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A relativistic formulation of the de la Peña-Cetto stochastic quantum mechanics

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A covariant generalization of a non-relativistic stochastic quantum mechanics introduced by de la Peña and Cetto is formulated. The analysis is done in space-time and avoids the use of a non-covariant time evolution parameter in order to search for Lorentz invariance. The covariant form of the set of iterative equations for the joint coordinate and momentum distribution function $Q(x, p)$ is derived and expanded in power series of the coupling of the particle with the stochastic forces. Then, particular solutions of the zeroth order in the charge of the iterative equations for $Q(x, p)$ are considered. For them, it follows that the space-time probability density $\rho(x)$ and the function $S(x)$ which gradient defines the mean value of the momentum at the space time point x, define a complex function $\psi(x)$ which exactly satisfies the Klein-Gordon (KG) equation. These results for the zeroth order solution reproduce the ones formerly and independently derived in the literature. It is also argued that when the KG solution is either of positive or negative energy, the total number of particles conserves in the random motion. Other solutions for the joint distribution function in lowest order, satisfying the positive condition are also presented here. The are consistent with the assumed lack of stochastic forces implied by the zeroth order equations. It is also argued that such joint distributions, after considering the action of the stochastic forces, might furnish an explanation of the quantum mechanical properties, as associated to ensembles of particles in which the vacuum makes such particles behave in a similar way as Couder's droplets moving over oscillating liquid surfaces. Some remarks on the solutions of the positive joint distribution problem proposed in the Olavos's analysis are also presented.

Keywords: Stochastic QED; Couder's experiments.

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1. Introduction

The search for stochastic descriptions of quantum mechanics and quantum fields theory has a large history. By example, in the works [1–6], it was considered that the random forces which determine the phase space density of particle, were given in a stochastic and relativistic invariant way. This defined the so called Stochastic Electrodynamics (SQED). Specifically, these forces were generated by an electromagnetic field configuration, obtained by exciting all the oscillation modes with one half a quanta of energy. In addition, the phases of the photon modes were assumed to be stochastically defined within the interval (0.2π) in a uniform way. This force had been argued to be invariant under Lorentz transformations [2, 5]. In addition, in reference [1], it was argued that the stochastic motion guided by those forces, when taken in the non-relativistic limit, leads to the Schrodinger equation. Further, in the series of works [8–13], it was proposed a solution for the a central open issue of the theory: the apparent lack of positive definiteness of the coordinate-momentum joint distribution function emerging in the previous discussions. More recently, and with the purpose of start considering a generalization of the results in Ref. 1, in Ref. 18 it was suggested that random forces, showing the same statistical distribution in all Lorentz frames, can be expected to imply the satisfaction of the Klein-Gordon equation, for which the non-relativistic limit is the Schrodinger one.

The present work is devoted to present a derivation of a relativistic version of the stochastic electrodynamics. That is, we intend to relativistically generalize the discussion done in Ref. 1. For this purpose we start from the relativistic formulation of the kinetic equations given in Ref. 7. The implementation of the stochastic force is assumed to coincide with the one employed in the stochastic electrodynamics [5]. Then, a formula for the equation satisfied by the fluctuation independent space-momenta joint distribution function is derived. It directly generalizes the expression obtained in [1] for the non-relativistic limit. A formula for the joint distribution is derived.

Then, the solution of the equations for the joint particle distribution is searched as expanded in a power series in the squared particle charge. Further, it is shown the existence of particular solutions for the joint distribution in the zeroth order in the charge. This implies that the density of particles at a given space time point is defined as a square of complex function satisfying the Klein-Gordon equation, in this considered interaction free approximation. This indicates that the Klein-Gordon equation (or its non-relativistic limit the Schrodinger one) can be described as satisfying some of the equations of SQED in the first approximations. It can be cited that recently, in Ref. 15 a derivation was presented of the Klein-Gordon equation, from a modified classical Hamilton-Jacobi equation for a particle interacting with random background forces. In addition an alternative derivation of the KG equations was also obtained in Ref. 13. The present discussion independently generalizes the kinetic discussion of the non-relativistic analysis done in [1]. The presentation here also clarifies the role of the positive and negative energy solutions, by showing that both of them separately imply the conservation of the total number of particles in the stochastic motions, assumed that external electromagnetic fields are absent.

By the side, it can be observed here that a special circumstance could help to overcome the known lack of strict positiveness of the ansatz for the joint distribution adopted in [1] and here. The KG equation, when seen as theory of particles lacks a standard definition for the position operator having eigen-functions like the Dirac's Delta function. The position operator for this theory shows Gaussian like spatial behavior with non vanishing values within an spatial neighborhood of the size of the Compton wavelength of the particle [17]. Then, it looks reasonable that a clear interpretation of the $\widetilde{Q}^0(x,p)$ as describing particles with a well defined position x, can run in troubles. Thus, one can imagine that a proper modification of the kinetic equations to take account of an extensive nature of the particles could lead to a consistent hidden variable interpretation of SQED in describing quantum mechanics. Assumed that above mentioned difficulty can be surmounted, an interesting extension of the work could seem feasible. For this, after including an external electromagnetic field, it seem possible to develop a picture in which both types of particles move randomly: one kind of them guided by the positive energy solutions and the other one (with opposite charges) moves as driven by the negative energy waves. The development of such a picture is an interesting envisioned extension of the work.

Further in the work, we also present some solutions for the zeroth order joint distribution function $Q(x,p)$ which have positive values in all the phase space points. They have this property independently of the assumption done in the Olavo's works [8] about the infinitesimal character of the of the Fourier conjugate variable of the particle momenta in the stochastic motions. It also can be stressed that the zeroth order equations for joint distribution function corresponds to the limit in which no stochastic action of the particles are effected. Therefore, these solutions could be more reasonable to be adopted in the zeroth order, since they describe uniform motions of free particles. Two kinds of localized solutions are found: One constructed as point-like localized spatial dependence which moves with the four-velocity associated to the 4-momentum of the particle. The other kind is attained by employing Yukawa like localized solutions of the Klein-Gordon equation in their construction. Both types of joint coordinate momentum distributions in turn strongly suggest the possibility of describing extended particles, showing the surprising experimental properties exhibited by droplets moving over oscillating liquid surfaces [19,20]. This idea comes from the suspicion about that after the action of stochastic forces (in higher orders in the coupling) both sorts of solutions might transform in extended wavepackets surrounding a stochastic mean position of the particle moving with constant velocity $v = p/\sqrt{p^2+m^2}$. Such outcome is suggested in reference [21], in which the existence of such configurations is argued from a given proof of a stochastic Noether theorem. The investigation of the scattering properties of such solutions on two slits screens and potential walls, by example is expected to be considered elsewhere.

In Sec. 2 we introduce the basic notions of the relativistic kinetic theory. Next, in Sec. 3, the relativistically invariant equations for the mean value of the distribution and its random fluctuations are written. Further, in Sec. 4, the momentum Fourier transform of the mean joint distribution is introduced and the equations for it, are written. Section 5 considers the equations following in the first order zeroth approximation in the coupling with the stochastic forces. It is exposed how solutions of the Klein-Gordon equation define particular solutions of the relativistic kinetic equations, determining a possible joint distribution function in the assumed zeroth order in the charge. Next, Sec. 6 discusses how these special solutions determine particle distributions which conserve the total number of particles when the KG waves are assumed to be alternatively positive or negative energy modes. Further, in Sec. 7 ,we present other special solutions for the equation for the joint distribution function, satisfying the positiveness condition and being consistent with the lack of action of the stochastic forces in zeroth order of their coupling. Also, their possible links with the Couder's experiments are identified. Finally, we advance some remarks linked with the argue presented in Olavo's analysis about the positiveness of the joint distribution function [9,12].

The results are reviewed and commented in the Summary section.

2. The equation for the joint distribution function

Let us start by writing the relativistic invariant equation for the density of points in phase space $R(x,p)$ for an ensemble of massive particles all evolving under the action of a stochastic 4-force $F^\mu(x,p)$ which was derived in Ref. 7

$$p^\mu \frac{\partial}{\partial x^\mu} R(x,p) + m\, F^\mu(x,p) \frac{\partial}{\partial p^\mu} R(x,p) = 0. \qquad (1)$$

The 4-coordinates x^μ will be considered in the metric

$$g^{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \qquad (2)$$

where the natural system of coordinates will be also employed, in which the light velocity $c = 1$ and the time is the x^0 coordinate and the Planck constant $\hbar = 1$. The four momentum as usual, is given in terms of the 3-velocity as

$$p^\mu = \frac{m(1, \overrightarrow{v})}{\sqrt{1-\overrightarrow{v}^2}}. \qquad (3)$$

In order to simplify the discussion, we will firstly consider that the external force vanishes. The particle density

$n(\overrightarrow{x},t)$ and the particle flow $\overrightarrow{j}(\overrightarrow{x},t)$ in this relativistically invariant case have the form

$$\begin{aligned} N^{\mu}(x) &= (n(\overrightarrow{x},t),\overrightarrow{j}(\overrightarrow{x},t)) \\ &= \int \frac{d\overrightarrow{p}}{p^0} p^{\mu}\, R(x,p), \end{aligned} \tag{4}$$

in which the integration is over all the 3-momenta. In general the conventions defined in Ref. 7 will be employed. It will be assumed that the momenta values are defined on the mass-shell

$$p^2 - m^2 = 0.$$

In the present work, the force will assumed to be stochastically defined as in Ref. 5. That force had been argued to be invariant under Lorentz transformations [2–5]. Therefore, the Eq. (1) also becomes relativistically invariant in form. As remarked before in Ref. 1, it was argued that the stochastic motion guided by that force, when taken in the non-relativistic limit, leads to the Schrodinger equation in the first steps of an iterative process of solution of the equations for the non relativistic distribution function. Therefore, as it was argued in Ref. 18, it can be suspected that the relativistic invariant motions determined by (1) could be related with the satisfaction of the Klein-Gordon equation, for which the non-relativistic limit is the Schrodinger one. This work is devoted to investigate this possibility. In order to make the discussion clearer let us argue in the next section that in the non-relativistic limit, the stochastic equations reproduce the ones employed in Ref. 1.

2.1. The non-relativistic limit of the equation

In this case since $(\overrightarrow{v}^2/c^2) \ll 1$ the momentum and the external force can be approximately given by

$$p^{\mu} = m(1,\overrightarrow{v}). \tag{5}$$

Then, after considering $\overrightarrow{v} = \overrightarrow{p}/m$ the Eq. (1) reduces to

$$\begin{aligned} &\frac{\partial}{\partial x^0} R(x,\overrightarrow{p}) + \frac{\overrightarrow{p}}{m}\cdot\frac{\partial}{\partial\overrightarrow{x}} R(x,\overrightarrow{p}) \\ &\quad + \overrightarrow{F}(x,p)\frac{\partial}{\partial\overrightarrow{p}} R(x,\overrightarrow{p}) = \frac{\partial}{\partial x^0} R(x,\overrightarrow{p}) \\ &\quad + \frac{\partial}{\partial\overrightarrow{x}}\left(\frac{\overrightarrow{p}}{m}\cdot R(x,\overrightarrow{p})\right) \\ &\quad + \overrightarrow{F}(x,p)\cdot\frac{\partial}{\partial\overrightarrow{p}} R(x,\overrightarrow{p}) = 0. \end{aligned} \tag{6}$$

Further, assuming that $\overrightarrow{F}(x,p) = \overrightarrow{F}(x)$, in other words that the force is independent of the momentum, leads to

$$\begin{aligned} &\frac{\partial}{\partial x^0} R(x,\overrightarrow{p}) + \frac{\partial}{\partial\overrightarrow{x}}\left(\frac{\overrightarrow{p}}{m}\cdot R(x,\overrightarrow{p})\right) \\ &\qquad + \frac{\partial}{\partial\overrightarrow{p}}\cdot(\overrightarrow{F}(x)\, R(x,\overrightarrow{p})) = 0. \end{aligned} \tag{7}$$

which is the same starting formula employed in [1]. However, in order to arrive to this expression it was assumed that the force is not momentum dependent. But, the stochastic electric force term employed in [1] is momentum independent, an thus it makes the non-relativistic equation employed in Ref. 1 and the one employed here, equivalent in the non-relativistic limit.

2.2. The adopted SQED relativistic random vacuum forces

Let us give a precise definition of the relativistic stochastic process under consideration. Note first that we had omitted a time t argument in the distribution in order to avoid the use of the non-relativistic invariant definition of the time. Therefore, the stochastic character of the process will be implemented by defining a large ensemble of particle trajectories in the phase space (x,p). Each of these trajectories will be defined by a solution of the Eq. (1) for a force given by a random realization of the relativistic invariant Lorentz force employed in SQED [5]

$$F^{\mu}(x,p) = \frac{q}{m} F^{\mu}_{\nu}(x) p^{\nu} = q\, f^{\mu}(x,p), \tag{8}$$

$$f^{\mu}(x,p) = F^{\mu}_{\nu}(x)\frac{p^{\nu}}{m}, \tag{9}$$

in which the stochastic space-time dependent field intensity $F_{\gamma\beta}(x)$ is given by

$$F_{\gamma\beta}(x) = \partial_{\gamma} A_{\beta}(x) - \partial_{\beta} A_{\gamma}(x), \tag{10}$$

$$A_{\beta}(x) = (A_0(x),\overrightarrow{A}(x)), \tag{11}$$

$$\begin{aligned} \overrightarrow{A}(x) = \sum_{\lambda=1}^{2}\int d\overrightarrow{k}\,\frac{1}{w_k}\overrightarrow{\epsilon}(\overrightarrow{k},\lambda) h(\overrightarrow{k},\lambda) \\ \times \sin(\overrightarrow{k}\cdot\overrightarrow{x} - w_k\, x^0 + \theta(\overrightarrow{k},\lambda)), \end{aligned} \tag{12}$$

where $w_k = |\overrightarrow{k}|$, $\overrightarrow{\epsilon}(\overrightarrow{k},\lambda)$ are two unit polarization vectors associated to the wave vector $\overrightarrow{k}$ and satisfying

$$\overrightarrow{\epsilon}(\overrightarrow{k},\lambda)\cdot\overrightarrow{\epsilon}(\overrightarrow{k},\lambda') = \delta_{\lambda\lambda'}, \quad \overrightarrow{k}\cdot\overrightarrow{\epsilon}(\overrightarrow{k},\lambda) = 0, \tag{13}$$

and the number h is defined as satisfying

$$\pi^2 h^2 = \frac{1}{2} w_{\overrightarrow{k}}. \tag{14}$$

Finally, the phases $\theta(\overrightarrow{k},\lambda)$ are defined as independent random (one for each value of $(\overrightarrow{k},\lambda)$) and uniformly distributed in the interval $(0,2\pi)$ [5]. In what follows, in place of the force expression, we will prefer to work with the defined above force per unit of charge

$$f^{\mu}(x,p) = \frac{1}{q} F^{\mu}(x,p) = F^{\mu}_{\nu}(x)\frac{p^{\nu}}{m}. \tag{15}$$

2.3. Definitions for the operators and their kernels in joint coordinate-momentum space

We will consider in what follows linear kernels in the space of functions $PS = \{g(x,p)\}$of the particle coordinates x and momenta p, which explicitly written will make the expression to appear as cumbersome. Then, for any of such kernels K, which action on functions of the space g is defined as

$$g'(x,p) = \int dx'\ dp'\ K(x,p;x',p')\ g(x',p'), \tag{16}$$

its compact operator expression will be defined according to the following equivalence rules

$$g(x,p) \equiv g, \tag{17}$$

$$\int dx'\ dp'\ K(x,p;x',p')\ g(x',p') \equiv \widehat{K}\ g. \tag{18}$$

The special Delta function kernel $\delta^{(8)}(x-x', p-p') = \delta^{(4)}(x-x')\ \delta^{(4)}(p-p')$ will be simply defined as the identity $\widehat{I},$ which will mean for the kernel associated to the inverse of K, the relation

$$\widehat{K}^{-1}\widehat{K} = \widehat{K}\ \widehat{K}^{-1} = \widehat{I}.$$

The local operators, like $L = p^\mu(\partial/\partial x^\mu)$ and $f = f^\mu(x,p)(\partial/\partial p^\mu)$ are also considered as kernels in the usual way

$$p^\mu \frac{\partial}{\partial x^\mu} \to p^\mu \frac{\partial}{\partial x^\mu}\delta^{(8)}(x-x', p-p') \equiv \widehat{L}, \tag{19}$$

$$f^\mu(x,p)\frac{\partial}{\partial p^\mu} \to f^\mu(x,p) \times \frac{\partial}{\partial p^\mu}\delta^{(8)}(x-x', p-p') \equiv \widehat{f}. \tag{20}$$

3. The equations for the joint distribution function

Now, we will apply the method of smoothing (See [16] and [1]) in order to reduce the Eq. (1) to a non random one for the coordinate-momenta joint distribution function over the defined ensemble of trajectories. The ensemble is generated by samples of the stochastic force, generated by the random phases of the electromagnetic modes $\theta(\overrightarrow{k}, \lambda)$ taken for all the values of momenta and polarization $(\overrightarrow{k}, \lambda)$. For the further analysis, the distribution function will be decomposed in its average coordinate-momenta joint distribution $Q(x, \overrightarrow{p})$ and its random fluctuations $\delta Q(x, \overrightarrow{p})$ as

$$R(x,p) = Q(x,p) + \delta Q(x,p), \tag{21}$$

with

$$Q(x, \overrightarrow{p}) = \widehat{P}\ R(x,p), \tag{22}$$

$$\delta Q(x, \overrightarrow{p}) = (1-\widehat{P})R(x,p), \tag{23}$$

where the $\widehat{P}$ is a projection operator satisfying $\widehat{P}^2 = 1$. After substituting these expression in equation (1) and applying alternatively $\widehat{P}$ or $(1-\widehat{P})$, the following two equations follow

$$L\ Q(x,p) + q\ \widehat{P}\ f^\mu(x,p)\frac{\partial}{\partial p^\mu}\delta Q(x,p) = 0,$$

$$L\ \delta Q(x,p) + q\ (1-\widehat{P}\ )f^\mu(x,p)\frac{\partial}{\partial p^\mu}Q(x,p) = 0, \tag{24}$$

$$L = p^\mu \frac{\partial}{\partial x^\mu}. \tag{25}$$

Consider now the retarded Green function in the space PS of the differential operator $L = p^\mu(\partial/\partial x^\mu)$, satisfying

$$p^\mu \frac{\partial}{\partial x^\mu} G(x,p;x',p') = \delta^{(4)}(x-x')\delta^{(4)}(p-p'). \tag{26}$$

In terms of its Fourier transform in the two arguments, the Green function G takes the form

$$\begin{aligned} G(x,p;x',p') &= \int\int \frac{dq}{(2\pi)^4}\ \frac{dz}{(2\pi)^4}\frac{1}{-i\ p^\mu q_\mu} \\ &\times \exp(-i(x-x')^\mu q_\mu - i(p-p')^\mu z_\mu) \\ &= \int\int \frac{dq}{(2\pi)^4}\ \frac{1}{-i\ p^\mu q_\mu} \\ &\times \exp(-i(x-x')^\mu q_\mu)\ \delta(p-p') \\ &= G(x-x'|p')\ \delta(p-p') \equiv \widehat{L}^{-1}. \end{aligned} \tag{27}$$

It is important to note here, that this expression for G indicates that the derivatives $\partial/\partial p^\mu$ do not commute with the operator G, since

$$\begin{aligned} \frac{\partial}{\partial p^\mu}G(x,p;x',p') &= G(x-x'|p')\ \frac{\partial}{\partial p^\mu}\delta(p-p') \\ &= -G(x-x'|p')\ \frac{\partial}{\partial p'^\mu}\delta(p-p') \\ &= G(x-x'|p')\ \delta(p-p')\frac{\partial}{\partial p'^\mu} \\ &+ \frac{\partial}{\partial p'^\mu}(G(x-x'|p'))\ \delta(p-p') \\ &\neq G(x-x'|p')\ \delta(p-p')\frac{\partial}{\partial p'^\mu}. \end{aligned} \tag{28}$$

Therefore, this non commutativity of the momentum derivative with the propagator G made difficulty to show in this relativistic case an important property derived in Ref. 1: the quadratic dependence in the Fourier transform variable z of the momentum p, of some relevant quantities in the discussion. This lack of commutativity led us, further ahead in this work, to consider the expansion in the coupling in the equations, in place of the expansion in powers of z. Now, acting with the product of G and $p^\mu(\partial/\partial x^\mu)$ on an arbitrary function $g(x,p)$ it follows

$$\int\int dx'dp'G(x,p;x',p')p'^\mu\frac{\partial}{\partial x'^\mu}g(x',p')$$
$$=\int dx'dp'\int\frac{dq}{(2\pi)^4}\,\frac{\delta(p-p')}{-i\,p^\mu q_\mu}\exp(-i(x-x')^\mu q_\mu)$$
$$\times\,p'^\mu\frac{\partial}{\partial x'^\mu}g(x',p')=g(x,p), \tag{29}$$

which implies

$$\int\int dx'dp'\;p^\mu\frac{\partial}{\partial x^\mu}G(x,p;x',p')$$
$$\equiv\delta(x-x')\delta(p-p'). \tag{30}$$

As in reference we will now define a compact notation in order to eliminate the cumbersome appearance determined by the kernel structure of the Green function. This notation is described by expressing the above relation in the form

$$\int\int dx'dp'p'^\mu\frac{\partial}{\partial x'^\mu}G(x,p;x',p')p'^\mu\frac{\partial}{\partial x'^\mu}$$
$$\equiv\widehat{L}\,\widehat{L}^{-1}=\widehat{I}\equiv\delta(x-x')\delta(p-p').$$

The use of these relations written above, after acting with the kernel G at the left of the relations (24) gives for the average of the distribution and its random part, the expressions

$$Q=-q\,\widehat{L}^{-1}\,\widehat{P}\,\widehat{f}\delta Q, \tag{31}$$
$$\delta Q=-q\,\widehat{L}^{-1}[\widehat{I}+q\,\widehat{L}^{-1}(I-\widehat{P}\,)\widehat{f}]^{-1}\widehat{P}\,\widehat{f}Q. \tag{32}$$

Then, the substitution of these relations in the first of the Eqs. (24) leads to the following equations for the joint distribution function $Q(x,p)$

$$\widehat{L}\,Q=q^2\widehat{P}\,\widehat{f}\,\widehat{L}^{-1}[\widehat{I}+q\,\widehat{L}^{-1}(\widehat{I}-\widehat{P}\,)\widehat{f}]^{-1}\widehat{f}\,Q,$$
$$=q^2\widehat{P}\,\widehat{f}\,\widehat{L}^{-1}\sum_{n=0}^{\infty}(-1)^n[q\,\widehat{L}^{-1}(\widehat{I}-\widehat{P}\,)\widehat{f}]^n\,\widehat{f}\,Q,$$
$$=q^2\widehat{P}\widehat{f}\widehat{L}^{-1}\sum_{m=0}^{\infty}\left(q^2\right)^m\,[\widehat{L}^{-1}(\widehat{I}-\widehat{P}\,)\widehat{f}]^{2m}\widehat{f}\,Q. \tag{33}$$

Therefore, the equation for the joint distribution function can be written in a compact form, which after expanded in powers of q^2, takes the form

$$\widehat{L}\,Q=q^2\,\widehat{J}\,(q^2)\,Q, \tag{34}$$
$$\widehat{J}\,(q^2)=\sum_{m=0}^{\infty}\left(q^2\right)^m\,\widehat{P}\,\widehat{f}\,\widehat{L}^{-1}[\widehat{L}^{-1}(\widehat{I}-\widehat{P})\,\widehat{f}]^{2m}\widehat{f}$$
$$=\sum_{m=0}^{\infty}\left(q^2\right)^m\,\widehat{J}^{\,m}(q^2). \tag{35}$$

In this relation it has been employed that the mean value of an odd number of the random force functions [1,5] vanishes. We recall that in that relation $\widehat{f}$ is the operator corresponding to the kernel

$$\widehat{f}\equiv f^\mu(x,p)\frac{\partial}{\partial p^\mu}. \tag{36}$$

4. The momentum Fourier transformed joint distribution function

Let us perform now the Fourier transformation of the joint distribution over the momentum variable as follows

$$Q(x,p)=\int dz\,\widetilde{Q}(x,z)\exp(-i\,p^\mu z_\mu), \tag{37}$$
$$\widetilde{Q}(x,z)=\int\frac{dp}{(2\pi)^4}Q(x,p)\exp(i\,p^\mu z_\mu). \tag{38}$$

It can be mentioned that this concept had been defined and employed, by example in Refs. 1 and 8. It is also named as the Characteristic Function. Then, after Fourier transforming the Eq. (34) the following equation for $Q(x,p)$ can be written

$$p^\mu\frac{\partial}{\partial x^\mu}Q(x,p)=q^2\sum_{m=0}^{\infty}\left(q^2\right)^m$$
$$\times\int dx'dp'\;J^m(q^2)(x,p;x',p')\;Q(x',p'), \tag{39}$$
$$\frac{\partial}{i\partial x^\mu\partial z_\mu}\widetilde{Q}(x,z)=q^2\sum_{m=0}^{\infty}\left(q^2\right)^m\int dx'd\,z'$$
$$\times\,J^{\,m}(q^2)\left(x,\frac{\partial}{i\,\partial z};x',\frac{\partial}{i\,\partial z'}\right)\;\widetilde{Q}(x',z'), \tag{40}$$

with the operator

$$J^m(q^2)\left(x,\frac{\partial}{i\,\partial z};x',\frac{\partial}{i\,\partial z'}\right)$$

operating in the space of function of the variables (x,z) is defined by

$$J^m(q^2)\left(x,\frac{\partial}{i\,\partial z};x',\frac{\partial}{i\,\partial z'}\right)$$
$$=\left(\widehat{P}\,\widehat{f}\,\widehat{L}^{-1}[\widehat{L}^{-1}(\widehat{I}-\widehat{P})\,\widehat{f}]^{2m}\widehat{f}\right)$$
$$\times\,(x,p;x',p')\Big|_{p\to\frac{\partial}{i\,\partial z},p'\to\frac{\partial}{i\partial z'}}. \tag{41}$$

Now, the mean value of a function of the coordinates and momenta at a specific space-time position x can be written in two forms as

$$\langle A(x,p)\rangle_x=\frac{1}{\rho_t(x)}\int dp\;\;A(x,p)\,Q(x,p)$$
$$=\frac{1}{\rho_t(x)}\left[A\left(x,\frac{\partial}{i\partial z}\right)\;\widetilde{Q}(x,z)\right]_{z=0}, \tag{42}$$

where the distribution function in the 3D-space points $\overrightarrow{x}$ and a given time x_0 is given by

$$\rho_t(x) = \int dp\, Q(x,p) = \sum_{m=0} \left(q^2\right)^m \rho^m(x) = \sum_{m=0} \left(q^2\right)^m \int dp\, Q^m(x,p) = \sum_{m=0} \left(q^2\right)^m \widetilde{Q}^m(x,0), \tag{43}$$

in which the general expressions for the distribution function have been expanded in series of the squared charge as follow

$$Q(x,p) = \sum_{m=0} \left(q^2\right)^m Q^m(x,p), \tag{44}$$

$$\widetilde{Q}^m(x,z) = \sum_{m=0} \left(q^2\right)^m \widetilde{Q}^m(x,z). \tag{45}$$

Therefore, the general equation (40) can be written in the form

$$\frac{\partial}{i\,\partial z_\mu \partial x^\mu} \widetilde{Q}^0(x,z) = 0, \tag{46}$$

$$\frac{\partial}{i\,\partial z_\mu \partial x^\mu} \widetilde{Q}^m(x,z) = q^2 \sum_{n=0}^{\infty} (q^2) \int dx' d\, z' J^{\,n}(q^2) \times \left(x, \frac{\partial}{i\partial z}; x', \frac{\partial}{i\partial z'}\right) \widetilde{Q}^{m-n-1}(x',z'),\ m \geq 1. \tag{47}$$

4.1. The interaction free limit $q^2 \to 0$

Let us consider now the satisfaction of the first of the iterative equations in which the random movement had been decomposed. In this case all the $Q^m(x,p)$ for $m \geq 1$ will vanish and thus $Q(x,p) = Q^0(x,p)$. It should be remarked here, that zeroth order equation for $Q^0(x,p)$ exactly coincides with one obtained in Ref. 13 for the relativistic characteristic function in the absence of interaction with the random force. Then, the total density reduces to

$$\rho_t(x) = \int dp\, Q^0(x,p) = \rho^0(x) = \int dp\, Q^0(x,p) = \widetilde{Q}^0(x,0). \tag{48}$$

The mean values in this limit have the expression

$$\langle A(x,p)\rangle_x = \frac{1}{\rho^0(x)} \int dp\, A(x,p) Q^0(x,p) = \frac{1}{\rho^0(x)} \left[A\left(x, \frac{\partial}{i\partial z}\right) \widetilde{Q}^0(x,z)\right]_{z=0}, \tag{49}$$

in which the distribution function has been expanded in series of the squared charge.

Let us define for what follow

$$\rho(x) = \rho^0(x). \tag{50}$$

Then, the equation for $\widetilde{Q}^0(x,z)$

$$\frac{\partial}{i\,\partial z_\mu \partial x^\mu} \widetilde{Q}^0(x,z) = 0, \tag{51}$$

can be derived after expanding the exponential in powers of z. A similar equation had been obtained also in Refs. 13 where relativistic expressions for the characteristic function, not coming from a stochastic discussion was independently considered.

It is helpful to write the Fourier transforms in the zero order in q^2

$$Q^0(x,p) = \int\int dz\, \widetilde{Q}^0(x,z) \exp(i\, p^\mu z_\mu), \tag{52}$$

$$\widetilde{Q}^0(x,z) = \int\int \frac{dp}{(2\pi)^4} Q^0(x,p) \exp(-i\, p^\mu z_\mu). \tag{53}$$

The mean value formula reduces to

$$\langle A(x,p)\rangle_x = \frac{1}{\rho(x)} \int dp\ \ A(x,p)\, Q^0(x,p) = \frac{1}{\rho(x)} \left[A\left(x, \frac{\partial}{i\partial z}\right)\ \widetilde{Q}^0(x,z)\right]_{z=0}, \tag{54}$$

which allows to write the lowest order equation in the form

$$\frac{\partial}{i\partial z_\mu \partial x^\mu} \widetilde{Q}^0(x,z) = -\frac{\partial}{\partial x^\mu} \iint \frac{dp}{(2\pi)^4} Q^0(x,p) p^\mu \exp(-ipz) = \frac{\partial}{\partial x^\mu} (\rho(x) \langle p^\mu \exp(-i\, p\, z)\rangle_x = 0. \tag{55}$$

We will now introduce new variables z_+ and z_- in substitution of the variables x and z. The change is defined as

$$z_\mu^+ = x_\mu + \beta\, z_\mu, \quad z_\mu^- = x_\mu - \beta\, z_\mu,$$
$$x_\mu = \frac{1}{2}(z_\mu^+ + \beta\, z_\mu^+), \quad z_\mu = \frac{1}{2\beta}(z_\mu^+ - \beta\, z_\mu^+), \tag{56}$$

and for the derivatives

$$\frac{\partial}{\partial z_\mu^+} \equiv \partial_+^\mu = \frac{1}{2}\left(\frac{\partial}{\partial x_\mu} + \frac{\partial}{\beta\partial\, z_\mu}\right),$$
$$\frac{\partial}{\partial z_\mu^-} \equiv \partial_-^\mu = \frac{1}{2}\left(\frac{\partial}{\partial x_\mu} - \frac{\partial}{\beta\partial\, z_\mu}\right), \tag{57}$$

$$\frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial z_\mu^+} + \frac{\partial}{z_\mu^-}\right),$$
$$\frac{\partial}{\partial z_\mu} = \beta\left(\frac{\partial}{\partial z_\mu^+} - \frac{\partial}{z_\mu^-}\right). \tag{58}$$

The above kind of transformations had been suggested by reference [1] and were also employed in reference [13, 14] in deriving the Klein-Gordon equation. These relations allow to derive the identities

$$\frac{\partial^2}{\partial z_\mu \partial z_\nu} = \beta^2 \frac{\partial^2}{\partial x_\mu \partial x_\nu} - 2\beta^2(\partial_-^\mu \partial_+^\nu + \partial_+^\mu \partial_-^\nu) \tag{59}$$

$$\frac{\partial^2}{\partial z_\mu \partial x^\mu} = \beta\left(\frac{\partial^2}{\partial z_\mu^+ \partial z^{\mu+}} - \frac{\partial^2}{\partial z^{\mu-} \partial z_\mu^-}\right). \tag{60}$$

This last equation permits to write the zeroth order equation for $\widetilde{Q}^0$ in the form

$$\frac{\partial}{i\partial z_\mu \partial x^\mu}\widetilde{Q}^0(x,z)=\frac{\beta}{i} \times\left(\frac{\partial^2}{\partial z_\mu^+\partial z^{\mu+}}-\frac{\partial^2}{\partial z^{\mu-}\partial z_\mu^-}\right)\widetilde{Q}^0(x,z)=0. \quad (61)$$

4.2. Satisfying the higher order equations

In order to directly satisfy the set of equations for $n\geq 3$ let us assume $\widetilde{Q}^0(x,z)$ in the form

$$\widetilde{Q}^0(x,z)\equiv\widetilde{Q}^0(z^+,z^-)=\Psi^*(z^+)\Psi(z^-). \quad (62)$$

It should be noted that this form is suggested by the analysis done in Ref. 1. This assumption directly led to the character of solutions of the Schrodinger equation to the entering function Ψ. Then we will follow this assumption in searching for a covariant generalization of the discussion in [1]. However, it can be noted that the satisfaction of the quantum equations as implied by the stochastic theory being constructed could be more naturally expected to appear after including the stochastic effects by considering higher than zeroth orders in the coupling with the stochastic forces. Therefore, in the last part of this work we also search for zeroth order joint distributions being positive definite and also consistent with a free motion of a localized particles when the random forces are absent in the zeroth order in the coupling expansion.

Then, substituting the above commented assumed form in the zeroth equation leads to

$$\frac{1}{\Psi^*(z^+)}\frac{\partial^2}{\partial z_\mu^+\partial z^{\mu+}}\Psi^*(z^+) -\frac{1}{\Psi(z^-)}\frac{\partial^2}{\partial z_\mu^-\partial z^{\mu-}}\Psi(z^-)=0. \quad (63)$$

But this relation is directly satisfied if Ψ obey the linear equation for any argument u and fixed value of the parameter M

$$\frac{\partial^2}{\partial u^\mu\partial u_\mu}\Psi(u)-M^2\Psi(u)=0. \quad (64)$$

Then, the satisfaction of this equation implies

$$\frac{\partial}{i\partial x^\mu\,\partial z_\mu\partial z_{\mu_1}\partial z_{\mu_2}...\partial z_{\mu_n}}\widetilde{Q}^0(x,z)=0, \quad m=1,2,...\infty, \quad (65)$$

a condition which will be helpful in the further discussion.

Thus, we had been able to find a solution of the equation describing the free approximation of the random process. The equation for $\widetilde{Q}^0$ can be also written as

$$\begin{aligned}\frac{\partial}{i\partial x^\mu\,\partial z_\mu}\widetilde{Q}^0(x,z)&=-\frac{\partial}{\partial x^\mu}\\ &\times\iint\frac{dp}{(2\pi)^4}Q^0(x,p)\,p^\mu\exp(-i\,p^\mu z_\mu)\\ &=-\frac{\partial}{\partial x^\mu}(\rho(x)\langle\,p^\mu\exp(-i\,p^\mu z_\mu)\rangle_x)=0, \quad (66)\end{aligned}$$

which after expanding the exponential in powers of $p.z$ gives the following set of equations

$$\frac{\partial}{\partial x^\mu}(\rho(x)\langle p^\mu\rangle_x)=0, \quad (67)$$

$$\frac{\partial}{\partial x^\mu}(\rho(x)\langle p^\mu p^\nu\rangle_x)=0, \quad (68)$$

$$\frac{\partial}{\partial x^\mu}(\rho(x)\langle p^\mu p^{\mu_1}p^{\mu_2}...p^{\mu_n}\rangle_x)=0,\ n=1,2,...\infty. \quad (69)$$

The last of these relations is directly implied by Eq. (65). Thus, let us study in what follows the satisfaction of the first two equations after the adopted ansatz

$$\widetilde{Q}^0(x,z)=\Psi^*(z_\mu^+)\Psi(z_\mu^-).$$

We will use now

$$\langle p^\mu p^\nu\rangle_x=-\frac{1}{\widetilde{Q}^0(x,0)}\left[\frac{\partial}{\partial z_\mu\partial z_\nu}\,\widetilde{Q}^0(x,z)\right], \quad (70)$$

and the general relations

$$\begin{aligned}\frac{\partial}{\partial z_\mu}\left(\frac{1}{\widetilde{Q}^0(x,z)}\frac{\partial}{\partial z_\mu}\,\widetilde{Q}^0(x,z)\right)&=-\frac{1}{(\widetilde{Q}^0(x,z))^2}\\ &\times\frac{\partial}{\partial z_\nu}\widetilde{Q}^0(x,z)\,\frac{\partial}{\partial z_\mu}\widetilde{Q}^0(x,z)\\ &+\frac{1}{\widetilde{Q}^0(x,z)}\frac{\partial}{\partial z_\nu\partial z_\mu}\widetilde{Q}^0(x,z), \quad (71)\end{aligned}$$

which after evaluated in $z=0$ permits to write

$$\begin{aligned}\langle p^\mu p^\nu\rangle_x&=\langle p^\mu\rangle_x\langle p^\nu\rangle_x-\left[\frac{\partial^2}{\partial z_\mu\partial z_\nu}\ln\ \widetilde{Q}^0(x,z))\right]_{z=0}\\ &=\langle p^\mu\rangle_x\langle p^\nu\rangle_x\\ &-\beta^2\left[\frac{\partial^2}{\partial x_\mu\partial x_\nu}\ln\widetilde{Q}^0(x,z)\right]_{z=0}+\sigma^{\mu\nu}, \quad (72)\end{aligned}$$

$$\sigma^{\mu\nu}=2\beta^2\left[(\partial_-^\mu\partial_+^\nu+\partial_+^\mu\partial_-^\nu)\ln\widetilde{Q}^0\ \right]_{z=0}, \quad (73)$$

where it was used relation (59).

However, the assumed form of the zeroth order distribution allows also to find

$$\begin{aligned}\sigma^{\mu\nu}&=2\beta^2\left[(\partial_-^\mu\partial_+^\nu+\partial_+^\mu\partial_-^\nu)\ln\widetilde{Q}^0\ \right]_{z=0}\\ &=2\beta^2\Big[(\partial_-^\mu\partial_+^\nu+\partial_+^\mu\partial_-^\nu)(\ln\Psi^*(z^+)\\ &+\ln\Psi(z^-)\Big]_{z=0}=0. \quad (74)\end{aligned}$$

Thus, the resting two equations which remaining to be verified in their compatibility with the ansatz can be written. as

$$\frac{\partial}{\partial x^\mu}(\rho(x)\langle p^\mu\rangle_x)=0,$$
$$\frac{\partial}{\partial x^\mu}\Bigg[\rho(x)(\langle p^\mu\rangle_x\langle p^\nu\rangle_x - \beta^2\Big[\frac{\partial^2}{\partial x_\mu\partial x_\nu}\ln\ \widetilde{Q}^0(x,z))\Big]_{z=0}\Bigg]=0. \quad (75)$$

The derivation of these equations was directly suggested by their non relativistic counterparts in [1]. They also coincide with the ones derived in Ref. 13. In this work an alternative relativist discussion of a similar ansatz for the characteristic function, not coming from an stochastic formulation, was earlier and independently presented.

5. The Klein-Gordon equation in the non-interacting limit

The $q^2=0$, will be called the "non interacting or free approximation". In this section we will study the compatibility of the two Eqs. (75) with the expression assumed for the zeroth order joint distribution function. After checking this, it will follow that in this free limit, the equations admit solutions for joint distribution function which are defined by waves solving the Klein-Gordon equations. This result directly generalizes the derivation of the Schrodinger equation in Ref. 1. Analogous derivation of the KG equation was formerly and independently given in Ref. 13. Firstly, let us search for solutions of the set of the two equations

$$\frac{\partial}{\partial x^\mu}(\rho(x)\langle p^\mu\rangle_x)=0, \quad (76)$$
$$\frac{\partial}{\partial x^\mu}\Bigg[\rho(x)\langle p^\mu\rangle_x\langle p^\nu\rangle_x - \beta^2\rho(x)\Big[\frac{\partial^2}{\partial x_\mu\partial x_\nu}\ln\rho(x)\Big]\Bigg]=0, \quad (77)$$

where it has been substituted $\rho(x)=\widetilde{Q}(x,0)$. Further, let search for solutions in which the mean momentum value at a given space-time point x is defined by

$$\langle p^\mu\rangle_x=\frac{\partial}{\partial x_\mu}S(x). \quad (78)$$

After substituting in (76) and (77), it follows

$$\frac{\partial}{\partial x^\mu}\left[\rho(x)\frac{\partial}{\partial x_\mu}S(x)\right]=0, \quad (79)$$
$$\frac{\partial}{\partial x^\mu}\left[\rho(x)\frac{\partial}{\partial x_\mu}S(x)\frac{\partial}{\partial x_\nu}S(x)\right] - \beta^2\rho(x)\left[\frac{\partial^2}{\partial x_\mu\partial x_\nu}\ln\rho(x)\right]=0. \quad (80)$$

The second equation can be transformed as follows

$$\frac{\partial}{\partial x_\mu}S(x)\frac{\partial}{\partial x^\mu\partial x_\nu}S(x) - \frac{\beta^2}{\rho(x)}\frac{\partial}{\partial x^\mu}\rho(x)\left[\frac{\partial^2}{\partial x_\mu\partial x_\nu}\ln\rho(x)\right]=0, \quad (81)$$
$$\frac{\partial}{\partial x_\mu}\left(\frac{1}{2}\frac{\partial}{\partial x_\mu}S(x)\frac{\partial}{\partial x^\mu}S(x)\right) - \frac{\beta^2}{\rho(x)}\frac{\partial}{\partial x^\mu}\left(\rho(x)\frac{\partial^2}{\partial x_\mu\partial x_\nu}\ln\rho(x)\right)=0. \quad (82)$$

Further, the density dependent term can be expressed as the divergence of a vector (as the first term also is) as follows

$$\frac{1}{\rho(x)}\frac{\partial}{\partial x^\mu}\left(\rho(x)\frac{\partial^2}{\partial x_\mu\partial x_\nu}\ln\rho(x)\right)=\frac{\partial}{\partial x^\nu}\left(\frac{1}{2}\frac{\partial}{\partial x^\mu}\ln\rho(x)\times\frac{\partial}{\partial x_\mu}\ln\rho(x)+\rho(x)\frac{\partial^2}{\partial x_\mu\partial x^\mu}\ln\rho(x)\right)=\frac{\partial}{\partial x^\nu}\left(-\frac{1}{2}\times\frac{\partial}{\partial x^\mu}\ln\rho(x)\frac{\partial}{\partial x_\mu}\ln\rho(x)+\frac{1}{\rho(x)}\frac{\partial^2}{\partial x_\mu\partial x^\mu}\rho(x)\right). \quad (83)$$

Therefore, the following relation arises

$$\frac{\partial}{\partial x_\nu}\left(\frac{1}{2}\frac{\partial}{\partial x_\mu}S(x)\frac{\partial}{\partial x^\mu}S(x)+\frac{\beta^2}{2}\frac{\partial}{\partial x^\mu}\ln\rho(x)\frac{\partial}{\partial x_\mu}\ln\rho(x) - \beta^2\left(\frac{1}{\rho(x)}\frac{\partial^2}{\partial x_\mu\partial x^\mu}\rho(x)\right)\right)=0. \quad (84)$$

This equation implies,

$$\frac{1}{2}\frac{\partial}{\partial x_\mu}S(x)\frac{\partial}{\partial x^\mu}S(x)+\frac{\beta^2}{2}\frac{\partial}{\partial x^\mu}\ln\rho(x)\frac{\partial}{\partial x_\mu}\ln\rho(x) - \beta^2\left(\frac{1}{\rho(x)}\frac{\partial^2}{\partial x_\mu\partial x^\mu}\rho(x)\right)=ctc. \quad (85)$$

Now, if we fix the constant to be a positive value, given by $ctc=m^2$, the searched joint distribution function satisfies the two equations

$$\frac{\partial}{\partial x^\mu}\left[\rho(x)\frac{\partial}{\partial x_\mu}S(x)\right]=0, \quad (86)$$
$$\frac{1}{2}\frac{\partial}{\partial x_\mu}S(x)\frac{\partial}{\partial x^\mu}S(x)+\frac{\beta^2}{2}\frac{\partial}{\partial x^\mu}\ln\rho(x)\frac{\partial}{\partial x_\mu}\ln\rho(x) - \beta^2\left(\frac{1}{\rho(x)}\frac{\partial^2}{\partial x_\mu\partial x^\mu}\rho(x)\right)=m^2. \quad (87)$$

5.1. The satisfaction of the Klein-Gordon equation

Consider now expressing the KG equation for a complex scalar field $\phi(x)$ in terms of the phase function $S(x)$ and a positive density $\rho(x)$ defined as follows

$$\phi(x) = (\rho(x))^{\frac{1}{2}} \exp(i\, S(x)). \tag{88}$$

Therefore, substituting in the KG equation it follow

$$(\partial^2 + m^2)\phi(x) = 0,$$
$$\left(\frac{\partial^2}{\partial x_\mu \partial x^\mu} + m^2\right)(\rho(x))^{\frac{1}{2}} \exp(i\, S(x)) = 0,$$

which after separating the real and imaginary parts and equalizing both of them to zero, leads to the two equations

$$\frac{\partial}{\partial x_\mu} S(x) \frac{\partial}{\partial x^\mu} \ln \rho(x) + \frac{1}{\rho(x)} \frac{\partial}{\partial x^\mu \partial x_\mu} S(x) = 0, \tag{89}$$

$$\frac{1}{2}\frac{\partial}{\partial x_\mu} S(x) \frac{\partial}{\partial x^\mu} S(x) + \frac{1}{4}\frac{\partial}{\partial x^\mu} \ln \rho(x) \frac{\partial}{\partial x_\mu} \ln \rho(x) - \frac{1}{2}\frac{1}{\rho(x)} \frac{\partial^2}{\partial x_\mu \partial x^\mu} \rho(x) = m^2. \tag{90}$$

It can be noted that Eqs. (89) and (90) become fully equivalent to (86) and (87) after assuming that the constant β^2 takes the value

$$\beta^2 = \frac{1}{2}. \tag{91}$$

Observe that this value coincides with the one before derived for the non-relativistic situation in [1]. Therefore, under the defined non-interacting approximation, it followed that the space time distribution

$$\rho(x) = \int dp\; Q(x,p) = \widetilde{Q}(x,0)$$

and the also space time function S (which determines the mean value of the omentum at a given space time point through

$$\langle p^\mu \rangle_x = \frac{\partial}{\partial x_\mu} S(x)$$

both define a complex function

$$\phi(x) = (\rho(x))^{\frac{1}{2}} \exp(iS(x))$$

satisfying the KG equation.

In order that the searched solution of the distribution

$$\widetilde{Q}^0(x,z) = \Psi^*(z^+)\Psi(z^-), \tag{92}$$

can be compatible with the solution for the KG waves generating the density $\rho(x)$ and the phase function S, the relation

$$\widetilde{Q}^0(x,0) = \Psi^*(z^+)\Psi(z^-)\big|_{z\to 0} = \Psi^*(x)\Psi(x) = \rho(x), \tag{93}$$

implies that the constant M defining the ansatz, and m defining the solution of the Hamilton-Jacobi equation, should coincide.

Up to now we have discussed the generalization of the de la Peña-Cetto derivation of the set of equations which should be satisfied by the joint distribution function in the non-relativistic limit of the SQED. We also derived a particular solution for the joint distribution function which implies the satisfaction of the Klein-Gordon equation.

6. About the role of positive energy solutions of the KG equations

Let us discuss an important physical question related with the obtained solutions. Since the particles which are assumed to undergo the random forces, are relativistic classical particles with rest mass m, it is natural to assume that the KG equation solution describing the stochastic motion should be expected to be a positive energy one. This circumstance is in certain form confirmed by a special property of the positive energy solutions (and also the negative energy ones): their total probability conserves in time, that is

$$\frac{\partial}{\partial x^0} \int d\overrightarrow{x}\, \rho(x) = \frac{\partial}{\partial x^0} \int d\overrightarrow{x}\, \phi^*(x)\phi(x) = 0. \tag{94}$$

This property can be easily derived after considering the following equations also satisfied by the positive or negative energy solutions

$$i\frac{\partial}{\partial x^0}\phi(x) = \pm\sqrt{m^2 - (\overrightarrow{\nabla})^2}\phi(x) = \pm\left(\sum_{m=0}^{\infty}(-1)^m \frac{((\overrightarrow{\nabla})^2)^m}{2^m}\right)\phi(x).$$

For proving the condition, consider that the spatial integral of the density for a positive energy (or a negative energy one) solution can be written and transformed in the way

$$\begin{aligned}\frac{\partial}{\partial x^0} \int d\overrightarrow{x}\, \rho(x) &= \frac{\partial}{\partial x^0} \int d\overrightarrow{x}\, \phi^*(x)\phi(x) \\ &= \int d\overrightarrow{x}\, \left(\frac{\partial}{\partial x^0}(\phi^*(x))\phi(x) - \phi^*(x)\frac{\partial}{\partial x^0}\phi(x)\right) \\ &= \mp\frac{1}{i}\int d\overrightarrow{x}\, \left(\left(\sum_{m=0}^{\infty}(-1)^m \frac{((\overrightarrow{\nabla})^2)^m}{2^m}\right)\phi^*(x)\right)\phi(x) \\ &\quad - \phi^*(x)\left(\sum_{m=0}^{\infty}(-1)^m \frac{((\overrightarrow{\nabla})^2)^m}{2^m}\phi(x)\right) = 0,\end{aligned} \tag{95}$$

in which there had been performed integration by parts over the all the derivatives forming the appearing series. Thus, either the positive or negative energy solutions define a stochastic motions conserving the total number of particles.

This result suggests the possibility of simultaneously consider two stochastic motions: both of them associated with positive energy solutions: but having opposite values of the charges. Let us indicate the positive energy solutions by $\phi_+(x)$ and the negative energy ones by $\phi_-(x)$. But we can also define positive energy functions starting from the negative waves by defining $\varphi_+(x) = \varphi_+(x^0, \overrightarrow{x}) = \phi_-(x^0, \overrightarrow{x})$. In this way all the solutions of the KG equations could participate in defining a combined stochastic process in which two kinds of particles participate: one kind with positive charges and the another with negative ones. We suspect that this construction can be extended to a full covariant stochastic theory of complex scalar particles which could appear to be a hidden variable theory for the quantum field theory of such particles. This question will be explored in extensions of this study. However, before considering this problem, the approach should made consistent, by finding a positive definite joint distribution function. The possibilities for this will be discussed in the next section.

In conclusion, for defining the found solution of the zeroth order equation for the joint distribution as a well defined hidden variable theory, it rest only (in the relativistic as a well as in the non-relativistic cases) to check wether or not, the derived joint distribution can obey the important positivity property, which is required by its character as a coordinate and momenta distribution of classical particles of mass m, all showing a relativistic momenta obeying the mass shell condition $p^2 - m^2 = 0$. In the coming sections we will remark on these questions.

7. Positiveness of the joint distribution and Olavo's analysis

Finally, in this section we want to discuss the question about the required positive character of the joint coordinate-momentum distribution, needed if at the end, the SQED approach can furnish a consistent hidden variable approach to quantum mechanics. It is known that the Fourier transformed joint distribution of the form $\widetilde{Q}^0(x,z) = \Psi^*(z^+)\Psi(z^-)$, does not give a solution to this difficulty, since in general, the joint coordinate-momentum distributions following from the inverse Fourier transform of $\widetilde{Q}^0(x,z)$(for all values of z) are not positive definite in general. Thus, the before discussed here solution for the joint distribution has a formal value, but for every solution of the KG equation does not furnish a physical positive joint distribution.

A step in the solution of this relevant interpretation problem, was given in Refs. 8 to 13. In these works it has been emphasized the idea about that by retaining only the second order in the expansion in the conjugate variable of the momenta in the Fourier transform of the joint distribution, a proper positive distribution is obtained [9, 12]. Thus, the infinitesimal values of these Fourier conjugate variable of the momentum was elevated to a central assumption. The consistency of this analysis had been argued by few alternative derivations [8–13].

It should recalled that we consider that the derivation of the Schrodinger (or the Klein-Gordon one) from the equations for $Q(x,p)$ in the absence of the vacuum stochastic force, is in some sense inconsistent with the spirit of the SQED, in which the stochastic action of the force is expected to define the quantum properties. What could be more natural is to start the iterative construction of higher order in the stochastic coupling, by employing also a zeroth order solutions for the joint distribution, but reflecting a free classical motion of the particle. This approach directly suggests that that SQED could be valid as describing quantum mechanical effects, but in a form compatible with the recent experimental results of Ives Couder. Those surprising findings show that mechanical systems, like liquid droplets moving on oscillating liquid surfaces, can exhibit quantum mechanical properties, as tunnel effects and double slit interference [19, 20].

Finally, in a last subsection we present some remarks on some special issues in the discussions given in [9, 12].

7.1. Positive zeroth order joint distributions and Couder's findings

In this section we will present a solution of the equation for the joint distribution $Q(x,p)$ being positive definite in phase space. These solutions strongly suggests a possible link with the recent studies on quantum mechanical properties in the movements of droplets [19, 20] . The picture could be as follows. After considering the action of stochastic forces on the free solutions to be presented below, the stochastic movements of the particle could form a localized standing wave which center of mass could be in uniform motions. The possible existence of such standing waves, showing counterparts moving with different velocities under a Lorentz transformation, had been suggested in a recent work [21]. In it a form of a stochastic Noether theorem had been introduced. The extender nature of these Lorentz invariant stochastic solutions can be imagined to show properties being similar to the Couder's droplets moving over liquid surfaces. If such is the case, the quantum particles of SQED could be imagined to describe the quantum properties in nature through the Couder's mechanism. We expect to study this possibility elsewhere.

The construction of the positive joint distributions is based in the Yukawa potential like solution of the Klein-Gordon equation in the presence of sources

$$\psi(\overrightarrow{x}) = \frac{1}{4\pi}\frac{\exp(-m\,r)}{r}, \tag{96}$$

$$r = \sqrt{x_1^2 + x_2^2 + x_3^2}, \tag{97}$$

satisfying the KG equation

$$(\partial^2 - m^2)\psi(\overrightarrow{x}) = -\delta^{(3)}(\overrightarrow{x}), \tag{98}$$

where $\delta^{(3)}$ is the three dimensional Dirac delta function. After performing a Lorentz transformation to a frame moving with velocity $\overrightarrow{v}$ along, let say, the x_1 axis, followed by a shift in the position of the origin of coordinates (at vanishing time t) to an arbitrary point $\overrightarrow{x}_k$, this Yukawa like solution becomes a "moving" one, of the form

$$\psi_{x_k,}(x,p) = \frac{1}{4\pi}\frac{\exp\left(-m\sqrt{\frac{\overrightarrow{p}^2+m^2}{m^2}\left(x_1-\frac{p_1}{\sqrt{\overrightarrow{p}^2+m^2}}t-x_1^k\right)^2+(x_2-x_2^k)_2^2+(x_3-x_3^k)}\right)}{\sqrt{\frac{\overrightarrow{p}^2+m^2}{m^2}\left(x_1-\frac{p_1}{\sqrt{\overrightarrow{p}^2+m^2}}t-x_1^k\right)^2+(x_2-x_2^k)_2^2+(x_3-x_3^k)}} \tag{99}$$

$$x=(t,\overrightarrow{x}),\quad p=(p^0,\overrightarrow{p})=\left(\frac{m}{\sqrt{1-v^2}},\frac{m\,\overrightarrow{v}}{\sqrt{1-v^2}}\right),\quad x_k=(0,\overrightarrow{x}_k). \tag{100}$$

In this change it had been used the relations

$$v=\frac{p_1}{\sqrt{\overrightarrow{p}^2+m^2}},\quad \sqrt{1-v^2}=\frac{m}{\sqrt{\overrightarrow{p}^2+m^2}}. \tag{101}$$

This function also can be expressed in a rotational invariant form as

$$\psi_{x_k,}(x,p)=\frac{1}{4\pi}\frac{\exp\left(-m\sqrt{\frac{\overrightarrow{p}^2+m^2}{m^2}\left(\left(\overrightarrow{x}-\frac{\overrightarrow{p}}{\sqrt{\overrightarrow{p}^2+m^2}}t-\overrightarrow{x}^k\right).\frac{\overrightarrow{p}}{|\overrightarrow{p}|}\right)^2+(\overrightarrow{x}-\overrightarrow{x}^k).\left(\overrightarrow{\overrightarrow{I}}-\frac{\overrightarrow{p}\,\overrightarrow{p}}{|\overrightarrow{p}|^2}\right).(\overrightarrow{x}-\overrightarrow{x}^k)}\right)}{\sqrt{\frac{\overrightarrow{p}^2+m^2}{m^2}\left(\left(\overrightarrow{x}-\frac{\overrightarrow{p}}{\sqrt{\overrightarrow{p}^2+m^2}}t-\overrightarrow{x}^k\right).\frac{\overrightarrow{p}}{|\overrightarrow{p}|}\right)^2+(\overrightarrow{x}-\overrightarrow{x}^k).\left(\overrightarrow{\overrightarrow{I}}-\frac{\overrightarrow{p}\,\overrightarrow{p}}{|\overrightarrow{p}|^2}\right).(\overrightarrow{x}-\overrightarrow{x}^k)}}, \tag{102}$$

where

$$\left(\overrightarrow{\overrightarrow{I}}-\frac{\overrightarrow{p}\,\overrightarrow{p}}{|\overrightarrow{p}|^2}\right)$$

is the projection tensor on the plane orthogonal to the velocity and $\overrightarrow{p}/|\overrightarrow{p}|$ is a unit vector in the direction of the velocity.

The function $\psi_{x_k,}(x,p)$ is positive definite in the phase space (x,p). Also, in the rest frame $p=(m,\overrightarrow{0})$, it directly satisfies the equation for the joint momenta-coordinate distributions, since it is time independent and the three velocity vanishes. Then

$$p^\mu\frac{\partial}{\partial x^\mu}\psi_{x_k,}(x,p)=0. \tag{103}$$

Having this equation a covariant form, it should be also valid after performing any Lorentz transformation of the coordinates and momenta. Now, we can define a set of N points $x_k, k=1,2,....,N$. Then, by superposing the functions of the type (102) for all the values of k, more general solutions can be constructed. They will describe a set of N localized solutions of the Klein-Gordon equation with sources. Also, the new distributions also will be positive quantities in all the mass-shell phase space. These solutions of the equations $p^\mu(\partial/\partial x^\mu)Q(x,p)=0$, showing a positive joint distribution have the forms

$$Q(x,p)=\sum_k\psi_{x_k,}(x,p), \tag{104}$$

where for each k value, the velocity for the distribution with index k, points in an arbitrary direction.

It can be noticed that the proposed solution is not rigorously point like. It has distributions showing a Compton wavelength extension. However, strictly point-like solutions for the joint distribution function are directly obtained in the form

$$Q(x,p)=\sum_k\Phi_{x_k,}(x,p), \tag{105}$$

$$p^\mu\frac{\partial}{\partial x^\mu}\Phi_{x_k,}(x,p)=0, \tag{106}$$

in which

$$\Phi_{x_k,}(x,p)=\frac{1}{4\pi}\delta^{(1)}\left(\sqrt{\frac{\overrightarrow{p}^2+m^2}{m^2}\left(x_1-\frac{p_1}{\sqrt{\overrightarrow{p}^2+m^2}}t-x_1^k\right)^2+(x_2-x_2^k)_2^2+(x_3-x_3^k)}\right) \tag{107}$$

$$x=(t,\overrightarrow{x}),\quad p=(p^0,\overrightarrow{p})=\left(\frac{m}{\sqrt{1-v^2}},\frac{m\,\overrightarrow{v}}{\sqrt{1-v^2}}\right),\quad x_k=(0,\overrightarrow{x}_k). \tag{108}$$

These joint distributions are positive in the whole phase space and solves the zeroth order equation. They represent the uniform motions of localized particles in the absence of stochastic perturbations, being more compatible with considered zeroth order equation under consideration. Below, we remark on the possible connections of these distributions with the Couder's findings of quantum mechanical properties droplets moving over oscillating liquid surfaces.

7.1.1. Possible links with the Couder's systems

Finally, let us here very roughly argue about the possible links of these special joint distributions with the Couder's experimental results [19, 20]. For this purpose, let us qualitatively discuss a situation in which one of the solution $\psi_{x_k},(x,p)$ with a given velocity, perpendicularly approaches a wall having two slits holes. Let us first note that the singularity of the solution is similar to the one in the Coulomb potential field and its classical energy is infinite. Thus, since we will assume that the particle has a finite mass $m,$ some negative contribution to the bound energy (which is concentrated in the singularity point) should cancel the infinite and positive contribution of the classical energy outside the point. Then, let us consider a sphere centered in the singularity at any instant, such that the field energy outside the sphere coincides with the total mass of the particle. Therefore, the contribution to the total energy inside the defined sphere, should vanish (the negative cohesive term should cancel the infinite positive energy laying outside the point, but inside the sphere). However, having not net mass, the system inside the sphere, might be suspected to weakly contribute to the free dynamics of the particle. Then, we have that outside the small sphere the system will satisfy the Klein-Gordon equation and also will move as a whole with constant velocity, Thus, the solution $\psi_{x_k},(x,p)$ might perhaps also be approximately represented by a wave packet solving KG equation, but with a momenta distribution showing non vanishing values only in a small neighborhood of the momenta component $p_1 = mv/\sqrt{1-v^2}$, in order that the packet shows a constant velocity. If this idea is valid (in spite of its very rough nature) then

$$
\begin{aligned}
\psi_{x_k},(x,p) &\simeq \int d\overrightarrow{p}\ f(\overrightarrow{p}) \exp\left(-i\sqrt{\overrightarrow{p}+m^2}t + \overrightarrow{p}.\overrightarrow{x}\ i\right) \\
&\simeq \int dp_1 dp_2 dp_3\ f(\overrightarrow{p}) \exp\left(-i\, p\left(\frac{\sqrt{\overrightarrow{p}+m^2}}{p}t - x_1\right)\right. \\
&\quad \left. + i\, p_2 x_2 + i\, p_3 x_3 \right) \\
&\simeq \int dp_1 dp_2 dp_3\ f(\overrightarrow{p}) \\
&\quad \times \exp(-i\, p(v\, t - x_1) + i\, p_2 x_2 + i\, p_3 x_3). \qquad (109)
\end{aligned}
$$

However, it should be noted that the singular solution is localized within region of the size of a Compton associated to the free mass parameter m. This is a small quantity for usual particles as, by example electrons. Thus, in order that the representation (109) could be valid, the momentum bandwidth of the integral in (109) should be larger than one over the spatial width of the singular solution $\delta x \simeq (1/m)$, that is, larger than m. In the relativistic limit $p_1 \gg m$, this condition can be satisfied. However, for mass parameters m larger than the electron's one, and in the non relativistic limit, this rule can not be imposed. Thus, we expect that the parameter m should have a small value in order to allow implementing the representation (109) in the non relativistic limit. Since the Couder's experiments are done for movements of the droplets over surfaces having massless propagating modes, even the vanishing mass parameter m could be allowed. As positive factor it can be noticed that after considering the effect of the stochastic forces, the wavepacket bound to the particle will be constituted by real waves stochastically excited by the random moving of the particle. Therefore, opening possibilities for showing Couder's like effects.

To end the argue, note that (109) is a wave corresponding to particles of mass m and wavelength $\lambda = 2\pi/p$. Therefore, assumed that the two slits to which the wave approaches, have a separation of a similar size to the wavelength λ, these modes should tend to be scattered by the action of the slits. Clearly, if the waves were completely free ones, this scattering should work. However, since the singularity is expected to maintain the structure of the particle when the scattering process occurs (as it happens in the Couder's experiments) the whole effect of such ”dispersion forces” could be suspected to be reduced to control the movement of the singularity (if it passes through the slits) to be pointing in the directions of the usual interference maxima. This argue also indicates (as noted above) that the results of Couder could perhaps be described by the found positive solutions of the equation for the joint distribution function after incorporating the action of the vacuum stochastic forces. The study of the two slits scattering on such configurations is expected to be considered elsewhere. The clear limitations of the above discussion for pure free solutions could be solved by the noted consideration of the stochastic forces. They have the chance of converting, the uniformly moving point like distribution functions in sorts of solitonic stochastic wavepackets showing similar properties as the Couder's droplets.

7.2. Remarks on the Olavo analysis

In the discussion given in [9, 12] it was obtained the general relation

$$
\begin{aligned}
&-\hbar^2\rho(x,t)\left(\frac{\partial^2 \ln Z(x,\delta x,t)}{\partial(\delta x)^2}\right)_{\delta q=0} \\
&= -\hbar^2\rho(x,t)\frac{\partial^2 \ln R(x;t)^2}{4\,\partial x^2} \qquad (110) \\
&= \int dp\ \left(p - \frac{\partial}{\partial x}S(x,t)\right)^2 F(x,p;t). \qquad (111)
\end{aligned}
$$

between the characteristic function Z and its inverse Fourier transform, the joint coordinate momentum distribution function F.

We intend to underline here that this exact equation (only assuming the product structure for $Z(x,\delta x,t) = \Psi^*(x + (\delta x/2), x - (\delta x/2))$) implies a strong link between the sign of the second derivative $(-\partial^2 \ln R(x;t)^2/\,\partial x^2)$ and the strict positivity of the joint distribution $F(x,p;t)$. This connection determines that when $(\partial^2 \ln R(x;t)^2/\,\partial x^2)$ in some spacial points turns to be negative, then $F(x,p,t)$ can not be strictly positive. But, it can be noted that it is possible to give initial conditions at some time t_o to defining arbitrary values for the spatial dependence of the wavefunctions. This property allows fixing the spatial dependence of the coordinates x for the density function in an arbitrary form. Therefore, the relation

$$-\frac{\partial^2 \ln R(x;t)^2}{\partial x^2} = \frac{1}{\rho(x,t)}\bigg(-\frac{\partial^2 \rho(x,t)}{\partial x^2} + \frac{1}{\rho(x,t)}\frac{\partial \rho(x,t)}{\partial x}\frac{\partial \rho(x,t)}{\partial x}\bigg) \tag{112}$$

can be always made negative, at a some time t_o and some point x_o, by fixing a minimum of the density at t_o and x_o. This conclusion means that solutions will always exist showing a non positive joint distribution function. Therefore, since this result is determined by the characteristic function up to second order in δx, that is in the limit considered in the Olavo's discussion, it is concluded that this approach still present a difficulty in consistently predict an always positive definite joint distribution function.

The above conclusion seems contradictory with the fact the in references [9, 12], it was argued the positivity of the joint distribution function. Let us remark on this point. In the notation of these works (in which the discussion was non relativistic), the characteristic function for infinitesimal values of the conjugate variable to the momenta δx was firstly obtained up to second order in the expansion over δx,in the form

$$Z_Q(x,\delta x;t) = R(x;t)^2 + \frac{i\,\delta x}{\hbar}R(x;t)^2\frac{\partial s(x;t)}{\partial x} \tag{113}$$

$$+\frac{(\delta x)^2}{2}\bigg[\frac{R(x;t)^2}{4}\frac{\partial^2 \ln R(x;t)^2}{\partial x^2} - \frac{R(x;t)^2}{\hbar^2}\left(\frac{\partial S(x;t)}{\partial x}\right)^2\bigg], \tag{114}$$

where $\rho(x,t) = R(x;t)^2$ is the particle density of the derived Schrodinger equation expressed as the absolute value of the wave function $R(x,t)$ and $\overline{p}(x,t) = \partial S(x;t)/\partial x$ is the gradient of the phase of the wavefunction $S(x,t)$. In a central step in the discussion in Ref. 9, after employing the assumed differential character of δx, the above expression was substituted by the following one.

$$Z_Q(x,\delta x;t) = R(x;t)^2 \exp\bigg[+\frac{i\,\delta x}{\hbar}\frac{\partial s(x;t)}{\partial x} + \frac{(\delta x)^2}{8}\frac{\partial^2 \ln R(x;t)^2}{\partial x^2}\bigg]. \tag{115}$$

This is a valid transformation up to the assumed second order in δx. In other words, a series of terms "completing" the appearing exponential function were added. Note that all such terms have at least a cubic dependence in δx. If the further use of the form of the characteristic function is reduced to evaluate its derivatives up to second order in δx, or integrals over δx, for small intervals in which the second order approximation gives a good approximation, the use of (115) is allowed. However, the obtaining of the expression for the joint distribution function in [9] rests in evaluating the Fourier inverse transformation

$$F(x,p;t) = R(x;t)^2 \int_{-\infty}^{\infty} d(\delta x)\exp\bigg(-i\,\delta x\bigg(p - \frac{\partial s(x;t)}{\hbar\,\partial x}\bigg) + \frac{(\delta x)^2}{8}\frac{\partial^2 \ln R(x;t)^2}{\partial x^2}\bigg)$$

$$= R(x;t)^2\sqrt{\pi}\,\frac{\exp\left(\frac{\left(p - \frac{\partial s(x;t)}{\hbar\,\partial x}\right)^2}{\frac{\partial^2 \ln R(x;t)^2}{2\,\partial x^2}}\right)}{\sqrt{-\frac{\partial^2 \ln R(x;t)^2}{8\,\partial x^2}}}, \tag{116}$$

which again is a positive definite expression only for the special class of wavefunctions having $-(\partial^2 \ln R(x;t)^2/8\,\partial x^2) > 0$, as argued before.

That is, even after assuming the modified expression for the characteristic function (115) as valid for finite values of δx, (as required to evaluate the inverse Fourier transform) the resulting expression can result as a non positive definite quantity for existing special wavefunctions.

Summary

We have presented a generalization of the non-relativistic stochastic quantum mechanics introduced by de la Peña and Cetto [1]. The discussion starts form the description of the random motions of a particle under the action of a relativistically invariant stochastic force defined in reference [5]. It is checked that in the non relativistic limit the starting equations reduce to the ones employed in [1]. Then, the set of equations for the joint distribution is expanded as a series of the particle charge. The free approximation, that is, the equation following in the zero order of the expansion in the charge is solved by considering the ansatz for the distribution adopted in [1]. After this, it is argued that the space-time probability distribution of the stochastic process $\rho(x)$ and a phase function $S(x)$ which gradient determines the momentum mean value at a definite space-time point, define a complex scalar function satisfying the KG equation through $\phi(x) = \sqrt{\rho(x)}\exp(i\ S(x))$. It is argued that the total number of particles $N = \int d\overrightarrow{x}\ \rho(x)$ determined by the space-time distribution conserves in time, if the KG solutions considered for determining the distributions are either positive or negative energy modes.

It is also addressed the fact that above mentioned ansatz solution, is associated to a joint distribution function possibly giving negative values. In this sense, it is remarked that there are special circumstances that could help to overcome this limitations. One of them is the following, the KG equation, when seen as theory of particles shows the interesting effect that it lacks a standard definition for the position operator eigenfunction as the Dirac's Delta function. In place of it, the appropriate position operator for this theory has a Gaussian like appearance showing non vanishing values in spatial regions of the size of the Compton wavelength of the particles [17]. Therefore, it looks reasonable that the proper interpretation of the $\widetilde{Q}^0(x,p)$ as describing particles with a well defined position x can show difficulties. This leads to the idea that a proper modification of the kinetic equations to take account of this extensive nature of the particles occasionally could still lead to a consistent equivalence of the SQED with QM and QFT.

The work is also presenting solutions for the joint distribution functions which obey the positive conditions required by a proper classical distribution. Such solutions in a sense, seems that can be more natural (in the zeroth order in the coupling with the stochastic forces) than the ansatz considered in [1]. It is also argued that these solutions could be related with systems of particles showing the Couder's experimental results, if the localized solutions employed for their construction can be approximately represented as Couder's like wavepackets after including the effects of the stochastic forces [19]. The investigation of this particular possibility is expected to be considered elsewhere.

They are exposed some remarks in connection the Olavo's discussion in reference [8–13] directed to define a positive definite joint distribution function. They identify some estimated limitations of the approach in connection with the central question of defining a positive result for this quantity.

Finally, assumed that the identified difficulties with positive condition of the joint distribution (the one adopted in [1]) can be properly solved, the discussion opens possibilities for the extension of the SQED analysis. Of particular interest is: The possibility of generalize the discussion to describe the stochastic evolution of two sets of particles: one described by the positive energy solutions and the other by the negative energy ones. After to also including the presence of an external electromagnetic fields, this construction seem to offer opportunities for describing the creation and annihilation particles by the electromagnetic field. The interaction terms, could result to be sources of the variation in the total numbers of positive or negative charged particles generated by the annihilation or creation of particles due to the action of the electromagnetic field. The search for the connection of this construction with the quantum field theory of the complex scalar field is a further question of interest to explore.

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