc\omega t-\frac{\sqrt{3}}{2}m\omega\csc\sqrt{3}\omega t\right)\right)\right], \tag{59}$$

Substituting Eq. (59) into Eq. (55), we obtain

$$C(x'_1,x'_2,t)=C(x'_2,t)\exp\left[\frac{i}{\hbar}\left({x'}_1^2\left(\frac{m\omega}{4}\cot\omega t+\frac{\sqrt{3}}{4}m\omega\cot\sqrt{3}\omega t\right)\right.\right.$$
$$\left.\left.+x'_1x'_2\left(\frac{m\omega}{2}\cot\omega t-\frac{\sqrt{3}}{2}m\omega\cot\sqrt{3}\omega t\right)\right)\right], \tag{60}$$

Thus, the Green function of Eq. (59) becomes

$$G(x_1,x_2,x'_1,x'_2,t)=C(x'_2,t)\exp\left[\frac{i}{\hbar}\left((x_1^2+x_2^2+{x'}_1^2)\left(\frac{m\omega}{4}\cot\omega t+\frac{\sqrt{3}}{4}m\omega\cot\sqrt{3}\omega t\right)\right.\right.$$
$$+(x_1x_2+x'_1x'_2)\left(\frac{m\omega}{2}\cot\omega t-\frac{\sqrt{3}}{2}m\omega\cot\sqrt{3}\omega t\right)-(x_1x'_1+x_2x'_2)\left(\frac{m\omega}{2}\csc\omega t-\frac{\sqrt{3}}{2}m\omega\csc\sqrt{3}\omega t\right)$$
$$\left.\left.-(x_1x'_2+x_2x'_1)\left(\frac{m\omega}{2}\csc\omega t-\frac{\sqrt{3}}{2}m\omega\csc\sqrt{3}\omega t\right)\right)\right], \tag{61}$$

Substituting Eq. (61) into Eq. (56), we get

$$C(x'_2,t)=C(t)\exp\left[\frac{i}{\hbar}\left({x'}_2^2\left(\frac{m\omega}{4}\cot\omega t+\frac{\sqrt{3}}{4}m\omega\cot\sqrt{3}\omega t\right)\right)\right], \tag{62}$$

So, the Green function in Eq. (61) can be written as

$$G(x_1,x_2,x'_1,x'_2,t)=C(t)\exp\left[\frac{i}{\hbar}\left((x_1^2+x_2^2+{x'}_1^2+{x'}_2^2)\left(\frac{m\omega}{4}\cot\omega t+\frac{\sqrt{3}}{4}m\omega\cot\sqrt{3}\omega t\right)\right.\right.$$
$$+(x_1x_2+x'_1x'_2)\left(\frac{m\omega}{2}\cot\omega t-\frac{\sqrt{3}}{2}m\omega\cot\sqrt{3}\omega t\right)-(x_1x'_1+x_2x'_2)\left(\frac{m\omega}{2}\csc\omega t+\frac{\sqrt{3}}{2}m\omega\csc\sqrt{3}\omega t\right)$$
$$\left.\left.-(x_1x'_2+x_2x'_1)\left(\frac{m\omega}{2}\csc\omega t-\frac{\sqrt{3}}{2}m\omega\csc\sqrt{3}\omega t\right)\right)\right], \tag{63}$$

To find $C(t)$, we must substitute the Green function of Eq. (63) into the Schrödinger equation

$$i\hbar\frac{\partial G(x_1,x_2,x_1',x_2',t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 G(x_1,x_2,x_1',x_2',t)}{\partial x_1^2} - \frac{\hbar^2}{2m}\frac{\partial^2 G(x_1,x_2,x_1',x_2',t)}{\partial x_2^2} + m\omega^2(x_1^2 - x_1x_2 + x_2^2)G(x_1,x_2,x_1',x_2',t). \quad (64)$$

After some algebra, we obtain an equation

$$\frac{dC(t)}{dt} = -C(t)\left(\frac{\omega}{2}\cot\omega t + \frac{\sqrt{3}}{2}\omega\cot\sqrt{3}\omega t\right). \quad (65)$$

Integrating Eq. (65) with respect to time, we obtain

$$C(t) = \frac{c}{(\sin\omega t\sin\sqrt{3}\omega t)^{1/2}}, \quad (66)$$

where C is a constant.

Substituting Eq. (66) into Eq. (63) and applying the initial condition in Eq. (35), the constant C is

$$C = \frac{3^{(1/4)}m\omega}{2\pi i\hbar}. \quad (67)$$

So, the Green function for a coupled harmonic oscillator can be expressed as

$$G(x_1,x_2,x_1',x_2',t) = \frac{m\omega}{2\pi i\hbar}\left[\frac{\sqrt{3}}{\sin\omega t\sin\sqrt{3}\omega t}\right]^{1/2}\exp\left[\frac{i}{\hbar}\left(\left(x_1^2+x_2^2+{x'}_1^2+{x'}_2^2\right)\left(\frac{m\omega}{4}\cot\omega t + \frac{\sqrt{3}}{4}m\omega\cot\sqrt{3}\omega t\right) + (x_1x_2 + x_1'x_2')\left(\frac{m\omega}{2}\cot\omega t - \frac{\sqrt{3}}{2}m\omega\cot\sqrt{3}\omega t\right) - (x_1x_1' + x_2x_2')\left(\frac{m\omega}{2}\csc\omega t + \frac{\sqrt{3}}{2}m\omega\csc\sqrt{3}\omega t\right) - (x_1x_2' + x_2x_1')\left(\frac{m\omega}{2}\csc\omega t - \frac{\sqrt{3}}{2}m\omega\csc\sqrt{3}\omega t\right)\right)\right], \quad (68)$$

which is the same form as the calculation of S. Pepore and B. Sukbot by the Schwinger method [6].

4. Conclusion

The method in calculating the Green functions with the aid of integrals of the motion presented in this article can be successfully applied in solving the dual damped oscillator and the coupled harmonic oscillator problems. This method has the crucial steps in deriving the integrals of the motions $\hat{q}_{10}$, $\hat{q}_{20}$, $\hat{p}_{10}$, and $\hat{p}_{20}$ and implying that the Green functions $G(x_1,x_2,x_1',x_2',t)$ is the eigenfunctions of the operators $\hat{q}_{10}$, $\hat{q}_{20}$, $\hat{p}_{10}$, and $\hat{p}_{20}$.

In fact, this method has many common features with the Schwinger method [3-6], but the Schwinger method uses the operator $\hat{q}(t)$ and $\hat{p}(t)$ in calculating the matrix element of Hamiltonian operator in deriving the Green function

$$G(x,x',t) = C(x,x') \times\exp\left(-\frac{i}{\hbar}\int_0^t \frac{\langle x(t\mid \hat{H}(\hat{x}(t),\hat{x}(0))\mid x'(0)\rangle}{\langle x(t)\mid x'(0)\rangle}dt\right). \quad (69)$$

In the Feynman path integral [1], the pre-exponential function $C(t)$ comes from sum over all fluctuating paths that depend on calculation of the functional integration while in the integrals of the motion method this term appears from solving the Schrodinger equation of Green function. In the Schwinger formalism [2-6], the pre-exponential function $C(t)$ arises from the commutation relation of $[\hat{x}(t),\hat{x}(0)]$. These different points of view may show the connection between classical mechanics and quantum mechanics. It can be conclude here that the integrals of the motion method in this paper seems to be more simple from the viewpoint of calculation.

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