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Weyl invariance in metric $f(R)$ gravity

J.M. Fernández Cristóbal
IES Giner de los Ríos. Segovia. Spain.
e-mail: jmariaffc@gmail.com

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We aim to derive the most general $f(R)$ gravity theory, including the matter, so that it be Weyl invariant. Making use of the mathematical equivalence of these theories with an type of scalar-tensor theory (which includes a scalar degree of freedom, ϕ) and by imposing the Weyl invariance for the pure gravity (under this label, we understand the part that does not involve fields of matter although it could include kinetic terms linked to ϕ) as well as for the matter sector, we obtain the fundamental equation that restricts the form of $V \doteq R\phi - f(R)$ (and, accordingly, of $f(R)$) so that the resulting action to be Weyl invariant in the Jordan frame. We show that this action is not other than the so-called gravity-dilaton action with one scalar field,Φ, which effective mass is R and Φ dependent. In the Einstein frame, the action becomes the Einstein-Hilbert action with the Ricci scalar being constant due to that the effective mass of scalar field in this frame vanish. So, we can assume that the Ricci scalar, in the Einstein frame, is the true Cosmological Constant. Therefore, is not preposterous to guess that, at least mathematically, all Weyl invariant metric $f(R)$ theory in the Jordan frame is equivalent, at classical level, to the Einstein gravity, in the Einstein frame, with a constant Ricci scalar. At quantum level, as it is known, both theories are not equivalent due to the presence of anomalies in one of the frames.

Keywords: Weyl invariance; metric f(R) gravity; Jordan frame; Einstein frame.

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1. Introduction

The $f(R)$ gravity [1,2] is a modification of the Einstein gravitational theory and it is essentially based on a generalization of the Einstein-Hilbert action:

$$S_{EH} \sim \int d^4x\sqrt{-g}R \longrightarrow \int d^4x\sqrt{-g}f(R) \qquad (1)$$

where $f(R)$ is a function of the Ricci scalar curvature R. Perhaps the more known $f(R)$ theory is the model of Starobinsky where $f(R) = R + (a/m^2)R^2$ [3]. The reasons for this (and others) modification of the Einstein General Relativity (GR) arise from diverse motivations and these are of different nature: astrophysical, cosmological [4-6], coming from high-energy physics, from the need to obtain a Quantum Gravity Theory [7] and so on (see [8,9]) for instance).

It is well known that there are three versions of $f(R)$ gravity, according to the variational principle used to derive the field equations: the *metric* $f(R)$ *gravity* [10] if the action is extreme with respect to the variation of the metric; the *Palatini* $f(R)$ *gravity* [11] if the action varies with respect to the metric and the connection, where both are considered as independent variables; finally *the metric-affine* $f(R)$ *gravity* [12] if the mechanism of Palatini is used but the action of matter is considered dependent of connection. For the Einstein-Hilbert action ($f(R) = R$)), the Palatini formulation and the metric gravity formulation are equivalent but this is not true for a $f(R)$ general theory.

Besides, both the Palatini action and the metric gravity action, can be written in the so-called Jordan frame^i, in which the scalar field is non-minimally coupled to the metric tensor, or in the so-called Einstein frame, in which it is minimally coupled to the metric tensor. The passage from one frame to the other is given by a conformal transformation $g_{\mu\nu} \longrightarrow \bar{g}_{\mu\nu} = f'(R)g_{\mu\nu} = \phi g_{\mu\nu}$ where the prime denote the derivative.

The issue of distinguishing if the Einstein frame and the Jordan frame are not but two different representations of the same physical theory or, contrary, they are two truly different theories, still has not been resolved. According to some authors, the Jordan frame is the physical frame [13]. For others [14,10] it is the Einstein frame because of its resemblance to General Relativity. There is also a third group integrated by the authors who claim the physical equivalence of both frames, at least at the classical level, since the conformal transformations do not change the mass ratios of elementary particles; therefore, those does not alter physics [15].

However, some authors claim a true physical difference between both frames, [16-18]. At the quantum level, the issue is still more complicated (see [19] for a very interesting general discussion).

On the other hand, from its birth, the local scale (conformal) invariant theories [20,21] have been considered in diverse contexts and to address different problems: cosmological [22-24] in the framework of the particle physics [25] or in quantum gravity [26,27].

In this paper, we consider the Weyl invariance (understood as a locally conformal symmetry^{ii}) of a general $f(R)$ theory and we establish the shape that their action must to have to be Weyl invariant. We show that this Weyl invariant action $f(R)$, in the Jordan frame, is equivalent mathematically to the so-called gravity-dilaton action. Given that, as it was said before, the passage from Jordan's frame to Einstein's frame is made by means of a particular conformal transformation in which a scalar field takes part, we ask ourselves if, for any $f(R)$ theory, it is possible to maintain the Weyl invariance (and, consequently, the conformal invariant because the Weyl invariance implies conformal invariance) in both frames or, on the contrary, the passage from a frame to another one implies to resign explicitly to her. We show that the latter is what actually happens due that actions $f(R)$ Weyl invariant in the Jordan frame are equivalent, in the Einstein frame, to the Einstein-Hilbert action with the Ricci scalar being constant.

2. The metric $f(R)$ gravity

Let us consider the covariant pure gravitational action, in the Jordan frame, for the n-dimensional $f(R)$ gravity theories:

$$S_g = \frac{1}{2\kappa^2}\int d^n x\sqrt{-g}f(R) \tag{2}$$

where $\kappa^2 = M_P^{-(n-2)} = 8\pi G_n$, being M_P the n-dimensional Planck mass ($c = \hbar = 1$), G_n the n-dimensional gravitational constant, $g = det(g_{\mu\nu})$ and R the Ricci scalar:

$$= \frac{1}{\sqrt{-g}}g^{\mu\nu}\partial_\lambda(\sqrt{-g}\Gamma^\lambda_{\mu\nu}) - \partial^2 ln\sqrt{-g} - g^{\mu\nu}\Gamma^\lambda_{\mu\alpha}\Gamma^\alpha_{\lambda\nu} \tag{3}$$

Obviously, for $f(R) = R$, the action (2) becomes the n-dimensional Einstein-Hilbert action.

It is well known (see [28,10,29]) that a Legendre transformation allows us to express the action (2) in another dynamically equivalent form in which the Lagrangian is linear in the scalar curvature and where an auxiliary dimensionless scalar degree of freedom (not matter field), ϕ, is added:

$$\phi(R) = \frac{df(R)}{dR} \Longrightarrow df(R) = d\left(R\phi(R)\right) - d\hat{V}(\phi(R)) \tag{4}$$

being:

$$R(\phi) = \frac{d\hat{V}(\phi)}{d\phi} \tag{5}$$

and where the condition $\phi(R) \neq Constant$, that is to say $(d^2f(R)/dR^2) \neq 0$, is assumed. Also, as it is usual, and due to diverse reasons (see [10], for example), we assume the condition $\phi > 0$.

In fact, $\hat{V}(\phi)$ is the Legendre transform of $f(R)$. Therefore the action (2) becomes:

$$S_g = \frac{1}{2\kappa^2}\int d^n x\sqrt{-g}\left[R\phi - \hat{V}(\phi)\right] \tag{6}$$

From ϕ we can define a scalar field (with $[\Phi] = M^{n-2/2}$) as:=

$$\phi = (\kappa\Phi)^2 \qquad V(\Phi) = \frac{1}{\kappa^2}\hat{V}(\kappa^2\Phi^2) \tag{7}$$

and, therefore, the action (6)(from now on $f(R) = R\Phi^2 - V(\Phi)$) is:

$$S_g = \frac{1}{2}\int d^n x\sqrt{-g}\left[R\Phi^2 - V(\Phi)\right] \tag{8}$$

being:

$$\Phi^2 = \frac{\partial f(R)}{\partial R} \qquad V(\Phi) = \Phi^2\left(R - \frac{f}{f'}\right)$$
$$R(\Phi) = \frac{1}{2\Phi}\frac{dV(\Phi)}{d\Phi} \tag{9}$$

The equivalent Lagrangian of the action (8) is named Helmholtz Lagrangian by analogy with the classic mechanic. We look for an action so that (8), and, accordingly the action (2), becomes invariant under the conformal transformation:

$$g_{\mu\nu} \longrightarrow \tilde{g}_{\mu\nu} = e^{2\sigma(x)}g_{\mu\nu}$$
$$\Longrightarrow \tilde{g} = e^{2n\sigma(x)}g \Longrightarrow \sigma = \frac{1}{2n}\log\frac{\tilde{g}}{g} \tag{10}$$

The purpose is to determine the form that $V(\Phi)$, and accordingly $f(R)$, must to have for to make it possible.

3. Weyl invariance

Under the conformal transformation (10), the connection and the Ricci scalar do it as:

$$\Gamma^\lambda_{\mu\nu} \longrightarrow \tilde{\Gamma}^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} + \delta^\lambda_\mu\nabla_\nu\sigma + \delta^\lambda_\nu\nabla_\mu\sigma - g_{\mu\nu}\nabla^\lambda\sigma \tag{11}$$

$$\tilde{\Gamma}^\lambda_{\mu\nu} \longrightarrow \Gamma^\lambda_{\mu\nu} = \tilde{\Gamma}^\lambda_{\mu\nu} - \delta^\lambda_\mu\nabla_\nu\sigma - \delta^\lambda_\nu\nabla_\mu\sigma + \tilde{g}_{\mu\nu}\nabla^\lambda\sigma \tag{12}$$

and:

$$R \longrightarrow \tilde{R} = e^{-2\sigma}\Big[R - 2(n-1)\nabla^2\sigma$$
$$- (n-1)(n-2)(\nabla\sigma)^2\Big] \tag{13}$$

where the tilde refers to the metric $\tilde{g}_{\mu\nu}$ and we denote $\nabla^2\sigma = g^{\mu\nu}\nabla_\mu\nabla_\nu\sigma \equiv \sigma_2$ and $(\nabla\sigma)^2 = g^{\mu\nu}\nabla_\mu\sigma\nabla_\nu\sigma \equiv \sigma_1^2$. Therefore:

$$\tilde{R} = e^{-2\sigma}\left[R - (n-1)\left(2\sigma_2 + (n-2)\sigma_1^2\right)\right] \tag{14}$$

or if $n \succ 2$:

$$\tilde{R} = e^{-2\sigma}\left[R - \frac{4(n-1)}{n-2}e^{-\frac{(n-2)\sigma}{2}}\nabla^2\left(e^{\frac{(n-2)\sigma}{2}}\right)\right] \tag{15}$$

So, for the action (8) to be Weyl invariant,$f(R)$ should be transformed under (10) as:

$$f(R) \longrightarrow \tilde{f}(\tilde{R}) = \left(\tilde{R}\tilde{\Phi}^2 - \tilde{V}(\tilde{\Phi})\right)$$
$$= e^{-n\sigma}f(R) = e^{-n\sigma}\left[R\Phi^2 - V(\Phi)\right] \tag{16}$$

In other words, under the Weyl transformation (10), $f(R)$ should be homogeneously transformed.

In a analogue manner, from the above equation and making use of (9) and (15), we get the way such that Φ and $V(\Phi)$

should be transformed in order to get one Weyl invariant action:

$$\tilde{\Phi}^2 = \frac{\partial \tilde{f}(\tilde{R})}{\partial \tilde{R}} = e^{-n\sigma}\frac{\partial f(R)}{\partial R}\frac{\partial R}{\partial \tilde{R}} = e^{-(n-2)\sigma}\Phi^2 \Longrightarrow \tilde{\Phi} = e^{-\frac{(n-2)\sigma}{2}}\Phi \quad (17)$$

and

$$\tilde{V}(\tilde{\Phi}) = e^{-n\sigma}V(\Phi) - \frac{4(n-1)}{n-2} \times e^{-\frac{(3n-2)\sigma}{2}}\Phi^2\nabla^2\left(e^{\frac{(n-2)\sigma}{2}}\right) \quad (18)$$

This equation provides restrictions for the shape of the scalar function $V(\Phi)$. After of some algebra (see the appendix),(18) can be rewritten in the form:

$$\tilde{V}(\tilde{\Phi}) - \frac{4(n-1)}{n-2}\tilde{\Phi}\tilde{\nabla}^2\tilde{\Phi} = e^{-n\sigma}\left[V(\Phi) - \frac{4(n-1)}{n-2}\Phi\nabla^2\Phi\right] \quad (19)$$

being $\tilde{\nabla}^2\tilde{\Phi} = \tilde{g}^{\mu\nu}(\partial_\mu\partial_\nu\tilde{\Phi} - \tilde{\Gamma}^\lambda_{\mu\nu}\partial_\lambda\tilde{\Phi})$

The Eq. (19) claims that $V(\Phi)-(4(n-1)/n-2)\Phi\nabla^2\Phi$ should be some function $\chi(\Phi)$ of such way that, under a Weyl transformation, it is transformed as:

$$\chi(\Phi) \overset{Weyl}{\longrightarrow} \tilde{\chi}(\tilde{\Phi}) = e^{-n\sigma}\chi(\Phi) \quad (20)$$

That is to say:

$$R\Phi^2 - f(R) - \frac{4(n-1)}{n-2}\Phi\nabla^2\Phi = \chi(\Phi) \quad (21)$$

should be transformed homogeneously.

4. Solutions

Taking into account dimensional arguments for $\chi(\Phi)$ ($[\chi(\Phi)] = M^n$), the most general polynomial expression which verifies (20) is:

$$\chi(\Phi) = \sum_{k=0}^{n} C_k m^k \Phi^{\frac{2(n-k)}{n-2}} \quad (22)$$

being C_k a dimensionless constant ($C_k \to \tilde{C}_k = C_k$) and $m^k \to \tilde{m}^k = e^{-k\sigma}m^k$

If we define $\xi \doteq (4(n-1)/n-2))$ and by using of the identity:

$$\int d^n x\sqrt{-g}\Phi\nabla^2\Phi = -\int d^n x\sqrt{-g}\partial_\mu\Phi\partial^\mu\Phi \quad (23)$$

then, the most general Weyl invariant action becomes:

$$S_g = \frac{1}{2}\int d^n x\sqrt{-g}\left[\xi\partial_\mu\Phi\partial^\mu\Phi + R\Phi^2 - \chi(\Phi)\right] \quad (24)$$

For $n = 4$, by considering only the terms that are invariant under $\Phi \to -\Phi$, and so (24) is finally:

$$S_g = \int d^4 x\sqrt{-g}\left[\frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi + \frac{1}{12}R\Phi^2 - \frac{\alpha}{2}m^2\Phi^2 - \frac{\omega}{4!}\Phi^4 - \beta m^4\right] \quad (25)$$

where $\alpha \doteq (C_2/6)$, $\beta \doteq (C_4/2)$ and $\omega \doteq (C_0/3)$

This action (except factors) was already considered in [32]. This action is the mathematical solution of (20). However, in order for make the whole theory self-consistent (that is to say, so that pure gravity plus the matter to be Weyl invariant), not all the terms are physically acceptable. If the Weyl invariance condition for matter is assumed also, as it will be seen below, the field equation of Φ along with the constraint of tracelessness of the energy-momentum tensor, imposed by the Weyl invariance, strongly reduces the number of allowed terms. In particular, the mass terms are removed. The same happens if there is no present matter (in the vacuum).

5. Fields equations

Let us consider the matter action:

$$S_{\text{matt}} = \int d^n x\sqrt{-g}\mathcal{L}_{\text{matt}}(\psi_i, g_{\mu\nu}) \quad (26)$$

being ψ_i the matter fields. The matter action is not generically Weyl invariant. Nevertheless we will assume as a working hypothesis that the matter (exotica or not) action of our model of the universe effectively is it. It is to say, it is invariant under the transformations:

$$g_{\mu\nu} \longrightarrow \tilde{g}_{\mu\nu} = e^{2\sigma(x)}g_{\mu\nu}$$
$$\psi_i \longrightarrow \tilde{\psi}_i = e^{-\lambda_i\sigma(x)}\psi_i \quad (27)$$

From the action $S_g + S_{\text{matt}}$ and by using (24),(26) and the identity

$$\nabla^2\Phi^2 = 2(\partial_\lambda\Phi\partial^\lambda\Phi + \Phi\nabla^2\Phi) \quad (28)$$

it is straightforward to derive the field equation for $g_{\mu\nu}$:

$$\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\Phi^2 + g_{\mu\nu}\left(2\Phi\nabla^2\Phi - \frac{2}{n-2}\partial_\varrho\Phi\partial^\varrho\Phi + \frac{1}{2}\chi(\Phi)\right) + \frac{2n}{(n-2)}\partial_\mu\Phi\partial_\nu\Phi - 2\Phi\nabla_\mu\nabla_\nu\Phi = T^{\text{matt}}_{\mu\nu} \quad (29)$$

where $\chi(\Phi)$ is given by (22) and being

$$T^{\text{matt}}_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_{\text{matt}}}{\delta g^{\mu\nu}} \overset{Weyl}{\longrightarrow} \tilde{T}^{\text{matt}}_{\mu\nu} = e^{-(n-2)\sigma}T^{\text{matt}}_{\mu\nu}.$$

By taking the trace in (29), one gets:

$$-R\Phi^2 + \xi\Phi\nabla^2\Phi + \frac{n}{n-2}\chi(\Phi) = \frac{2}{n-2}T^{\text{matt}} \quad (30)$$

On the other hand, by varying the action (24) with respect to Φ, we obtain the field equation :

$$\left(\xi\nabla^2 - R + \frac{1}{2\Phi}\chi'(\Phi)\right)\Phi = 0 \tag{31}$$

where the prime denote the derivative w.r.t. Φ.

Putting together (30) and (31), the following equation is derived:

$$(2-n)\Phi\chi'(\Phi) + 2n\chi(\Phi) = 4T^{\text{matt}} \tag{32}$$

It is well known (see [31], for example) that one important features of the Weyl invariance of any action S is that the trace of their associated energy-momentum tensor $T_{\mu\nu} = (2/\sqrt{-g})(\delta S/\delta g^{\mu\nu})$ is identically zero, on the equations of motions (on shell). It is to say, $g^{\mu\nu}T_{\mu\nu} = 0$. Indeed because of the assumed invariance of S_{matt} under the transformation (27):

$$\delta_W S_{\text{matt}} = -2\sigma g^{\mu\nu}\frac{\delta S_{\text{matt}}}{\delta g^{\mu\nu}} - \lambda_i\sigma\frac{\delta S_{\text{matt}}}{\delta\psi_i}\psi_i = 0 \tag{33}$$

For any matter field (it is to say, for all i) its field equation is $(\delta S_{\text{matt}}/\delta\psi_i) = 0$, and then $T^{\text{matt}}_{\mu\nu}g^{\mu\nu} = 0$. Obviously for the case in which there are not matter (the vacuum) also $T^{\text{matt}} = 0$.

In any case, whenever $T^{\text{matt}} = 0$, the solution of (32) is $\chi(\Phi) \sim \Phi^{(2n/n-2)}$. In this case, the scalar field equation, (31), is simply the trace of the gravitational field equation (29), as it is well known.

Therefore, the most general Weyl invariant pure gravity action,(24), physically plausible, namely compatible with the Weyl invariance of S_{matt}, finally, is given by the expression:

$$S_g = \frac{1}{2}\int d^n x\sqrt{-g}\left[\xi\partial_\mu\Phi\partial^\mu\Phi + R\Phi^2 - C_n\Phi^{\frac{2n}{n-2}}\right] \tag{34}$$

This action describes the so-called dilaton gravity and, was already considered in [27,19,23] and, for $n = 4$, derived in [21].

The (31) is the one of an scalar field:

$$\left(\nabla^2 + m^2_{\text{eff}}\right)\Phi = 0 \overset{Weyl}{\longrightarrow} \times e^{-\frac{n+2}{2}\sigma}\left(\tilde{\nabla}^2 + \tilde{m}^2_{\text{eff}}\right)\tilde{\Phi} = 0 \tag{35}$$

with the effective mass squared:

$$m^2_{\text{eff}}(\Phi) = \frac{1}{\xi}\left(\frac{n}{n-2}C_n\Phi^{\frac{4}{n-2}} - R\right) \overset{n=4}{\longrightarrow} \frac{1}{6}(2C_4\Phi^2 - R) \tag{36}$$

6. Einstein frame

The action (34), in which the scalar field is non-minimally coupled to the metric, is named Jordan frame action. By carrying out the following transformation:

$$g_{\mu\nu} \to \bar{g}_{\mu\nu} = (\kappa\Phi)^{\frac{4}{n-2}}g_{\mu\nu} \tag{37}$$

this action turns into other one in which the scalar field appears minimally coupled to the metric. Making use of (37,15), we arrive at the Ricci scalar in Einstein frame :

$$\bar{R} = (\kappa\Phi)^{-\frac{4}{n-2}}\left[R - \xi\Phi^{-1}\nabla^2\Phi\right] \tag{38}$$

The resultant action in the so-called Einstein frame (the bar refers to Einstein frame) is finally:

$$S_{\bar{g}} = \frac{1}{2\kappa^2}\int d^n x\sqrt{-\bar{g}}\left[\bar{R} - \frac{C_n}{\kappa^{\frac{4}{n-2}}}\right] \tag{39}$$

where the identity (23) was used. As it is seen from (39), the scalar field does not appear in the action when this is expressed in the Einstein frame. What happens truly is that the transformation (17) is verified and then the field Φ is fixed:

$$\Phi \longrightarrow \bar{\Phi} = (\kappa\Phi)^{-1}\Phi = \frac{1}{\kappa} \tag{40}$$

Given that (35) should be verify too, $m^2_{\text{eff}}(\bar{\Phi}) = 0$ and then

$$\bar{R} = \frac{n}{n-2}\frac{C_n}{\kappa^{\frac{4}{n-2}}} \tag{41}$$

Therefore, the action in Einstein frame finally becomes :

$$S_{\bar{g}} = \frac{1}{n\kappa^2}\int d^n x\sqrt{-\bar{g}}\bar{R} \tag{42}$$

being $\bar{R}$ constant.

Making use of the (39) and (41), (29) becomes:

$$\bar{R}_{\mu\nu} = \frac{\bar{R}}{n}\bar{g}_{\mu\nu} \tag{43}$$

This is the field equation of a maximally symmetric spacetime of constant curvature:

$$K = \frac{\bar{R}}{n(n-1)} = \frac{1}{\kappa^2}\frac{C_n}{(n-2)(n-1)}. \tag{44}$$

It to say, $\bar{R}$ seems the true cosmological constant.

7. Conclusions

We have considered the behavior of metric $f(R)$ gravity (with matter added) under Weyl transformations. Taking into account the well known property that states the mathematical equivalence of this theory with a scalar-tensor theory, we studied the behavior under the Weyl transformation of the action

$$\int d^n x\sqrt{-g}[R\Phi^2 - V(\Phi)]$$

and we derived the necessary condition for the action to be Weyl invariant. The most general Weyl invariant action, in the Jordan frame, includes both the degrees of freedom of the gravitational field as well as the scalar field whose effective mass (by this we understand the parameter that is involved in the Klein-Gordon equation) consists of two components: a term of self-interaction of the scalar field and the scalar curvature of the space-time. The Weyl invariant action for the metric $f(R)$ theory we obtained is the so-called dilaton gravity action. When we carry out the transformation $\bar{g}_{\mu\nu} \sim \Phi^2 g_{\mu\nu}$, going from Jordan frame to Einstein frame, the scalar field becomes massless and then $\bar{R}$ turns out to be constant and it arise as the true cosmological constant. In this way, the symmetry is broken and the action, in Einstein frame, is the Einstein- Hilbert action with a pure cosmological term. All this gives us reasons to assume that, despite how surprising it may seem, every invariant Weyl metric $f(R)$ gravity theory in Jordan frame is mathematically equivalent to a theory of type Einstein-Hilbert, in the Einstein frame, with only one cosmological term where the Ricci scalar is the cosmological constant.

If the physical equivalence is given too, it is something that should be studied in depth.

Appendix

A.

Starting from (18), we can see that the Weyl invariance condition is given by (19).

The (18) can be expressed as:

$$\tilde{V}(\tilde{\Phi}) = e^{-n\sigma}\left[V(\Phi) - \frac{4(n-1)}{n-2}\tilde{\Phi}\Phi\nabla^2\left(\Phi\tilde{\Phi}^{-1}\right)\right] \quad \text{(A.1)}$$

$$\nabla^2\left(\Phi\tilde{\Phi}^{-1}\right) = \tilde{\Phi}^{-1}\nabla^2\Phi - 2\tilde{\Phi}^{-2}\nabla\Phi\nabla\tilde{\Phi} + 2\tilde{\Phi}^{-3}\Phi\nabla\tilde{\Phi}\nabla\tilde{\Phi} - \tilde{\Phi}^{-2}\Phi\nabla^2\tilde{\Phi} \quad \text{(A.2)}$$

and:

$$\Phi\tilde{\Phi}\nabla^2\left(\Phi\tilde{\Phi}^{-1}\right) = \Phi\nabla^2\Phi + 2\tilde{\Phi}^{-1}\Phi\nabla\tilde{\Phi} \times \left(\tilde{\Phi}^{-1}\Phi\nabla\tilde{\Phi} - \nabla\Phi\right) - \tilde{\Phi}^{-1}\Phi^2\nabla^2\tilde{\Phi} \quad \text{(A.3)}$$

$$\tilde{\Phi}^{-1}\Phi\nabla\tilde{\Phi} = -\frac{n-2}{2}\Phi\nabla\sigma + \nabla\Phi \quad \text{(A.4)}$$

and therefore:

$$2\tilde{\Phi}^{-1}\Phi\nabla\tilde{\Phi}\left(\tilde{\Phi}^{-1}\Phi\nabla\tilde{\Phi} - \nabla\Phi\right) = \frac{(n-2)^2}{2} \times (\nabla\sigma)^2\Phi^2 - (n-2)\Phi\nabla\sigma\nabla\Phi \quad \text{(A.5)}$$

On the other hand:

$$-\tilde{\Phi}^{-1}\Phi^2\nabla^2\tilde{\Phi} = -e^{n\sigma}\tilde{\Phi}\tilde{\nabla}^2\tilde{\Phi} - \frac{(n-2)^2}{2}(\nabla\sigma)^2\Phi^2 + (n-2)\Phi\nabla\sigma\nabla\Phi \quad \text{(A.6)}$$

Finally:

$$\Phi\tilde{\Phi}\nabla^2\left(\Phi\tilde{\Phi}^{-1}\right) = \Phi\nabla^2\Phi - e^{n\sigma}\tilde{\Phi}\tilde{\nabla}^2\tilde{\Phi} \quad \text{(A.7)}$$

Therefore (29) become:

$$\tilde{V}(\tilde{\Phi}) = e^{-n\sigma}\left[V(\Phi) - \frac{4(n-1)}{n-2} \times \left(\Phi\nabla^2\Phi - e^{n\sigma}\tilde{\Phi}\tilde{\nabla}^2\tilde{\Phi}\right)\right] \quad \text{(A.8)}$$

which is no other than (19).

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i. Here,the word “frame” denotes a choice of dynamical variables, not a choice of a reference frame in space-time.

ii. On the difference between Weyl and conformal invariance, see [31].

1. Th.P. Sotiriou, V. Faraoni, *Rev. Mod. Phys.* **82** (2010) 451-497. arXiv:0805.1726 [gr-qc]
2. A. De Felice, and S. Tsujikawa, *“f(R) Theories”*, Living Rev. Relativity, **13**, lrr-2010-3.http://www.livingreviews.org/lrr-2010-3 (2010)
3. A. Starobinsky, *A. Phys. Lett. B* **91** (1980) 99.
4. S. Nojiri and S.D. Odintsov, *Int. J. Geom. Meth. Mod. Phys.* **4** (2007) 115.
5. S.M. Carroll, V. Duvvuri, M. Trodden, M.S. Turner, *Phys. Rev. D* **70** (2004) 043528. astro-ph/0306438.
6. S. Capozziello, V.F. Cardone, A. Troisi, *Phys. Rev. D* **71** (2005) 043503 astro-ph/0501426.
7. I.L. Buchbinder, S.D. Odintsov, I.L. Shapiro, *Effective action in quantum gravity* (IOP Publishing, Bristol UK)(1992) p. 413
8. H.-J. Schmidt, *Int. J. Geom. Meth. Phys.* **4** (2007) 209 .
9. C.M. Will, *Theory and Experiment in Gravitational Physics*. (Cambridge. Univ. Press, N. York, 1981).
10. G. Magnano and L.M. Sokolowsky, *Phys. Rev. D* **50** (1994) 5039-5059.
11. S. Capozzielo, and M. Francaviglia, *Gen. Rel. Grav.* **40** 357 (2008).
12. T.P. Sotiriou, *Modified Actions for Gravity: Theory and Phenomenology* . Ph. D. thesis, SISSA, eprint 0710.4438, (2008).
13. J.D. Barrow and K. Maeda, *Nucl. Phys. B* **341** (1990) 294.
14. L. Amendola, D. Bellisai and F. Occhionero, *Phys. Rev. D* **47** (1993) 4267.
15. N. Makino and M. Sasaki, *Prog. Theor. Phys*. **86** (1991) 103; R. Holman, E.W. Kolb, S. Vadas, Y. Wang and E. Weinberg, *Phys.*

Lett. B **237** (1990) 37; B.A. Campbell, A.D. Linde and K. Olive, *Nucl. Phys. B* **355** (1991) 146; J.A. Casas, J. Garcia-Bellido and M. Quiros, *Nucl. Phys. B* **361** (1991) 713; J.A. Casas, J. Garcia-Bellido and M. Quiros, *Class. Quantum Grav.* **9** (1992) 1371; L. Garay and J. Garcia-Bellido, *Nucl. Phys. B* **400** (1993) 416.

16. S. Capozziello, S. Nojiri, S.D. Odintsov, and A. Troisi, *Phys. Lett. B* **639** (2006).
17. G. Domenech, A. Naruko, and M. Sasaki, *JCAP.* **1510** (2015) 067.
18. S. Bahamonde, S.D. Odintsov, V.K. Oikonomou, and W. Matthew, *Correspondence of f(R) Gravity Singularities in Jordan and Einstein Frames*. gr-qc: 1603.05113v1.
19. M. Herrero-Valea, *Phys. Rev. D* **93** (2016) 105038.
20. H. Weyl, *Z. Phys*. **56** (1929) 330 [Surveys *High Energ.Phys.* **5** (1986) 261].
21. A. Padilla, D. Stefanyszyn, M. Tsoukalas, *Phys. Rev. D* **89** (2014) 065009.
22. I. Bars, S.H. Chen, P.J. Steinhardt, and N. Turok, *Physics Letters B* **715** (2012) 278.
23. E. Alvarez, S. González-Martin, and M. Herrero-Valea, *JCAP.* **03** (2015) 035.
24. R. Kallosh and A. Linde, *JCAP.* **1307** (2013) 002.
25. K.A. Meissner and H. Nicolai, *Physics Letters B* **648** (2007) 312.
26. F. Englert, C. Truffin, and R. Gastmans, *Nucl. Phys. B* **117** (1976) 407.
27. E. Alvarez, M. Herrero-Valea, and C.P. Martín, *High Energy Phys.* **10** (2014) 115.
28. T.P. Sotiriou, *Class. Quant. Grav.* **23** (2006) 5117.
29. A. Hindawi, B.A. Ovrut, and M. Waldram, *Phys. Rev. D* **53** (1996) 5039-5059.
30. L. Amendola, D. Polarski, and S. Tsujikawa, *Phys. Rev. Lett.* **98** (2007) 131302. astro-ph/0603703.
31. Karananas, G.K. Monin, *A. Physics Letters B* **757** (2016) 257-270.
32. M.P. Dabrowski, J. Garecki and D.B. Blashke, *Annalen Phys.* **18** (2009) 12-22.