Fractal imbibition in Koch's curve-like capillary tubes

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Fractal dimension effects in capillary imbibition process are analytically studied. The fractal formulation of tortuous flow with the assumption of a fractal tortuous path introduced by Cai and Yu is used to analyse the capillary rise through the tubes with deterministic fractal geometry. Capillary rise in Koch's curve-like tubes was investigated. A new permeability parameter that takes into account the tortuosity of the flow path is deduced, and a geometrical relationship for fractal dimension of flow tortuosity (d_τ) in porous media is obtained. The equilibrium height time as a function of fractal dimension of the flow tortuosity in capillary tubes with tortuous path was derived.

Keywords: Spontaneous imbibition; linear fractal; Koch curve; fractal dimension of flow tortuosity.

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1. Introduction

Spontaneous capillary imbibition is a transport phenomenon present in a variety of technological applications such as oil recovery, building materials, soil science, textile and hydrology [1]. Due to this diversity of applications, an infinity of theoretical [2-6] and experimental [7-14] studies have been carried out based on the pioneering works of Lucas [15] and Washburn [16] in order to understand the imbibition mechanism and the related phenomena (for review, see [1,17,18], and references therein).

It was experimentally demonstrated that the imbibition speed becomes slower than that of Lucas-Washburn Regime ($x \propto \sqrt{t}$) [7-14] as a consequence of the heterogeneity of flow in porous media due to the complex network of randomly distributed pores that connect to each other forming tortuous capillaries through which fluid flows. This tortuous path has a fractal character and the Lucas-Washburn equation is scarce to model imbibition process in permeable materials that possess fractal architecture.

Li and Zhao [19] added a fractal parameter to the classical model of spontaneous imbibition to describe the heterogeneity of the porous medium as $x \propto t^{d_f-2}$ where $2 < d_f < 3$ is the fractal dimension of the medium. However, Cai and Yu [5] presented a fractal formulation of tortuous flow through the assumption of a fractal tortuous path given by $x \propto t^{1/d_\tau}$, where d_τ is the *fractal dimension of flow tortuosity* to depict the previous experimental results for a scaling exponent within the range of $0.25 \leq \delta = 1/d_\tau < 0.5$. The mentioned fractal approach is used in this work to study of liquid travel distance in the Euclidean case regarding the fractal case and find the time when the equilibrium height is achieved as a function of the fractal dimension of flow tortuosity in tubes with the shape of the Koch curve, Modified Koch curve, Minkowski curve and the "Carpintieri curve" [20].

The objective of this work is to investigate the effects of the fractality in spontaneous imbibition processes on capillaries with paths similar to theoretical curves with exact fractal dimension.

The rest of the Letter is organized as follows. In Sec. 2 the mathematical tools needed in this paper were studied and defined. In Sec. 3 a new permeability parameter and an analytical model to describe the capillary rise by spontaneous imbibition on tubes with deterministic fractal geometry are established. In Sec. 4 an illustrative example in order to discuss some physical implications has been solved. In Sec. 5 the main findings and conclusions are outlined.

2. Basic tools in the fractal imbibition

In what follows, some basic concepts of fractal imbibition in highly tortuous capillaries are presented.

2.1. Governing equations in the Euclidean space

The governing equation of the liquid rise in a vertical capillary tube embedded in the Euclidean space E^1, when the capillary forces are dominant and the radius is very small, is given by a balance in the capillary, viscous and gravitational forces [21,22], respectively represented as:

$$\frac{4\sigma\cos\theta}{\Phi} = \frac{32\mu}{\Phi^2} x\frac{dx}{dt} + \rho g x, \tag{1}$$

where σ is the surface tension of the liquid having viscosity μ, θ is the contact angle between the liquid and the capillary surface, Φ is the capillary diameter, x denotes the distance penetrated by the liquid, t is the imbibition time, ρ is the fluid density and, g is the gravitational acceleration. The process of the liquid rise in a vertically straight capillary tube is obtained from Eq. (1) as $dx/dt = \mathcal{K}(P_c - \rho g x)/\mu x$, where $\mathcal{K} = \Phi^2/32$ is the intrinsic permeability of the tube [1,23].

When an equilibrium is reached, the force that drives the capillary rise $P_c = 4\sigma\cos\theta/\Phi$ equals the weight of the column of liquid, $\rho g x$; therefore the net force on the liquid vanishes and the rising of the liquid stops.

Liquid-air interface equilibrium height is defined as $x_{eq} = P_c/\rho g$; and the liquid rise in the capillary tube between the time of initial contact and the final equilibrium is described by the relation [1,23]:

$$\frac{dx}{dt} = \frac{\mathcal{K}}{\mu}\rho g\left(\frac{x_{eq}}{x} - 1\right). \tag{2}$$

When the gravitational forces can be considered very small, for example when the capillary tube is in the upright position and the penetration of the liquid x is very small, or the gravitational force is equal to zero (when the capillary tube is in the horizontal position), the above mentioned equation reduced to $dx/dt = \Phi\sigma\cos\theta/8\mu x$ and integrating it with the initial conditions $x = 0$, $t = 0$, the following Lucas-Washburn relation holds:

$$x(t) = \left(\frac{2\mathcal{K}P_c}{\mu}\right)^{\frac{1}{2}}(t)^{\frac{1}{2}}, \tag{3}$$

when the gravitational forces are included, integrating Eq. (2) with the initial conditions $x = 0$, $t = 0$, we get:

$$x(t) \sim x_{eq}\left(1 - e^{-t/t_{eq}}\right), \tag{4}$$

where $t_{eq} = -128\sigma\cos\theta\mu/\Phi^3\rho^2 g^2$.

2.2. Governing equations in fractal space

The actual tortuous path of a flow in a porous medium is defined as $x_\tau = \tau\, x$, where τ is the tortuosity. In the [24] a scaling relationship for a flow through heterogeneous media was developed for the actual length x_τ versus the scale of observation ϵ given by $x_\tau = \epsilon^{1-d_\tau}x^{d_\tau}$ where d_τ is the fractal dimension of the flow tortuosity. Expression mentioned implies the property of self-similarity, which means that the value of d_τ is constant for a range of length scales $\varepsilon_c \le x \le \ell$, where $\varepsilon_c > 0$ is the lower cutoff with value of the order of an average pore radius and ℓ is the upper cutoff that describes the straight-line distance that the particle travels between the starting and the ending points of fractal (tortuous) path (see Fig. 1). Hence the diameters of capillaries are analogous to the length scale ϵ [25,26] and the fractal scaling relationship between the diameter Φ and the length of capillaries x and x_τ in a porous medium can be written as:

$$x_\tau = x\left(\frac{x}{\Phi}\right)^{d_\tau - 1}, \tag{5}$$

where

$$1 \le d_\tau = 1 + \frac{\ln\tau}{\ln(x/\Phi)} < 2. \tag{6}$$

When $d_\tau = 1$ the capillary tube is straight and the one-dimensional Euclidean case is presented, meanwhile for $d_\tau = 2$ the capillary tube is a highly tortuous line so irregular that fills a two-dimensional space. The capillaries with tortuous path are embedded in the Euclidean space E^2.

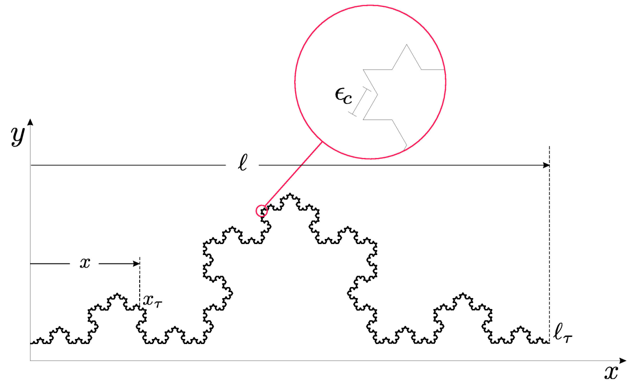

FIGURE 1. Fractal parameters of scale invariance for d_τ. Note that $\ell_\tau \to \infty$ as $\epsilon_c \to 0$.

Differentiating Eq. (5) with respect to time t for a single capillary results in [24]:

$$v_\tau = d_\tau\Phi^{1-d_\tau}x^{d_\tau - 1}v, \tag{7}$$

where $v_\tau = dx_\tau/dt$ is the actual velocity of liquid through distance of a tortuous capillary; $v = dx/dt$ is the straight-line imbibition velocity; and as for the Euclidian case $d_\tau = 1$, it holds that $v_\tau = v$. The scaling ratio between both velocities can be rewritten as

$$\frac{dx_\tau}{dt} = d_\tau\Phi^{1-d_\tau}x^{d_\tau - 1}\frac{dx}{dt}. \tag{8}$$

When a wetting liquid is contacted with a tortuous capillary of any shape (Euclidean or Fractal), the capillary rise is described as [5,16]:

$$\frac{dx_\tau}{dt} = \frac{\mathcal{K}}{\mu}\frac{x}{x_\tau}\rho g\left(\frac{x_{eq}}{x} - 1\right). \tag{9}$$

Substituting Eqs. (5) and (8) in (9) the following differential equation that governs the capillary rise in a tortuous capillary is obtained :

$$\frac{dx}{dt} = \frac{\mathcal{K}}{\mu}\frac{\rho g}{d_\tau\, x^{2d_\tau - 2}\Phi^{2-2d_\tau}}\left(\frac{x_{eq}}{x} - 1\right). \tag{10}$$

If the gravitational force is negligible, Eq. (10) is reduced to:

$$\frac{dx}{dt} = \frac{\mathcal{K}}{\mu\, d_\tau\Phi^{2-2d_\tau}}\frac{P_c}{x^{2d_\tau - 1}}, \tag{11}$$

with the initial conditions $x = 0$, $t = 0$. Integration leads to Lucas-Washburn-Cai Equation [5]:

$$x(t) = \left(\frac{2\mathcal{K}P_c}{\Phi^{2-2d_\tau}\mu}\right)^{\frac{1}{2d_\tau}} t^{\delta = \frac{1}{2d_\tau}}, \tag{12}$$

and when $d_\tau = 1$ the last expression is reduced to the classical of Lucas-Washburn equation.

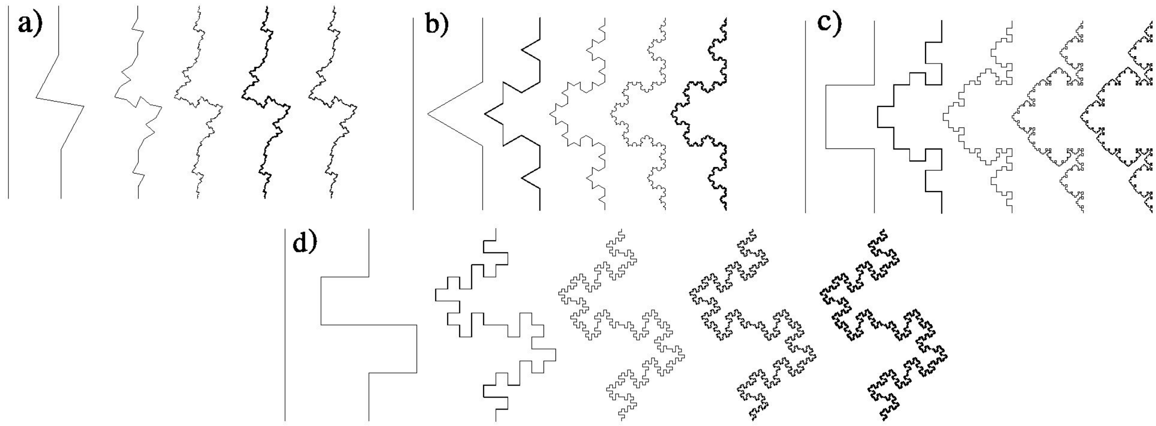

FIGURE 2. Capillary tubes with deterministic fractal geometries for first 6 iterations. a) Modified Koch curve [27]; b) Classic Koch curve; c) Carpintieri curve; d) Cuadratic Koch curve or "Minkowski sausage".

TABLE I. Properties of capillary tubes with deterministic fractal geometries, unitary porosity, constant diameter $\Phi = 0.001$ cm and vertical height $x = 64$ cm.

Parameter	Fractal curve			
	Koch	MK[a]	Carpintieri	Minkowski
d_f	001.261	001.161	001.340	0001.500
d_{τ_0}	001.000	001.000	001.000	0001.000
d_{τ_1}	001.025	001.020	001.036	0001.062
d_{τ_2}	001.051	001.040	001.073	0001.125
d_{τ_3}	001.077	001.060	001.109	0001.187
d_{τ_4}	001.103	001.090	001.146	0001.250
d_{τ_5}	001.129	001.100	001.183	0001.313
x_{τ_0}	064.000	064.000	064.000	0064.000
x_{τ_1}	085.330	080.000	096.000	0128.000
x_{τ_2}	113.770	100.000	144.000	0256.000
x_{τ_3}	151.703	125.000	216.000	0512.000
x_{τ_4}	202.271	156.250	324.000	1024.000
x_{τ_5}	269.695	195.310	486.000	2048.000

[a] Modified Koch curve.

3. Capillary rise on the linear fractals

Fractal propierties of capillary tubes with deterministic fractal geometries are obtained (see Fig. 2) for the first six iterations of each linear fractal including their fractal dimension, d_f, as it is shown in Table I.

The generalized permeability for a tortuous capillary tube is given by [28-31]:

$$\mathcal{K} = \frac{\Phi^2}{32}\frac{\phi}{\tau}. \tag{13}$$

However, the fractal approach to capillary cylinder permeability with fractal geometry is obtained inserting Eq. (5) in Eq. (13):

$$\mathcal{K}_\tau = \frac{\Phi^{d_\tau+1}}{32x^{d_\tau-1}}, \tag{14}$$

and for capillary tubes with unitary porosity, $\phi = 1$. Where $\mathcal{K}_\tau$, is a permeability ratio for permeable media with scale invariance in materials where a pre-fractal pore network exists [32].

It was shown that the equilibrium height x_{eq} is the same for both, straight tubes and tubes with tortuous path. Substituting x_{eq}, τ and $\mathcal{K}_\tau$ in Eq. (10) and the following expression that describes the capillary rise by spontaneous imbibition in tubes with fractal geometry is obtained:

$$\frac{dx}{dt} = \frac{\mathcal{K}_\tau \rho g}{\mu\, d_\tau x^{d_\tau-1}\Phi^{1-d_\tau}}\left(\frac{x_{eq}}{x} - 1\right). \tag{15}$$

When $d_\tau = 1$, Eq. (15) reduces to Eq. (2). In the first stage of imbibition the liquid rise in the capillary tube is given by $x(t) \propto t^\delta$ where the time exponent $\delta = 1/2d_\tau$ and the second stage of imbibition can be found solving numerically Eq. (15). The fractal dimension of flow tortuosity increases with the increase of the capillary tubes tortuosity, meanwhile in the case of tubes with a shape similar to linear fractals shown in Table I, the tortuosity increases with the increase of the fractal curve iteration (see Fig. 3) as described in Eq. (6), or it can be calculated by the following relation:

$$d_\tau = \lambda_i \ln\tau + 1, \tag{16}$$

where $\lambda_i = 0.2327709525\Phi^{0.1667743386}$ (see the insert of Fig. 3 where the slope λ_i depends on the capillary diameter by the power law).

4. Example in Koch's curve-like capillary tubes

In this section an illustrative example to clarify the physical implications of the introduced models is presented.

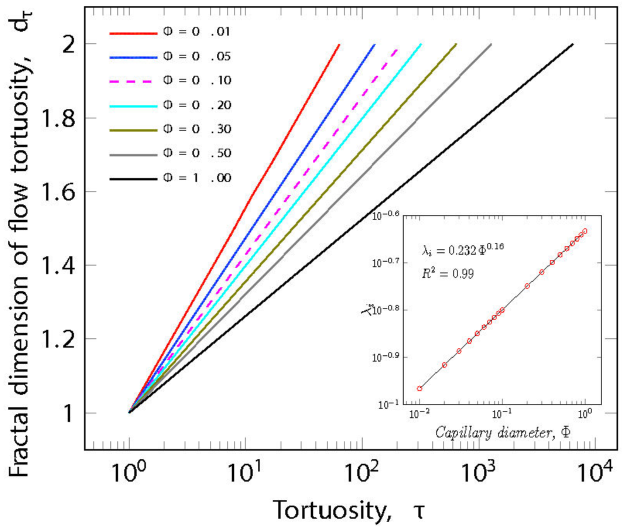

FIGURE 3. Semi-log plot τ *vs.* d_τ for different capillary diameters of linear fractals with logarithmic fittings, where $d_\tau = slope\ \ln(\tau) + 1$ represented by straight lines. Dashed line shows a graph obtained from linear fractals (that is in agreement with Fig. 1 of Ref. 5. Insert shows data of Fig. 3 *versus* capillary diameter, fitted curve for linear regression analysis.

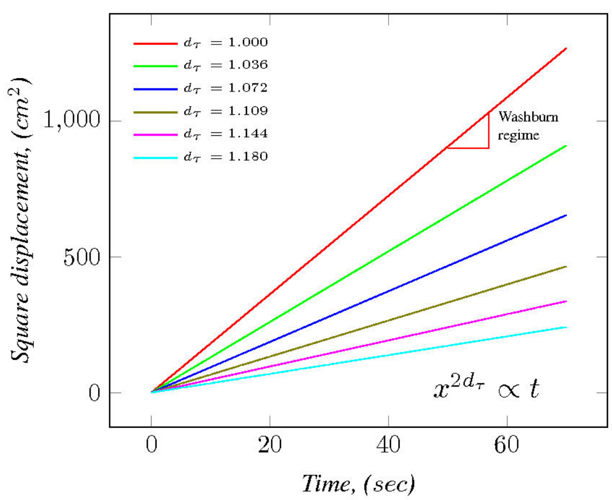

FIGURE 4. Comparison between the Euclidean case and fractal case in the first stage of imbibition. Graph shows Washburn Regime in Euclidean case ($d_\tau = 1$) while the fractal case ($d_\tau > 1$) evolves slower then the Washburn regime.

Consider a tortuous flow path, similar to a classic Koch curve, shown in Figure 1.b, with the following mechanical properties: $\theta = 0°$, $\rho = 0.998$ g/cm^3, $\mu = 0.01$ dina sec/cm^2, $\sigma = 725$ dina/cm and $x = 64$ cm for the first ($i = 0, 1, \cdots, 5$) iterations. The results of the first stage of spontaneous imbibition, are in agreement with the time scaling exponent $\delta = 0.5 d_\tau^{-1}$ as it is shown in Fig. 4. The influence of the fractality is reflected only by the fractal dimension of flow tortuosity, meanwhile the fractal dimension of the deterministic fractal-like capillary tube does not influence on the behaviour of the flow.

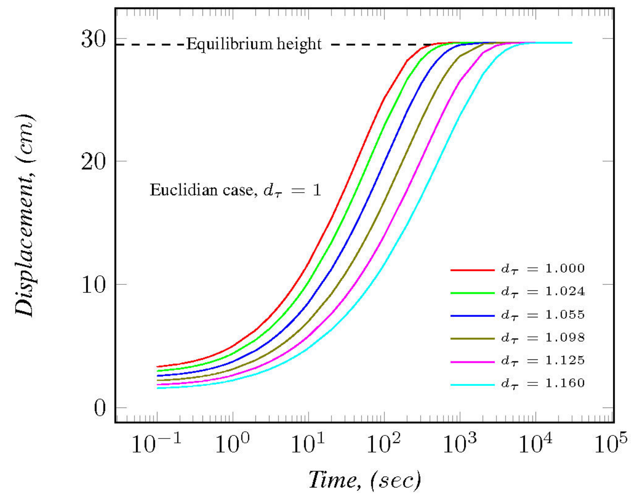

FIGURE 5. Comparison between Euclidean case and fractal case in the second stage of imbibition. Travel distance as function of time, for different iterations of the basic Koch's curve-like capillary tube.

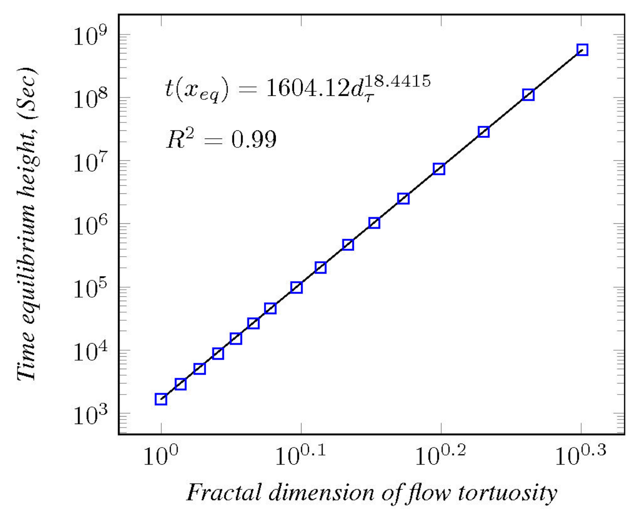

FIGURE 6. Time equilibrium height *versus* fractal dimension of flow tortuosity for $1 \leq d_\tau < 2$.

The flow behaviour in the second stage of imbibition, is obtained solving Eq. (15), where the equilibrium height is always the same and does not depend on the fractal dimension of the flow tortuosity (see Fig. 5). However, the time that fluid requires to reach the equilibrium height depends directly on d_τ, as:

$$t(x_{eq}) \propto d_\tau^{\alpha}, \tag{17}$$

where $\alpha = 18.44$, as in Fig. 6.

The previous results are common for ideal liquids, when the contact angle is equal to zero. The capillary rise is directly affected by the hydrostatic effect depending on the contact angle. This effect is calculated with the information of Fig. 4 according to Lucas-Washburn Eq. (3) and such effects are shown in Fig. 7. In an upcoming report the experimental results of fractal imbibition in tortuous Koch's curves-like capillary tubes will be given.

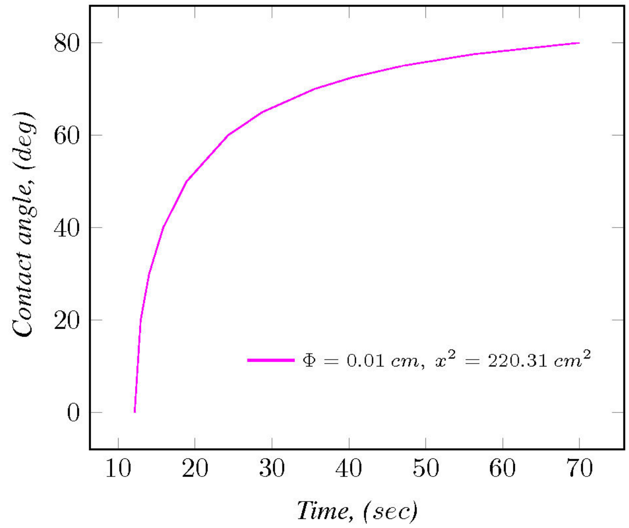

FIGURE 7. Contact angle as time function deduced from the Figure 4 data according to Eq.(3) and the parameters from Table I.

5. Conclusions

In this paper the permeability relation for flow paths in capillary tubes with fractal geometry was generalized. This relation was used in Eq. (10) (Jian Chao Cai Equation) to describe the spontaneous fractal imbibition model given by Eq. (15). Also it was found that the fractal dimension of flow tortuosity increases, as the tortuosity and the capillary diameters of the cylinders increases, and does not depend on the fractal dimension of capillary tube (see Eq. (16)). An illustrative example of capillaries with the shape of linear fractal of Koch curve was presented. Results obtained are in agreement with the previously reported findings standard calculations and with the Lucas-Washburn-Cai equation, showing that as the fractality increases the penetration distance decreases. Finally, it was found that the necessary time to reach the equilibrium height of the tortuous capillary tube in the second stage of spontaneous imbibition is a function of the *fractal dimension of flow tortuosity* such that $t(x_{eq}) \propto d_\tau^\alpha$.

This results provide a more detailed description of the physical phenomena of the spontaneous imbibition in linear fractal-like capillary tubes.

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