Lorentzian surfaces and the curvature of the Schmidt metric

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The b-boundary is a mathematical tool used to attach a topological boundary to incomplete Lorentzian manifolds using a Riemaniann metric, called the Schmidt metric, on the frame bundle. In this paper we give the general form of the Schmidt metric in the case of Lorentzian surfaces. Furthermore, we write the Ricci scalar of the Schmidt metric in terms of the Ricci scalar of the Lorentzian manifold and give some examples. Finally, we discuss some applications to general relativity.

Keywords: Gravitational singularities; Lorentzian surfaces; Schmidt metric; general relativity; b-boundary.

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1. Introduction

One of the biggest surprises that General Relativity (GR) has given us is that under certain circumstances the theory predicts its own limitations. There are two physical situations where we expect the theory to break down. The first one is the gravitational collapse of certain massive stars when their nuclear fuel is spent. The second one is the far past of the universe when the density and temperature were extreme. In both cases, we expect that the geometry of spacetime will show some pathological behaviour.

The nature of a gravitational singularity is a delicate issue. It might be tempting to define a gravitational singularity following other physical theories (such as electromagnetism) as the location where the relevant physical quantities are undefined. However, in the gravitational case this prescription does not work due to the identification of the spacetime background with the gravitational field. As a result, the concepts of 'spatial location' and 'temporal duration' have meaning only in the domain where the gravitational field is defined. This represents a problem because the size, place and shape of singularities can not be straightforwardly characterised by any physical measurement.

The first mathematical description of a gravitational singularity comes from Penrose and Hawking seminal theorems. They characterised singularities as obstructions to geodesic completeness and managed to show that this happens under certain conditions [1]. Broadly speaking, the theorems establish that a spacetime $(\mathcal{M}, g)$ that satisfies simultaneously:

- a condition on the curvature,
- an appropriate initial or boundary condition,
- and a global causal condition,

must be geodesically incomplete [2].

One would like to attach a boundary to the incomplete spacetime to understand the singularity better. The procedure to attach a boundary to a Lorentzian manifold can be done in several nonequivalent ways. In this work we will focus on the b-boundary method [3]. This method allows a classification of singularities in terms of parallel propagated frames, it distinguishes between points at infinity and points at a finite distance, and it generalizes the idea of affine length to all curves regardless of them being geodesic or not. Other common techniques to attach boundaries to Lorentzian manifolds are conformal boundaries [1, 4] and the causal boundaries [1, 5] which we describe below. In addition, we would like to mention other constructions such as the a-boundary [6, 7] and boundaries constructed from light rays [8].

The conformal boundary allows us to study the structure of the metric at "infinity". The idea of conformal compactification is to bring points at "infinity" on a non-compact pseudo-Riemannian manifold $(\mathcal{M}, g)$ to a finite distance (in a new metric) by a conformal rescaling of the metric $\tilde{g} = \Omega^2 g$. This precise definition of conformal compactification only applies to an asymptotically simple spacetime. $(\mathcal{M}, g)$ is asymptotically simple, if there are another smooth Lorentz manifold and associated metric $(\tilde{\mathcal{M}}, \tilde{g})$ such that:

- $\mathcal{M}$ is an open sub-manifold of $\tilde{\mathcal{M}}$ with smooth boundary $\partial\mathcal{M}$ called the conformal boundary;
- there exists a smooth scalar field Ω on $\tilde{\mathcal{M}}$ such that $\tilde{g} = \Omega^2 g$ on $\mathcal{M}$, and $\Omega = 0, d\Omega \neq 0$ on $\partial\mathcal{M}$;
- every null geodesic in $\mathcal{M}$ acquires a future and past endpoints on $\partial\mathcal{M}$.

This technique has the evident drawback that it can only be applied to this kind of spacetimes [4]. Moreover, notice that in Minkowski spacetime the conformal boundary is given by $\partial\mathcal{M} = \mathscr{I}^- \cup \mathscr{I}^+$ (where $\mathscr{I}^-$ corresponds to null-past infinity and $\mathscr{I}^+$ corresponds to null-future infinity) while i^o, i^+, i^- which correspond to spacelike infinity, future timelike infinity, and past timelike infinity respectively do not belong to the conformal boundary (the thorough reader can find in [1] formal definitions of $\mathscr{I}^-, \mathscr{I}^+, i^o, i^+, i^-$). The reason for this is because $\partial\mathcal{M}$ is not a smooth manifold at these points. Despite this, the conformal boundary has been successfully applied to study isolated systems in General Relativity [4] and to the AdS-CFT correspondence [9, 10].

On the other hand, the causal boundary of a spacetime consists on attaching a boundary that depends only on the causal structure. However, this implies on this particular construction that one is not able to distinguish between boundary points and points at infinity. Moreover one has to assume that $(\mathcal{M}, g)$ is strongly causal. This construction relies on indecomposable past sets (IP) and indecomposable future sets (IF), which we now define. An open set U is an IP if it satisfies $I^-(U) \subset U$ and cannot be expressed as the union of two proper-open subsets V and W, satisfying $I^-(V) \subset V$ and $I^-(W) \subset W$ respectively. Similarly using I^+ one can define IF. The class of IPs can be divided into two classes: proper IPs (PIPs) which are of the form $I^-(p)$ for $p \in \mathcal{M}$, and terminal IPs (TIPs) which are not formed by the history of any point in $\mathcal{M}$. We shall denote by $\overbrace{\mathcal{M}}$ the set of all IPs of the space $(\mathcal{M}, g)$ and $\underbrace{\mathcal{M}}$ the set of all IFs of the space $(\mathcal{M}, g)$. Originally, one defines a suitable topology on $\overbrace{\mathcal{M}}, \underbrace{\mathcal{M}}$ to identify IFs and IPs and one can form a space $\mathcal{M}^* = \mathcal{M} \bigcup \Delta$ where Δ is called the causal boundary [1]. However, this topology presents some problems that have led to several redefinitions. A full revision of the causal boundary and its relationship with the conformal boundary can be found in [5]. Also its relation with boundaries in Riemannian and Fislerian manifolds can be found in [11].

There is also the b-boundary. This is a method developed by Schmidt, which allows one to attach a boundary $\partial\mathcal{M}$ called the *b-boundary* to any incomplete spacetime $\mathcal{M}$ (or even to any manifold with a connection). The procedure consists on constructing a Riemannian metric for the frame bundle $L(\mathcal{M})$ or the orthonormal bundle $O(\mathcal{M})$, called the Schmidt metric. This metric is then used to generalise the idea of affine length to all curves. This generalisation is important because it helps to unify some elements of Riemannian geometry with Lorentzian geometry. For example, while only in the Riemannian case geodesic completeness implies that every curve is complete; the notion of b-completeness implies completeness of every curve in both signatures. The definition of a curve we are using here is a piecewise-C^1 curve.

The structure of this paper is as follows. In Sec. 2 we give a general overview of the mathematical preliminaries needed. In Sec. 3 we describe how the Schmidt metric and the b-boundary are constructed following the procedure by Schmidt [3]. In Sec. 4 we discuss the geometry of the orthonormal-bundles for $1+1$ conformally flat spacetimes. In the last section, namely Sec. 5, we discuss the results in the context of gravitational singularities in general relativty.

2. Preliminaries

As a first step, let us present some of the required concepts of differential geometry. We present the basic concept of fibre bundles, G-principal bundles, solder forms and connections. The manifolds we consider in this paper are paracompact, C^∞, connected, and Hausdorff.

2.1. Fibre Bundles and G-Principal Bundles

A *Fibre bundle* with fibre $\mathcal{F}$ is a manifold E with a surjetive map $\pi : E \to \mathcal{M}$ where there is a neighbourhood $\mathcal{U}$ at each point p of $\mathcal{M}$ such that $\pi^{-1}(\mathcal{U})$ is isomorphic to $\mathcal{U} \times \mathcal{F}$, *i.e.*, for each point $p \in \mathcal{U}$ there is a diffeomorphism ϕ_p of $\pi^{-1}(p)$ onto $\mathcal{F}$ such that the map $\psi(\overline{p}) = (\pi(\overline{p}), \phi_{\pi(\overline{p})})$ is a diffeomorphism. We call $\mathcal{M}$ the base space of the fibre bundle E.

A *G-principal bundle* P over a manifold $\mathcal{M}$ is a fibre bundle where the fibre is a lie group G with a continuous right action R_g that acts freely:

$$(\overline{p}, g) \in P \times G \to R_g(\overline{p}) \in P$$

and satisfies that $\mathcal{M}$ is the quotient space of P by the equivalence relation induced by G [12].

Let $\mathcal{M}$ be a n-dimensional manifold. A *frame* $\{\mathbf{E}^a\}_p$ at point p is an ordered basis of T_p. Let $L(\mathcal{M})$ be the set of all frames $\{\mathbf{E}^a\}$ at all points on $\mathcal{M}$ with the projection π sending a frame at p to p. Then the *general linear group* $GL(n, \mathbb{R})$ has a natural action on $\{\mathbf{E}^a\}$, i. e., given $(\{\mathbf{E}^a\}, A^b_a)$ the action of $A^b_a \in GL(n, \mathbb{R})$ on $\{\mathbf{E}^a\}$ is $\{\mathbf{E}^b = A^b_a \mathbf{E}^a\}$. If $\{x^a\}$ are coordinates on $\mathcal{M}$ and we choose the frame $\{(\partial/\partial x^a)\}$, then it can be shown that the coordinates (x^a, β^a_b) are a local coordinate system of $L\mathcal{M}$, where β^a_b represent the ab element for the change of basis matrix β between $\{(\partial/\partial x^a)\}$ and any other frame $\{\mathbf{E}^b\}$. In fact this choice makes $L(\mathcal{M})$ a G-principal bundle called the *frame bundle*. Moreover, if we have a metric in $\mathcal{M}$ and we restrict the frames to just orthonormal frames, we obtain another G-principal bundle called the *orthonormal frame bundle* $O(\mathcal{M})$. The associated Lie Group to $O(\mathcal{M})$ is then the *orthonormal group*, $SO(n, \mathbb{R})$ or $SO^+(1, n, \mathbb{R})$. This last definition is signature dependent.

Every tangent space $T_{\overline{p}}P$ of a G-principal bundle P has a subspace called the *vertical subspace* $V_{\overline{p}}$. This subspace is given by the kernel of the differential $D\pi$ restricted at $\overline{p}$. Explicitly,

$$V_{\overline{p}} = \{\mathbf{X} \in T_{\overline{p}}P | D\pi_{\overline{p}}(\mathbf{X}) = 0 \in T_{\pi(p)}\mathcal{M}\}.$$

2.2. Solder form and Connections

The *solder form* of a frame bundle $L(\mathcal{M})$ is the map:

$$\theta : TL(\mathcal{M}) \to \mathbb{R}^n : Q \to \overline{p}^{-1}(D\pi(Q))$$

where Q is an element of $T_{\overline{p}}L\mathcal{M}$ and $\overline{p}$ is a linear map from $\mathbb{R}^n$ to $T_{\pi(\overline{p})}$ that sends the canonical vectors to a choice of basis on $T_{\pi(\overline{p})}$. The solder form for the orthonormal bundle $O(\mathcal{M})$ is defined similarly. Notice that $V_{\overline{p}} \subset ker(\theta)$.

A *connection* $\overline{\nabla}$ on a G-principal bundle is an assignment of a subspace $H_{\overline{p}}$ called the *horizontal subspace* of $T_{\overline{p}}(P)$ for all $\overline{p}$ in P, such that:

- $T_{\overline{p}}P = H_{\overline{p}} \oplus V_{\overline{p}}$, and
- $H_{\overline{p}g} = D_{\overline{p}}R_g(H_{\overline{p}})$ for every $\overline{p} \in P$ and $g \in G$.

A *connection form* ϖ of a connection $\overline{\nabla}$ in a G-principal bundle is a C^∞ map

$$\varpi : TP \to \mathfrak{g}$$

with the following properties:

- if $\varpi(X) = 0$ then $X \in H_{\overline{p}}$ for some $\overline{p}$ in P,
- for all g in G and all C^∞ maps $X : P \to TP$ $\varpi(DR_g(X)) = ad_*(g^{-1})\varpi(X)$, and
- for all $\vec{g} \in \mathfrak{g}, \varpi(X^*_{\overline{p}}) = \vec{g}$ where X^*_p is the tangent vector at $t = 0$ of a curve given by $\gamma(t) = R_{\exp t\vec{g}}(\overline{p})$.

Let us remind the reader that connections and connection forms uniquely determine one another.

In coordinates, the connection form ϖ is written as $\varpi = \sum_{a,b} \varpi^a_{\ b}\mathbf{F}^a_{\ b}$ where

$$\varpi^a_{\ b} = \sum_c \left((\beta^{-1})^a_{\ c} d\beta_b^{\ c} + \sum_{d,e} (\beta^{-1})^a_{\ c}\Gamma^c_{\ de}\beta^e_{\ b}dx^d \right), \quad (1)$$

where $(\beta^{-1})^a_{\ c}$ is the inverse of the matrix $\beta^a_{\ c}$ and $\Gamma^a_{\ bc}$ are the Christoffel symbols.

The solder form θ is then given by $\theta = \sum_a \theta^a \mathbf{e}^a$ where

$$\theta^a = \sum_c (\beta^{-1})^a_{\ c} dx^c, \quad (2)$$

and $\mathbf{e}^a$ is the natural basis of $\mathbb{R}^n$ [12].

3. The Schmidt metric

If one thinks of a singularity in classical Newtonian gravity, the statement that the gravitational field is singular at a certain location is unambiguous. As an example, take the gravitational potential of a spherical mass M in Cartesian coordinates

$$V(t, x, y, z) = \frac{GM}{\sqrt{x^2 + y^2 + z^2}},$$

where G is the gravitational constant, and the potential exhibits a singularity at the point $x = y = z = 0$, for any time t in $\mathbb{R}$. The location of the singularity is well defined because the coordinates have an intrinsic character which is independent of V.

However, in the case of GR the prescription given above can not work. This is due to the identification of the background spacetime with the gravitational field. Hence, only in the regions where the gravitational field is defined it is meaningful to talk about locations. Consider the spacetime with the line element

$$ds^2 = -\frac{1}{t^2}dt^2 + dx^2 + dy^2 + dz^2,$$

defined on the manifold $\{(t, x, y, z) \in \mathbb{R}\backslash\{0\} \times \mathbb{R}^3\}$. If we say that there is a singularity at the point $t = 0$, we will be speaking too soon for two reasons. The first one is that $t = 0$ is not part of the manifold. It makes no sense to talk about $t = 0$ as a location where the field diverges. The second thing is that the lack of an intrinsic meaning of the coordinates in GR must be taken seriously. By making the coordinate transformation $\eta = \log(t)$, we obtain the line element

$$ds^2 = -d\eta^2 + dx^2 + dy^2 + dz^2$$

on $\mathbb{R}^4$, which is an isometric extension of the previously defined spacetime. This spacetime is, of course, Minkowski spacetime which is non-singular [1].

Another idea is trying to define a singularity in terms of invariant quantities, such as invariant scalars. The reason for this is that if these quantities diverge then it matches our physical idea that objects must suffer stronger and stronger deformations as we encounter the singularity. These scalars are usually constructed from contractions of the Riemann tensor and its derivatives. Unfortunately, these scalars are not well-suited to define the complete geometry. Consider the metric

$$ds^2 = dudv + H_{ij}(u)x^i x^j du^2 - dx^i dx^i,$$

given in the coordinates (u, v, x^1, x^2) and where $H(u)$ is C^1. This spacetime is known as a pp-wave spacetime and it can be shown that every polynomial curvature-scalar vanishes, despite the fact that in general the spacetime is not flat [13].

A more troublesome feature of using scalars for defining singularities is that they are 'too local' in the sense that they are evaluated at given points. Therefore, if the point is removed, the scalar cannot be computed directly and we need an approximation procedure.

A precise mathematical way to approximate the "missing points" is to use convergent sequences of points on the manifold. In this case the formal statement is: "The sequence $\{R(x_n)\}$ diverges while the sequence $\{x_n\}$ converges to y", where $R(x_n)$ is some scalar curvature invariant evaluated at x_n in $\mathcal{M}$ and y is some point not necessarily in $\mathcal{M}$.

In Riemannian geometry, the notion of distance allows us to define Cauchy sequences $\{x_n\}$ and therefore a notion of convergence. Moreover, if every Cauchy sequence converges in $\mathcal{M}$ then every geodesic can be extended indefinitely. This means we can take the domain of every geodesic to be $\mathbb{R}$. In this case, we say that $\mathcal{M}$ is *geodesically complete*. In fact, the converse is also true: if $\mathcal{M}$ is geodesically complete then $\mathcal{M}$ is metrically complete, *i. e.*, every Cauchy sequence converges to a point in $\mathcal{M}$ [14]. This allows us to use Cauchy sequences or sequences of points along geodesics as our sequences of points.

The Riemannian case is an useful example, but as soon as we move to Lorentzian geometry, which we take as the correct geometrical setting for GR, the previous discussion cannot be used as stated. The reason is that Lorentzian metrics do not have a distance function defined and, therefore, Cauchy sequences cannot be defined. Thus, one is restricted to the notion of geodesically complete manifolds in the Lorentzian case.

Moreover, the existence of three kinds of vectors available in any Lorentzian metric defines three nonequivalent notions of geodesic completeness —depending on the character of the tangent vector of the curve— spacelike completeness, null completeness and timelike completeness, which are, unfortunately, not equivalent. It is possible to construct spacetimes with the following characteristics [15, 16, 19]:

- timelike complete, spacelike and null incomplete,
- spacelike complete, timelike and null incomplete,
- null complete, timelike and spacelike incomplete,
- timelike and null complete, spacelike incomplete,
- spacelike and null complete, timelike incomplete, or
- timelike and spacelike complete, null incomplete.

Furthermore, there are examples of a geodesically null, timelike and spacelike complete spacetimes with an inextendible timelike curve of finite length [16, 19]. A particle following this trajectory will experience bounded acceleration and in a finite amount of proper time its spacetime location would stop being represented as a point in the manifold.

In order to overcome this, Schmidt provided an elegant way to generalise the idea of affine length to all curves, regardless of such curves being geodesic or not. This construction in the case of incomplete curves allows to attach to the spacetime $\mathcal{M}$ a topological boundary $\partial\mathcal{M}$ called the b-boundary. The procedure for constructing the Schmidt metric consists in building a Riemannian metric in the frame bundle $L(\mathcal{M})$. We use the solder form θ on $L(\mathcal{M})$ and the connection form ϖ on $L(\mathcal{M})$ associated to the Levi-Civita connection ∇ on $\mathcal{M}$ to do this. Explicitly, the Schmidt metric is given by

$$\overline{g}(\mathbf{X},\mathbf{Y}) = \theta(\mathbf{X})\cdot\theta(\mathbf{Y}) + \varpi(\mathbf{X})\bullet\varpi(\mathbf{Y}), \qquad (3)$$

where $\mathbf{X},\mathbf{Y}\in T_{\overline{p}}P$ and $\cdot,\bullet$ are the inner products in $\mathbb{R}^n$ and $\mathfrak{g}\cong\mathbb{R}^{n^2}$ respectively. The construction of the Schmidt metric is more general and can be applied to any manifold with a connection. This connection does not necessarily need to be a metric compatible connection [3]. However, as mentioned above, we will use the Levi-Civita connection because we will always assume a metric on the manifold.

Let $\gamma:[a,b]\to\mathcal{M}$ be a piecewise-C^1 curve through p in $\mathcal{M}$. A curve $\overline{\gamma}:[a,b]\to L(\mathcal{M})$ in $L(\mathcal{M})$ is called the *lift of the curve* γ if it satisfies $\pi(\overline{\gamma})=\gamma$ and $D\pi(\dot{\overline{\gamma}})=\dot{\gamma}$. The length of $\overline{\gamma}$ with respect to the Schmidt metric is

$$L_{\overline{\gamma}}(b) = \int_a^b \|\dot{\overline{\gamma}}(\eta)\|_{\overline{g}}\, d\eta,$$

which is called the *generalised affine-length* of γ. We can then use this to re-parametrise γ which generalises the notion of an affine parameter. In the case where γ is a geodesic parametrised by $L_{\overline{\gamma(t)}}$, it is parametrised with respect to an affine parameter. If every curve in a spacetime $\mathcal{M}$ with finite generalised-affine-length has endpoints, we call this spacetime *b-complete*. If it is not b-complete, we say that the spacetime is *b-incomplete.*

Notice that if there is a curve γ in $\mathcal{M}$ that has finite affine-length and no endpoint, then the lift curve $\overline{\gamma}$ cannot have an endpoint. Otherwise, if $\overline{p}$ is the endpoint of $\overline{\gamma}$, $\pi(\overline{p})=p$ would be an endpoint of γ contradicting the incompleteness of γ. The previous remark shows that geodesic incompleteness implies b-incompleteness. The converse is not true as Geroch's example [16] shows a b-incomplete spacetime that is geodesically complete. Therefore, b-incompleteness is a generalisation of geodesic incompleteness.

Now given an incomplete spacetime $\mathcal{M}$, using the Riemannian metric $\overline{g}$ on $L(\mathcal{M})$, we can 'Cauchy complete' $L(\mathcal{M})$. Let us denote by $\overline{L(\mathcal{M})}$ the Cauchy completion of $L(\mathcal{M})$.

We define the quotient space $\overline{\mathcal{M}}=\overline{L(\mathcal{M})}/G^+$, where G^+ is the connected component of the identity of $GL(n;\mathbb{R})$ under the equivalence of orbits, *i.e.*, $(\overline{p},g)\in\overline{L(\mathcal{M})}\sim(\overline{q},g')\in\overline{L(\mathcal{M})}$ if $\overline{p}=\overline{q}$ and there is $h\in GL(n;\mathbb{R})$ such that $g=hg'$. This quotient induces a topology in $\overline{\mathcal{M}}$ by taking the finest topology that makes the map $\pi:\overline{L(\mathcal{M})}\to\overline{\mathcal{M}}$ continuous and, therefore $\overline{\mathcal{M}}$ is a topological space. However, it does not imply that $\overline{\mathcal{M}}$ is a manifold. Finally, we can characterise the b-boundary as the set $\partial\mathcal{M}=\overline{\mathcal{M}}\backslash\mathcal{M}$.

We repeat the same construction for subgroups of $GL(n;\mathbb{R})$. In particular, a common choice in the Lorentzian case is the subgroup of all Lorentz transformations preserving both orientation and direction of time, which is called

the proper orthochronous Lorentz group, and it is denoted by $SO^+(1,n)$. In a completely analogous way we can form the quotient $\overline{\mathcal{M}} = \overline{O(\mathcal{M})}/SO^+(1,n;\mathbb{R})$ and define the b-boundary as the set $\partial\mathcal{M} = \overline{\mathcal{M}}\backslash\mathcal{M}$. The completion using $SO^+(1,n)$ is homeomorphic to the completion using $GL(n;\mathbb{R})$ [17]. The advantage of this construction is that $O(\mathcal{M})$ is a manifold of dimension $n + (n(n-1)/2)$ instead of the $n + n^2$ dimensions of $L(\mathcal{M})$. Also, the construction can be carried in a manifold with a Riemannian metric, in that case $\overline{\mathcal{M}}$ is homeomorphic to the Cauchy completion of $\mathcal{M}$ [23]. This reinforces the conviction that the b-boundary is a natural way to attach boundaries to manifolds with connections.

4. The Schmidt metric of 1+1 spacetimes

In this section, we locally construct the Schmidt metric for general 1+1 spacetimes. Moreover, we find a relationship between the scalar curvature of the Schmidt metric on ($O(\mathcal{M})$, $\overline{g}$) and the scalar curvature of ($\mathcal{M}$, g). Finally, we give several explicit examples.

Notation: We use overlines to denote the Riemannian geometric quantities that belongs to $O(\mathcal{M})$ while the geometric quantities without any overline belong to the Lorentzian manifold $\mathcal{M}$.

4.1. The Schmidt metric for 1+1 conformal spacetimes

Let $\mathcal{M}$ be a 2-D manifold with a Lorentzian metric g and an orthonormal bundle $O(\mathcal{M})$. Then, we can find coordinates (v,w) which locally transform the line element of the metric g to the following form [20]:

$$ds^2 = \Omega^2(v,w)(-dv^2 + dw^2). \tag{4}$$

An orthonormal basis is then given by the vector fields

$$E_1 = \frac{1}{\Omega}\frac{\partial}{\partial v} \quad \text{and} \tag{5}$$

$$E_2 = \frac{1}{\Omega}\frac{\partial}{\partial w}. \tag{6}$$

The orthonormal basis prescribed above is not unique. Any other orthonormal basis is of the form

$$\tilde{E}_1 = \cosh\chi\frac{1}{\Omega}\frac{\partial}{\partial v} + \sinh\chi\frac{1}{\Omega}\frac{\partial}{\partial w} \tag{7}$$

$$\tilde{E}_2 = \cosh\chi\frac{1}{\Omega}\frac{\partial}{\partial w} + \sinh\chi\frac{1}{\Omega}\frac{\partial}{\partial v} \tag{8}$$

for some $\chi \in \mathbb{R}$.

Let us notice that the coefficients of such a basis with respect to $(\partial/\partial v, \partial/\partial w)$ define an unique non-singular matrix β with inverse β^{-1}:

$$\beta = \frac{1}{\Omega}\begin{pmatrix}\cosh\chi & \sinh\chi \\ \sinh\chi & \cosh\chi\end{pmatrix}, \quad \text{and}$$

$$\beta^{-1} = \Omega\begin{pmatrix}\cosh\chi & -\sinh\chi \\ -\sinh\chi & \cosh\chi\end{pmatrix}. \tag{9}$$

These matrices are important in the sense that they are useful to define local coordinates on $O(\mathcal{M})$ as follows:

$$\left\{v, w, \frac{1}{\Omega}\left(\cosh\chi\frac{\partial}{\partial v} + \sinh\chi\frac{\partial}{\partial w}\right),\right.$$
$$\left.\frac{1}{\Omega}\left(\cosh\chi\frac{\partial}{\partial w} + \sinh\chi\frac{\partial}{\partial v}\right) | (v,w) \in \mathcal{M}, \chi \in \mathbb{R}\right\}. \tag{10}$$

As stated in Sec. 3, the Schmidt metric $\overline{g}$ for any $\mathbf{X}, \mathbf{Y} \in TO(\mathcal{M})$ on $O(\mathcal{M})$ is given by

$$\overline{g}(\mathbf{X},\mathbf{Y}) : \varpi(\mathbf{X})\cdot\varpi(\mathbf{Y}) + \theta(\mathbf{X})\cdot\theta(\mathbf{Y}) \tag{11}$$

where ϖ is the connection form on $O(\mathcal{M})$ and θ the solder form.

Now let us consider a curve $\overline{\gamma}(s)$ in $O(\mathcal{M})$ given by $\overline{\gamma} : s \in [a,b] \mapsto (v(s), w(s), \beta^a_c(s))$ and evaluate $\theta(\dot{\overline{\gamma}})$ and $\varpi(\dot{\overline{\gamma}})$. Explicitly we have

$$\theta(\dot{\overline{\gamma}}) = \Omega\begin{pmatrix}\dot{v}\cosh\chi - \dot{w}\sinh\chi \\ -\dot{v}\sinh\chi + \dot{w}\cosh\chi\end{pmatrix}, \tag{12}$$

and

$$\varpi(\dot{\overline{\gamma}}) = \begin{pmatrix}0 & \dot{\chi} + \frac{1}{\Omega}((\partial_v\Omega)\dot{w} + (\partial_w\Omega)\dot{v}) \\ \dot{\chi} + \frac{1}{\Omega}((\partial_v\Omega)\dot{w} + (\partial_w\Omega)\dot{v}) & 0\end{pmatrix}, \tag{13}$$

where we have used 1 and 2. Then, the line element for the Schmidt metric using a general inner product can be written as:

$$ds^2 = \Omega^2(v,w)[(a_{11}\cosh^2\chi + a_{22}\sinh^2\chi)dv^2 - 2(a_{11}+a_{22})\sinh\chi\cosh\chi dvdw + (a_{22}\cosh^2\chi + a_{11}\sinh^2\chi)dw^2$$
$$+ 2a_{12}(\cosh 2\chi dvdw - \sinh 2\chi(dv^2 + dw^2))] + (b_{22} + 2b_{24} + b_{44})\left(d\chi + \frac{1}{\Omega}\left(\frac{\partial\Omega}{\partial v}dw + \frac{\partial\Omega}{\partial w}dv\right)\right)^2. \tag{14}$$

where $A = (a_{ij}), B = (b_{ij})$ are symmetric matrixes with positive eigenvalues.

It can be shown that using two different inner products produce two unifomly equivalent metrics [18].

In application it is commonly used the Euclidean innner product which give the line element for the Schmidt:

$$ds^2 = \Omega^2(v,w)(\cosh(2\chi)(dv^2+dw^2) - 2\sinh(2\chi)dvdw) + \left(d\chi + \frac{1}{\Omega}\left(\frac{\partial\Omega}{\partial v}dw + \frac{\partial\Omega}{\partial w}dv\right)\right)^2. \quad (15)$$

We avoid quoting long tensorial expressions for the curvature tensors and give only the result for the Ricci scalar of (15) in terms of Ω and its derivatives, but we have that $\overline{R}$ is given by

$$\overline{R} = -\frac{1}{2\Omega^8}\left((\Omega_{ww}-\Omega_{vv})\,\Omega - \left(\Omega_w^2 - \Omega_v^2\right)\right)^2 - 2. \quad (16)$$

Taking into account that

$$R = -\frac{2}{\Omega^4}((\Omega_{ww}-\Omega_{vv})\Omega - (\Omega_w^2-\Omega_v^2)) \quad (17)$$

This means that Eq. (16) becomes

$$\bar{R} = -\frac{1}{8}R^2 - 2 \quad (18)$$

Notice that in (18) is the relationship between both scalar curvatures. As direct consequence, we can establish the negativity of the Ricci scalar for any Schmidt metric in the Lorentzian signature. In Sec. 5 we give counterexamples that such a condition does not hold in the Riemannian case. Also, Eq. 18 has been obtained using the Levi-Civita connection. Therefore, using another connection, even in the Lorentzian case, may not hold.

Now we calculate such scalar curvatures for some physical spacetimes.

4.2. The Schmidt metric of Minkowski spacetime

We can write the Schmidt metric in the form

$$ds^2 = (dt^2+dx^2)\cosh(2\chi) - 2dtdx\sinh(2\chi) + d\chi^2. \quad (19)$$

Now let us consider the change of coordinates: $t = u+\tilde{v}, x = u-\tilde{v}$ and write:

$$ds^2 = 2(\cosh 2\chi + \sinh 2\chi)du^2 + 2(\cosh 2\chi - \sinh 2\chi)d\tilde{v}^2 + d\chi^2 \quad (20)$$

or in an equivalent manner

$$ds^2 = 2e^{2\chi}du^2 + 2e^{-2\chi}d\tilde{v}^2 + d\chi^2. \quad (21)$$

We explicitly calculate $\overline{R}_{ab}$ and get

$$\overline{R}_{\chi\chi} = -2, \quad (22)$$

and all other components are zero. The Ricci scalar is then

$$\overline{R} = -2. \quad (23)$$

Hence, the geometry in the bundle is not flat even if Minkowski spacetime is flat.

4.3. The Schmidt metric of Friedmann-Robertson-Walker (FRW) spacetime

For simplicity, let us consider the case of the $1+1$ FRW cosmological model which can be obtained from the 4-dimensional one by collapsing two spatial coordinates. This is equivalent to considering the injection map

$$h : (\eta, x) \to (\eta, x, y_0, z_0) : \mathcal{M} \to \mathcal{N} = \mathcal{M}\times\Sigma, \quad (24)$$

where Σ is a suitable two dimensional manifold. This way the four dimensional metric reduces to

$$ds^2 = \eta^q(-d\eta^2 + dx^2), \quad (25)$$

for any value of $q > 0$. The Ricci scalar corresponding to the spacetime described by (25) is

$$R = -q\eta^{-2-q}. \quad (26)$$

From Eq. (15) the Schmidt metric in $O(\mathcal{M})$, for our case study, takes the form

$$ds^2 = \eta^q(\cosh(2\chi)(d\eta^2+dx^2) - 2\sinh(2\chi)d\eta dx) + \left(d\chi + \frac{q}{\eta}dx\right)^2. \quad (27)$$

Using computer algebra we calculated the Ricci tensor for the line element (27). In components it is given by

$$\begin{aligned}
\overline{R}_{\eta\eta} &= \frac{1}{8}q\eta^{-4-q}(4\eta^{2+q} - q\cosh(2\chi)),\\
\overline{R}_{xx} &= \frac{1}{32}q\eta^{-2(3+q)}(q^3 - 16\eta^{4+2q} - 16q\eta^{4+2q}\\
&\quad - 4q(5+2q)\eta^{2+q}\cosh(2\chi)),\\
\overline{R}_{\chi\chi} &= -2 + \frac{1}{8}q^2\eta^{-2(2+q)},\\
\overline{R}_{\eta x} &= \frac{1}{8}q^2(3+q)\eta^{-4-q}\sinh(2\chi),\\
\overline{R}_{\eta\chi} &= \frac{1}{4}q(2+q)\eta^{-3-q}\sinh(2\chi),\\
\overline{R}_{x\chi} &= \frac{1}{16}q\eta^{-5-2q}(q^2 - 16\eta^{4+2q}\\
&\quad - 4(2+q)\eta^{2+q}\cosh(2\chi)).
\end{aligned}$$

And the Ricci scalar is

$$\overline{R} = -2 - \frac{1}{8}q^2\eta^{-2(2+q)}, \quad (28)$$

which can equivalently be obtained from Eq. (18).

4.4. The Schmidt metric of De Sitter and Anti-De Sitter spacetimes

Let us now consider the De Sitter and Anti-De Sitter models and study the behaviour of the corresponding curvature scalars. First, consider the De Sitter case. The two dimensional De Sitter spacetime for closed spatial sections is defined with the line element

$$ds^2 = -d\tau^2 + \alpha^{-2}\cosh^2(\alpha\tau)d\omega^2.$$

To obtain the conformal form we make the change $\tan(\eta/2) = \tanh(\alpha\tau/2)$, which leads to

$$ds^2 = \frac{1}{\alpha^2\cos^2(\eta)}(-d\eta^2 + d\omega^2). \tag{29}$$

In these coordinates (η, ω) De Sitter space is conformal to the static Einstein universe [1]. The Ricci scalar for (29) is then

$$R = 2\alpha^2. \tag{30}$$

Using Eq. (15) we get

$$\begin{aligned} ds^2 = {} & \frac{1}{\alpha^2\cos^2(\eta)}(\cosh(2\chi)(d\eta^2 + d\omega^2) \\ & - 2\sinh(2\chi)d\eta d\omega) + (d\chi - \tan(\eta)d\omega)^2. \end{aligned} \tag{31}$$

The Ricci tensor is computed by taking $\Omega = 1/\alpha\cos(\eta)$. The non-vanishing components are:

$$\begin{aligned} \overline{R}_{\eta\eta} &= -\left(1 + \frac{\alpha^2}{2}\cosh(2\chi)\right)\sec^2(\eta), \\ \overline{R}_{\omega\omega} &= \frac{\alpha^4}{4}(1 + (-1 + 4\alpha^{-4})\cos(2\eta) \\ &\quad - 2\alpha^{-2}\cosh(2\chi))\sec^2(\eta), \\ \overline{R}_{\chi\chi} &= -2 + \frac{\alpha^4}{2}, \\ \overline{R}_{\eta\omega} &= \alpha^2\cosh(\chi)\sec^2(\eta)\sinh(\chi), \\ \overline{R}_{\omega\chi} &= \left(-2 + \frac{\alpha^4}{2}\right)\tan(\eta). \end{aligned}$$

Thus the Ricci scalar is

$$\overline{R} = -2 - \frac{\alpha^4}{2}.$$

Notice that in the limit as $\alpha \to 0$ we recover the Minkowski limit once again.

Now let us look at the Anti-De Sitter spacetime. The two dimensional Anti-De Sitter metric has the line element

$$ds^2 = \frac{1}{\alpha^2 y^2}(-dt^2 + dy^2)$$

with $y > 0$. The Ricci scalar is

$$R = -2\alpha^2. \tag{32}$$

We identify $\Omega = 1/\alpha y$ and using Eq. (15) we obtain

$$\begin{aligned} ds^2 = {} & \frac{1}{\alpha^2 y^2}(\cosh(2\chi)(dt^2 + dy^2) \\ & - 2\sinh(2\chi)dtdy) + \left(d\chi - \frac{1}{y}dt\right)^2. \end{aligned}$$

Where the non-vanishing components of the Ricci tensor are

$$\begin{aligned} \overline{R}_{tt} &= \frac{-2 + \alpha^4 - \alpha^2\cosh(2\chi)}{2y^2}, \\ \overline{R}_{yy} &= -\frac{2 + \alpha^2\cosh(2\chi)}{2y^2}, \\ \overline{R}_{\chi\chi} &= -2 + \frac{\alpha^4}{2}, \\ \overline{R}_{ty} &= \frac{\alpha^2\cosh(\chi)\sinh(\chi)}{y^2}, \\ \overline{R}_{t\chi} &= -\frac{-4 + \alpha^4}{2y}, \end{aligned}$$

and its trace is given by

$$\overline{R} = -2 - \frac{\alpha^4}{2}.$$

Notice that for spacetimes that behave asymptotically as Anti-De Sitter spacetime, the curvature would behave similarly as in the Anti-De Sitter case as one approaches the asymptotic region. Moreover, in many applications such as in the AdS/CFT correspondance one uses a conformal compactification. In those cases it is neccesary to compute the curvature again because the curvature is not a conformal invariant.

5. Discussion

In our exposition, we obtained Eq. (15) which is the line element of the Schmidt metric for all $1+1$ Lorentzian manifolds $\mathcal{M}$. This line element determines, via the curvature, all the local isometric invariants. If $\partial\mathcal{M} = \emptyset$, then the 3-manifold corresponds to the orthonormal bundle where the fibres of the bundle are $SO^+(1,1) \cong \mathbb{R}$. Therefore, $O(\mathcal{M})$ is not compact. If $\partial\mathcal{M} \neq \emptyset$, then the 3-manifold is not necessarily a G-bundle (the group may not act freely or transitively). In fact there are general geometric conditions on the curvature to guarantee that the fibres above a boundary point are degenerate [24]. This is, for example, the case when $\mathcal{M}$ is the Friedmann-Robertson-Walker spacetime. Then $\overline{O(\mathcal{M})}$ is not a G-bundle as the fibre over the singularity is a point instead of a copy of $SO(1,1)^+(\mathcal{M})$. Moreover, if there is a singularity not only in the past but also in the future both singularities are identified as the same boundary point [23]. The degeneracy of the fibre also affects the topology of $\overline{\mathcal{M}}$, which in the Friedmann-Robertson-Walker case is no longer Hausdorff. In fact this topological behaviour is expected in general spacetimes when the fibre totally degenerates such as in

Schwarzschild and Kasner metrics [24]. However, there has been mathematical developments which allow to circumvent this undesirable situation by taking a canonical minimum refinement of the topology in the completion M which $T2$-separates the spacetime M and its boundary ∂M [25]. Notice that this result does not guarantee that points which one may consider physically different such as the initial and final singularity in a closed Friedmann-Robertson-Walker scenarios are not identified. In order to achieve such separation, certain modifications to the completion process have been suggested [23]. The b-boundary has also given some results that link the geometry of principal bundles with that of the base manifold [26] and with non-commutative geometry [27]. Moreover, it has been shown that in four dimensions the Friedmann-Robertson-Walker and Schwarzschild b-completion $\partial\mathcal{M}$ is a point [21, 23, 28].

The notion of b-incomplete spaces allows us to describe incomplete curves in manifolds with connections. Our initial motivation to study this, was to develop the language to describe pathologies in the geometry as we approach points that in some sense are "boundary points" of the manifold. One can describe how the main manifestation of gravity in GR, the curvature of the manifold, can behave along b-incomplete curves. This is the scheme proposed by Ellis and Schmidt to classify singularities [29, 30]. In particular, they defined that if $p \in \partial\mathcal{M}$ and there is some scalar constructed from the tensors g_{ab}, $R^a_{\ bcd}$ and r-covariant derivatives of $R^a_{\ bcd}$ that does not behave in a C^0 way, then p is a C^r *scalar* singularity. Using this definition we have the following result

Theorem 1 *Let $\overline{\gamma}(s) : [0, a] \to O(\mathcal{M})$ be a lift from a curve $\gamma(t) : [0, b] \to \mathcal{M}$ such that $\gamma(b) \in \partial\mathcal{M}$. If $\overline{R} \to -\infty$ as $s \to a$ then $|R| \to \infty$ as $t \to b$ and $\gamma(b)$is a scalar singularity.*

The proof follows directly from Eq. 18.

Notice that the hypothesis of this theorem together with the hypothesis of any of the Hawking and Penrose theorems gives a singularity theorem in which it is guaranteed that curvature blow up exist. This is in contrast with the usual singularity theorem in which only geodesic incompleteness can be shown.

The theorem above and the singularity theorems implicitly assume a characterisation of singularities in terms of incomplete curves. This notion of singularity captures the idea that there are 'obstructions' within the history of point-like observers. In the future one would like extending those theorems to relate these obstructions to curvature blow-up and ill-possessedness of initial value problems of field equations. This approach constitutes most of the research program on the Strong Cosmic Censorship conjecture [31, 32], the idea behind generalised hyperbolicity [33–36] and field regularity [37–42].

Appendix

A. The Riemannian case

As it was mentioned in Sec. 5, in the Riemannian case there are three conformally distinctly connected Riemann surfaces (the disc, the plane and the sphere). Moreover, in this case the fibres in $O(\mathcal{M})$ are $SO(n, \mathbb{R})$ which is a compact group. Below we give the Schmidt metric for the general case of $1 + 1$ Riemannian manifolds and compute the curvature scalar for the disc, the sphere and the hyperbolic plane.

In the Riemannian case it is a well know fact that if $\mathcal{M}$ is a 2-D manifold with a Riemannian metric we can find coordinates (v, w) which transform locally the line element of the metric to a conformally flat form. Therefore, we have

$$ds^2 = \Omega^2(v, w)(dv^2 + dw^2). \tag{A.1}$$

An orthonormal basis is given by the vector fields

$$E_1 = \frac{1}{\Omega}\frac{\partial}{\partial v}, \quad \text{and} \tag{A.2}$$

$$E_2 = \frac{1}{\Omega}\frac{\partial}{\partial w}. \tag{A.3}$$

Any other orthonormal basis is constructed as a linear combination of (A.2) as

$$\tilde{E}_1 = \cos\chi\frac{1}{\Omega}\frac{\partial}{\partial v} + \sin\chi\frac{1}{\Omega}\frac{\partial}{\partial w}, \qquad \text{and} \tag{A.4}$$

$$\tilde{E}_2 = \cos\chi\frac{1}{\Omega}\frac{\partial}{\partial w} + \sin\chi\frac{1}{\Omega}\frac{\partial}{\partial v} \tag{A.5}$$

for some $\chi \in \mathbb{R}$. Let us notice that the coefficients of basis (A.4) with respect to $\frac{\partial}{\partial v}, \frac{\partial}{\partial w}$ define a unique matrix β and its inverse β^{-1}:

$$\beta = \frac{1}{\Omega}\begin{pmatrix} \cos\chi & -\sin\chi \\ \sin\chi & \cos\chi \end{pmatrix} \quad \text{and}$$

$$\beta^{-1} = \Omega\begin{pmatrix} \cos\chi & \sin\chi \\ -\sin\chi & \cos\chi \end{pmatrix}. \tag{A.6}$$

Notice the main difference with the Lorentzian case in the definition of the matrix β.

The Schmidt metric $\overline{g}$ on $O(\mathcal{M})$ is given by

$$\overline{g}(X, Y) : \varpi(X) \cdot \varpi(Y) + \theta(X) \cdot \theta(Y), \tag{A.7}$$

for $X, Y \in TO(\mathcal{M})$. where

$$\theta(\dot{\gamma}) = \Omega\begin{pmatrix} \dot{v}\cos\chi - \dot{w}\sin\chi \\ \dot{v}\sin\chi + \dot{w}\cos\chi \end{pmatrix} \tag{A.8}$$

and

$$\varpi(\dot{\gamma}) = \begin{pmatrix} 0 & -\left(\dot{\chi} + \frac{1}{\Omega}((\partial_v\Omega)\dot{w} + (\partial_w\Omega)\dot{v})\right) \\ \dot{\chi} + \frac{1}{\Omega}((\partial_v\Omega)\dot{w} + (\partial_w\Omega)\dot{v}) & 0 \end{pmatrix} \tag{A.9}$$

giving the line element for the Schmidt metric:

$$ds^2 = \Omega^2(v,w)(dv^2 + dw^2) + \left(d\chi + \frac{1}{\Omega}\left(\frac{\partial\Omega}{\partial v}dw + \frac{\partial\Omega}{\partial w}dv\right)\right)^2 \quad \text{(A.10)}$$

The plane

The euclidean metric on the plane is given by the line element

$$ds^2 = dv^2 + dw^2 \quad \text{(A.11)}$$

which is characterised by $R = 0$.

Then using Eq. (A.10) we have that the line element for the corresponding Schmidt metric is

$$ds^2 = dv^2 + dw^2 + d\chi^2 \quad \text{(A.12)}$$

which is just the flat metric in $O(\mathcal{M})$ so we have $\overline{R} = 0$ which violates the bound given by Eq.18.

The sphere

The round metric on the sphere is given by the line element

$$ds^2 = d\theta^2 + \sin^2(\theta)d\varphi^2. \quad \text{(A.13)}$$

Eq. (A.13) can be expressed in terms of isothermal coordinates (v, w) as

$$ds^2 = \frac{1}{\cosh^2 v}(dv^2 + dw^2). \quad \text{(A.14)}$$

This metric is characterised by $R = 1$.

In a similar manner as we did for the plane metric we use Eq. (A.10) to get the line element for the Schmidt metric:

$$ds^2 = \frac{1}{\cosh^2 v}(dv^2 + dw^2) + (d\chi - \tanh(v)dw)^2, \quad \text{(A.15)}$$

with curvature scalar $\overline{R} = 3/2$. Notice that in this case the curvature scalar is positive which for Lorentzian manifolds can not happen as a result of Eq.18.

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